Math 3150 – HW 3 Solutions

June 5, 2018

3.8, 3.9, 3.12, 3.13, 3.16, 3.21

1. 3.8

   Make graphs of the periodic extensions on the region $x \in [-3\pi, 3\pi]$ of the following functions f defined on $x \in [-\pi, \pi]$. Be sure to indicate the extensions exact value at all points in $x \in [-3\pi, 3\pi]$, particularly at jump discontinuities (if any). Also, determine if the extension is continuous or not on the real line.

   (a) $f(x) = \sin(x)$
   (b) $f(x) = \sin(x/2)$
   (c) $f(x) = \cos(x/4)$
   (d) $f(x) = |x|$
   (e) $f(x) = x$
   (f) $f(x) = 1$
   (g) $f(x) = e^x$
   (h) $f(x) = e^{|x|}$

   Solution:

   To make the periodic extensions, basically just place repeat the function from $[-\pi, \pi]$ to $[\pi, 2\pi]$ and so on. Those with values $f(-\pi) \neq f(\pi)$ will have jumps and therefore be discontinuous. If done correctly, the resulting extensions are: (a) continuous, (b) discontinuous, (c) continuous, (d) continuous, (e) discontinuous, (f) continuous, (g) discontinuous, (h) continuous.

2. 3.9

   Solution:

   1. yes, yes, yes, yes
   2. yes, yes, yes, yes
   3. yes, yes, yes, yes
   4. yes, no, yes, yes
   5. yes, yes, yes, yes
   6. yes, yes, yes, yes
   7. yes, yes, yes, yes
   8. yes, no, yes, yes

3. 3.12

   Consider the function f defined in example 3.2. Using the mean value convergence property of Fourier series, indicate the value of the Fourier series of f at the points $-\pi, 0, \pi/2, 3\pi/4, \pi$.

Solution:

- $x = -\pi$:
$$\hat{f}(-\pi) = \frac{1}{2}\left(\lim_{x\to-\pi^+}(x+\pi)e^{(x+\pi)} + \lim_{x\to-\pi^-}(-1)\right) = -\frac{1}{2}.$$
- $x = 0$:
$$\hat{f}(0) = \frac{1}{2}\left(\lim_{x\to0^+} e^{-2x}\sin(20x^2) + \lim_{x\to0^-}(x+\pi)e^{(x+\pi)}\right) = \frac{\pi e^{\pi}}{2}.$$
- $x = \pi/2$:
$$\hat{f}(\pi/2) = \frac{1}{2}\left(\lim_{x\to(\pi/2)^+}(1) + \lim_{x\to(\pi/2)^-} e^{-2x}\sin(20x^2)\right) = \frac{1}{2}\left(1 + e^{-\pi}\sin(5\pi^2)\right).$$
- $x = 3\pi/4$:
$$\hat{f}(3\pi/4) = \frac{1}{2}\left(\lim_{x\to(3\pi/4)^+}(-1) + \lim_{x\to(3\pi/4)^-}(1)\right) = 0.$$
- $x = \pi$:
$$\hat{f}(\pi) = \frac{1}{2}\left(\lim_{x\to(\pi)^+}(x+\pi)e^{(x+\pi)} + \lim_{x\to(\pi)^-}(-1)\right) = \frac{1}{2}\left(2\pi e^{2\pi} - 1\right).$$

4. 3.13

Consider the function f defined on $x \in [-\pi, \pi]$:

$$f(x) = \begin{cases} 27, & x \neq 1 \\ 3, & x = 1 \end{cases}.$$

Find the Fourier series $\hat{f}(x)$, specifying every Fourier coefficient exactly. What is the value of $\hat{f}$ at $x = 1$?

