Name:________________________________ Quiz Score:_____/10

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the box to the left of the question. If there are any other issues, please ask the instructor.

[2] 1. Consider the differential operator L, defined by

$$Lu := u''(x).$$

If I asked you to find the eigenvalues and eigenfunctions of this operator, you should refuse, as the question as asked is ill-posed. Why? What fundamental difference between ODEs and PDEs does this illustrate?

Solution: In ODEs, we had a differential equation like

$$\frac{dy}{dx} = ay'' + by' + cy,$$

and we would solve the characteristic polynomial $0 = ar^2 + br + c$ to find the roots (often you called these r, but they're really eigenvalues!). From there, you would find the corresponding eigenfunctions $v_1(x), v_2(x)$ and then then your solution is a linear combination of these

$$y(x) = c_1 v_1(x) + c_2 v_2(x).$$

Some initial conditions (say $y(0) = 1, y'(0) = 1$) would then give us c_1, c_2. In other words, *for ODEs: we find the eigenvalues and eigenfunctions, and then apply boundary (initial) conditions.*

In contrast, *for PDEs, the boundary conditions determine the eigenvalues/eigenfunctions. They are not an afterthought. They directly determine the fundamental behavior.* Therefore, without BCs, this question does not make sense.

rubric: 1 pt for mentioning boundary conditions, 1 pt for mentioning that ODE eigenvalues/functions do not involve the BCs.

[6] 2. Find all the eigenvalues and eigenfunctors of the differential operator L

$$Lu := u''(x)$$

with boundary values

$$u'(0) = 0, \qquad u'(\pi) = 0.$$

Note: this question is *not* ill-posed.

Solution: While this is definitely a calculation you will forget in future years, it is fundamental for this class. We'll use the eigenvalues and eigenfunctions to construct a solution to PDEs later. This was the most straightforward problem I could find of this type.

Consider our eigenvalue equation (by definition)

$$Lu = \lambda u \qquad \leftrightarrow \qquad u''(x) = \lambda u(x),$$

we know the characteristic polynomial for this ODE is $r^2 = \lambda$ so it has roots

$$r = \pm\sqrt{\lambda}.$$

We then reduce to our three classic scenarios: i) if $\lambda = 0$, these are just zero, ii) if $\lambda > 0$, these are real. iii) if $\lambda < 0$, these are complex. We must investigate each.

case 1, $\lambda = 0$:

As I mentioned in class, this case is usually best to start with because it's easiest. If $\lambda = 0$, $r = 0$, so our solution becomes

$$u = c_1 x + c_2.$$

We now apply the boundary conditions, noting that $u'(x) = c_1$. Therefore, if we apply the left boundary $u'(0) = 0$, we immediately have $c_1 = 0$. What about the second boundary? Well, if $c_1 = 0$, $u'(\pi) = 0$ is automatically satisfied.

Therefore, we've found a solution $u(x) = c_2 = \text{constant}$ that satisfies our eigenvalue equation *and* boundaries *and* is not zero. Therefore, $\lambda = 0$ is an eigenvalue with corresponding eigenfunction $u = \text{constant}$.

case 2, $\lambda > 0$: If our roots are real, our solutions looks like

$$u = c_1 e^{\omega x} + c_2 e^{-\omega x},$$

where I've used the shorthand $\omega := \sqrt{\lambda}$. Taking a derivative, we have

$$u = \omega c_1 e^{\omega x} - c_2 \omega e^{-\omega x}.$$

Applying the left boundary $u'(0) = 0$, we have

$$0 = c_1 \omega e^0 - c_2 \omega e^0 = \omega(c_1 - c_2).$$

Clearly we don't want $\omega = 0$ (that was case 1), so $c_1 - c_2 = 0$ and therefore $c_1 = c_2$. Applying the right boundary $u'(\pi) = 0$, we next have

$$0 = c_1 \omega e^{\omega \pi} - c_1 \omega e^{-\omega \pi}.$$

Multiplying both sides by $e^{\omega \pi}$, we get

$$0 = c_1 \omega e^{2\omega \pi} - c_1 \omega = c_1 \omega \left(e^{2\omega \pi} - 1 \right).$$

The only way the thing in parentheses is equal to zero is if $\omega = 0$, which again, we do not want. Therefore, $c_1 = 0$ and $u = 0$ overall, meaning this is *not* a scenario in which we care about, and the operator does *not* have $\lambda > 0$ eigenvalues.

case 3, $\lambda < 0$: In this final scenario, our roots are complex so the solutions are

$$u = c_1 \cos(\omega x) + c_2 \sin(\omega x),$$

where here $\omega := \sqrt{-\lambda}$ since λ <0. Taking a derivative, we have

$$u = -c_1 \omega \sin(\omega x) + c_2 \omega \cos(\omega x).$$

Applying the left boundary $u'(0) = 0$, we have

$$0 = -c_1 \omega \sin(0) + c_2 \omega \cos(0) = c_2 \omega$$

since $\sin 0 = 0$, so we immediately conclude $c_2 = 0$. Applying the next boundary, we have

$$0 = -c_1 \omega \sin(\omega \pi)$$

Clearly we don't want $c_1 = 0$ or $\omega = 0$, meaning we must have

$$\sin(\omega \pi) = 0.$$

When is $\sin(x) = 0$? When $x = 0, \pi, 2\pi = n\pi$ for $n = 0, 1, 2$ and so on. Therefore,

$$\omega \pi = n\pi \quad \implies \quad \omega = n,$$

but remember $\omega = \sqrt{-\lambda}$ so we have

$$\sqrt{-\lambda} = n \quad \implies \quad \lambda = -n^2,$$

because we assumed $\lambda < 0$. Therefore we have infinitely many eigenvalues $\lambda_0 = 0, \lambda_1 = -1, \lambda_2 = -4, \lambda_3 = -9$ and corresponding eigenfunctions

$$u_n = \cos(\sqrt{\lambda_n} x) = \cos(nx).$$

rubric: 1 pt for writing the eigenvalue equation, 3 points for identifying structure of the problem (split into 3 cases), 2 pts for correctly characterizing all the cases.

[2] 3. Explain two reasons that we defined the inner product $\langle u, v\rangle$ between two abstract objects u, v. That is, what two things do we gain?

Solution: Again, this wasn't meant to be a mind-reading question, but rather distill the inner product to its basic properties. The two big things are that the inner product gives us notions of **length/distance** and **angle**.

Specifically, we defined the norm (magnitude) as

$$\|u\| = \sqrt{\langle u, u\rangle},$$

which allows us to define the distance between two objects u, v as

$$d(u, v) := \|u - v\|.$$

The second big gain we get is that we can now talk about angles, specifically the angle θ between two objects u, v is defined to be

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\|v\|}.$$

From this, we can talk about (the very important!) property of orthogonality, which is simply $\langle u, v\rangle = 0$.

rubric: 1 pt for each idea (distance/magnitude/length, angle)