Name:________________________________ Quiz Score:_____/10

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the square to the left of the question. If there are any other issues, please ask the instructor.

[7] 1. Consider the wave equation

$$u_{tt} = u_{xx}, \qquad x \in (-\infty, \infty)$$

subject to the initial conditions

$$u(x,0) = u_0(x) = \begin{cases} 1 - x^2 & -1 < x < 1 \\ 0 & \text{otherwise,} \end{cases}, \qquad u_t(x,0) = v_0(x) = 0.$$

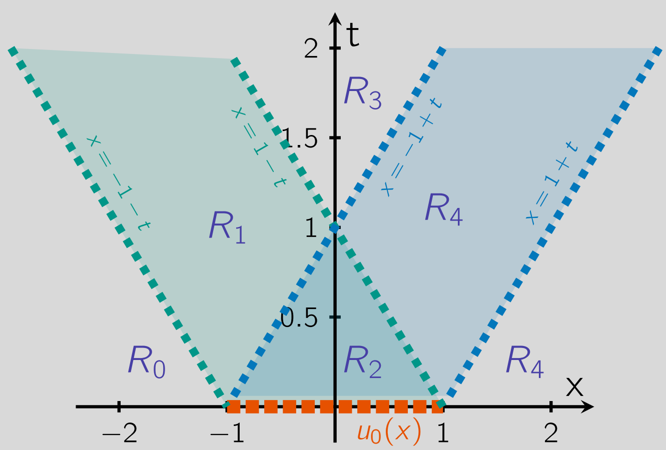
Using D'Alembert's formula, plot some snapshots that depict the behavior of solutions of this PDE.

Solution: In the case that $v_0 = 0$ (and $c = 1$ from the PDE), we see that D'Alembert's reduces to

$$u(x,t) = \frac{1}{2}\left[u_0(x-t) + u_0(x+t)\right].$$

We could try to grind out what this works out to be from the definition of u_0, but it's easier to take a step back and think about what the formula tells us: the solution is the (average) of two waves, one of which moves to the left with velocity 1 and the other moves to the right with velocity 1.

Thus, since we only start with information from $[-1, 1]$, we just need to track how this "window" moves, which is entirely told to us by the two wave solutions. This is what I attempt to depict below.



There's a lot going on in this picture, but let's try to digest it. We know that one of our waves travels left. That is the region in green. It starts at $[-1, 1]$ and travels with a constant rate (slope) 1 to the left.

The same can be said for the blue region: this is our rightward traveling wave, with again (slope = c) velocity 1.

From this, it's clear that 5 regions emerge. We need to just figure out what goes on in each of these regions to fully construct our solution.

We can first identify what goes on in regions R_0, R_3, R_4 quite easily. Since our solution is non-zero in the blue and green, we can safely say

$$u(x,t) = 0, \qquad \text{for } (x,t) \in \{R_0, R_3, R_4\}.$$

We could write down explicit descriptions of these regions if we wanted but this is fine enough for me.

In R_1, we see that this corresponds to just our leftmost traveling wave, meaning here we have

$$u(x,t) = u_0(x-t) = 1-(x-t)^2, \text{ for } \qquad (x,t) \in R_1.$$

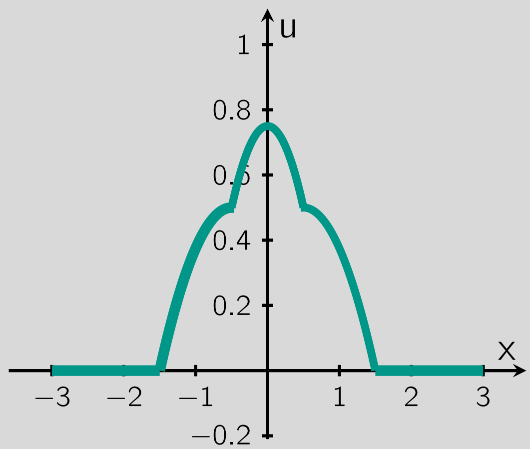
Similarly, for R_2, we have

$$u(x,t) = u_0(x+t) = 1-(x+t)^2, \text{ for } \qquad (x,t) \in R_2.$$

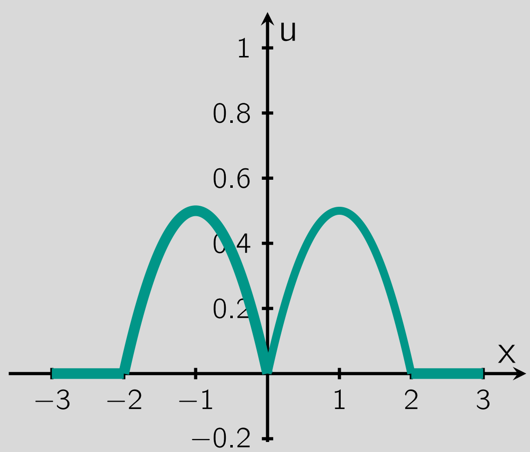
Lastly, in R_2, we see that the waves haven't separated yet, so we get the superposition of the two

$$u(x,t) = \frac{1}{2}\left[u_0(x+t) - u_0(x-t)\right] = \frac{1}{2}\left[\left\{1-(x+t)^2\right\} + \left\{1-(x-t)^2\right\}\right] \qquad \text{for } (x,t) \in R_2.$$

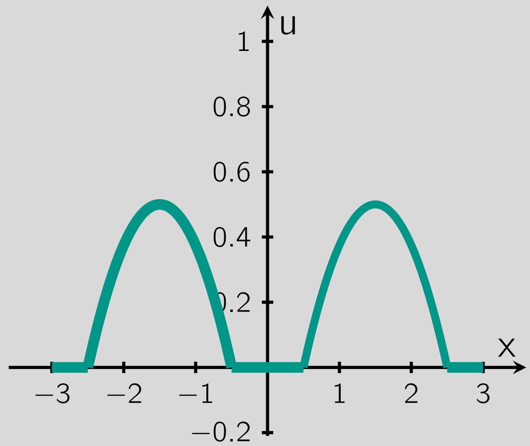
We can then plot this for a few snapshots of t. Let's first include one that incorporates R_3, so say $t = 0.5$. Note that to plot it for a single t slice, we get 3 different scenarios for the different x values, depending if it's in R_1, R_2, R_4, but we know the behavior in each.



Next, we see at $t = 1$, this is the exact moment the two waves split apart



And then say, at $t = 1.5$ and beyond they are clearly distinct



and they continue to travel apart.

3 2. Explain the notions of **domain of dependence** and **domain of influence** in the context of the wave equation. Why is this significant?

Solution: These notions were basically explained in the previous problem. **The domain of dependence** says that, given some point at some time, say (x_0, t_0), there is some (finite) domain of initial points that it depends on. For instance, we see that if chose the point $x = 2$ $t = 1$ in the previous problem, the value entirely depends on $u_0(1)$. In general, the domain of dependence follows the traveling wave with speed c.

The domain of influence is really the same idea, backward. If we say looked at some initial point $x = 1$, we could see which future points this one influences, which is along the line $x = 1 - t$. In the previous picture, the shaded regions are the domains of influence of the initial data.

D'Alembert's formula

$$u(x, t) = \frac{1}{2}\left[u_0(x - ct) + u(x + ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(\xi)\, \mathrm{d}\xi.$$