Solution: The Fourier series convergence theorem tells us that the Fourier series converges pointwise to the function $f(x)$ everywhere that $f(x)$ is continuous. Thus, for all $x \in [-\pi, \pi] \neq 1$, $\hat{f}(x) = a_0 = 27$. At the discontinuity $x = 1$, the Fourier series takes the value

$$\frac{1}{2}\left(\lim_{x\to(1)^+} f(x) + \lim_{x\to(1)^-} f(x)\right) = 27.$$

Therefore, $\hat{f}(x) = 27$. Alternatively, students may actually use the Fourier series definition to compute the coefficients. If they do so, they should get $a_0 = 27,\ a_n = 0,\ b_n = 0$.

5. 3.16

Find the Fourier series of $f(x)$ on the interval $x \in [-\pi, \pi]$.

$$f(x) = \begin{cases} \sin(x), & x \in [0, \pi] \\ -\sin(x), & x \in [-\pi, 0) \end{cases}.$$

Solution: The Fourier series takes the form

$$\hat{f}(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right),$$

with

$$a_0 = \frac{1}{2\pi}\left(\int_{-\pi}^{0}(-\sin(x))dx + \int_0^{\pi}(\sin(x))dx\right) = \frac{2}{\pi}.$$

$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^{0}(-\sin(x)\cos(nx))dx + \int_0^{\pi}(\sin(x)\cos(nx))dx\right)$$

$$b_n = \frac{1}{\pi}\left(\int_{-\pi}^{0}(-\sin(x)\sin(nx))dx + \int_0^{\pi}(\sin(x)\sin(nx))dx\right).$$

First, notice that if $n = 1$, then $a_1 = 0$ since $\sin(x)$ is orthogonal to $\cos(x)$. To calculate a_n for $n \geq 2$ it is helpful to introduce the change of variables $\hat{x} = -x$ for the first integral to try to make the two look the same. Doing this gives us:

$$a_n = \frac{1}{\pi}\left(\int_{\pi}^{0}(\sin(-\hat{x})\cos(-n\hat{x}))d\hat{x} + \int_0^{\pi}(\sin(x)\cos(nx))dx\right)$$

Apply what we know about the definition of even and odd functions, and switching the bounds of integration on the first integral, we get:

$$a_n = \frac{1}{\pi}\left(\int_0^{\pi}(\sin(\hat{x})\cos(n\hat{x}))d\hat{x} + \int_0^{\pi}(\sin(x)\cos(nx))dx\right)$$

$$= \frac{2}{\pi}\left(\int_0^{\pi}(\sin(x)\cos(nx))dx\right) = \frac{1}{\pi}\left(\int_0^{\pi}(\sin(x-nx)+\sin(x+nx))dx\right)$$

$$= \frac{1}{\pi}\left(\frac{-\cos((1-n)x}{1-n} - \frac{\cos((1+n)x}{1+n}\bigg|_0^{\pi}\right)$$

$$= \frac{\cos(\pi n)+1}{1-n^2}$$

Now, we can calculate b_n, again using the change of variables $\hat{x} = -x$ in the first integral to get:

$$b_n = \frac{1}{\pi}\left(\int_{\pi}^{0}(\sin(\hat{x})\sin(n\hat{x}))d\hat{x} + \int_0^{\pi}(\sin(x)\sin(nx))dx\right)$$

$$= \frac{1}{\pi}\left(-\int_0^{\pi}(\sin(\hat{x})\sin(n\hat{x}))d\hat{x} + \int_0^{\pi}(\sin(x)\sin(nx))dx\right) = 0.$$

So, the Fourier series is

$$\hat{f}(x) = \frac{2}{\pi} + \sum_{n=2}^{\infty}\frac{\cos(\pi n)+1}{1-n^2}\cos(nx).$$

6. 3.21

Solution:

7. We know that $f(x) = x^2$ on $[0, \pi]$ is half a parabola. Therefore, the even extension is just the same parabola extended to $[-\pi, \pi]$, so

$$f_{even}(x) = x^2, \qquad x \in [-\pi, \pi].$$

The odd extension is a little weirder, but basically we just need to make a mirror copy that goes negative, so

$$f_{odd}(x) = \begin{cases} x^2 & x \in [0, \pi] \\ -x^2 & x \in [-\pi, 0]. \end{cases}$$

The nice thing about defining these (and the whole point of doing so), is constructing the cosine and sine series of the original $f(x)$ just boils down to constructing the full Fourier series of the even and odd extensions.

That is, we want to construct the full Fourier series defined by

$$\hat{g}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

where $g(x)$ is either $f_{odd}(x)$ or $f_{even}(x)$.

Sine series: We know that $\sin(x)$ is an *odd* function, so that's the indicator of which extension to use. Thus, we can now forget everything about extensions and just compute the Fourier series of $f_{odd}(x)$ on $[-\pi, \pi]$. Because we've cooked up f_{odd} to be odd, $a_n = 0$ for all n since we don't need even functions to build an odd one. This leaves us to solve for b_n

$$\begin{aligned} b_n &= \frac{\langle f_{odd}(x), \sin(nx)\rangle}{\langle \sin(nx), \sin(nx)\rangle} \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_{odd}(x)\sin(nx)\,\mathrm{d}x. \end{aligned}$$

Here's where we can use a trick we did in class: because f_{odd} is odd and $\sin(x)$ is odd, from $[-\pi, 0]$ we have a negative times a negative, so this flips back to positive and the answer is the same as $[0, \pi]$. Thus,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_{odd}(x)\sin(nx)\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_{0}^{\pi} f_{odd}(x)\sin(nx)\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_{0}^{\pi} x^2\sin(nx)\,\mathrm{d}x. \end{aligned}$$

This is kinda annoying to integrate, but we can do it easily by parts twice by taking $u = x^2$ and $\mathrm{d}v = \sin(nx)\,\mathrm{d}x$

$$\begin{aligned} \int_0^{\pi} x^2\sin(nx)\,\mathrm{d}x &= [-x^2\cos(nx)/n]_{x=0}^{x=\pi} - \int_0^{\pi} -\frac{2x}{n}\cos(nx)\,\mathrm{d}x \\ &= -\frac{\pi^2}{n}\cos(n\pi) - [-\frac{2x}{n^2}\sin(nx)]_{x=0}^{\pi} + \int_0^{\pi}\frac{2}{n^2}\sin(nx)\,\mathrm{d}x \\ &= \frac{\left(2-\pi^2n^2\right)\cos(\pi n) + 2\pi n\sin(\pi n) - 2}{n^3}. \end{aligned}$$

Note, however, $\cos(\pi n) = (-1)^n$ and $\sin(n\pi) = 0$ so we have

$$b_n = \frac{2\left((-1)^n\left(2-\pi^2n^2\right)-2\right)}{\pi n^3}$$

and therefore the Fourier sine series is

$$\hat{f}_{\sin} = \sum_{n=1}^{\infty} b_n\sin(nx) = \sum_{n=1}^{\infty}\frac{2\left((-1)^n\left(2-\pi^2n^2\right)-2\right)}{\pi n^3}\sin(nx).$$

Cosine series:

The cosine series is almost identical, except now we use cosines rather than sines, because we'll be constructing the Foruier series of the *even* extension. In other words, $b_n = 0$ for all n in this case.

Thus,

$$a_0 = \langle f_{even}, 1\rangle / \langle 1, 1\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{even}(x) 1 \, \mathrm{d}x = \frac{2\pi^2}{3}.$$

For a_n we can use a similar trick to before, noting that an even function times an even function is still even, so integrating from $[-\pi, \pi]$ is the same as twice the integral from $[0, \pi]$.

Therefore,

$$\begin{aligned} a_n &= \frac{\langle f_{even}, \cos nx\rangle}{\langle \cos nx, \cos nx\rangle} \\ &= \frac{2}{\pi}\int_0^{\pi} x^2 \cos(nx) \, \mathrm{d}x. \end{aligned}$$

Again, we could do integrating by parts or use Mathematica/Maple, and we find

$$a_n = \frac{4(-1)^n}{n^2},$$

so our Fourier cosine series of $f(x)$ is

$$\hat{f}_{\cos} = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx).$$