Name:______________________ Quiz Score:____/10

Answer each question completely in the area below. Show all work and explain your reasoning. If the work is at all ambiguous, it is considered incorrect. No phones, calculators, or notes are allowed. Anyone found violating these rules will be asked to leave immediately. Point values are in the box to the left of the question. If there are any other issues, please ask the instructor.

[2] 1. Consider the two functions

$$\phi_1(x) = 1, \qquad \phi_2(x) = x - \frac{1}{2}.$$

over the interval $x \in [0, 1]$ with the standard inner product.

Are ϕ_1, ϕ_2 orthogonal? Why or why not?

Solution: Here, we use the definition of orthogonality: the inner product between the two objects equaling zero. Computing this inner proudct:

$$\langle \phi_1, \phi_2 \rangle = \int_0^1 1 \cdot (x - \frac{1}{2}) = \left[\frac{x^2}{2} - \frac{1}{2}x \right]_{x=0}^{x=1} = 0.$$

Therefore, ϕ_1, ϕ_2 are indeed orthogonal. **rubric:** 1 pt for using the inner product equaling zero, 1 pt for finding it working out correctly.

[4] 2. Suppose we want to approximate

$$f(x) = \sqrt{x}$$

using a linear combination of our functions

$$\hat{f}(x) = c_1\phi_1(x) + c_2\phi_2(x).$$

What are the best choices for c_1, c_2?

Solution: We know that the orthogonal project formula gives us the best values of c_j, which takes the form

$$c_j = \frac{\langle f, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle}.$$

In this particular problem, we find

$$\begin{aligned} c_1 &= \frac{\langle f, \phi_1 \rangle}{\langle \phi_1, \phi_1 \rangle} \\ &= \frac{\int_0^1 \sqrt{x} \cdot 1 \, dx}{\int_0^1 1 \cdot 1 \, dx} \\ &= \frac{2/3}{1} = \frac{2}{3}. \end{aligned}$$

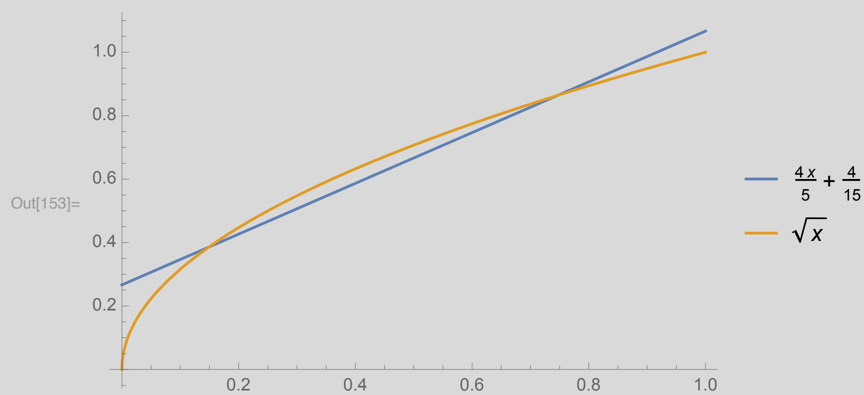
For the second coefficient,

$$c_2 = \frac{\langle f, \phi_2 \rangle}{\langle \phi_2, \phi_2 \rangle} = \frac{\int_0^1 \sqrt{x} \cdot (x - 1/2)\,\mathrm{d}x}{\int_0^1 (x-1/2)^2\,\mathrm{d}x} = \frac{1/15}{1/12} = \frac{12}{15}.$$

Putting this all together, we have

$$\hat{f} = \frac{2}{3}1 + \frac{12}{15}(x - 1/2) = \frac{4}{15} + \frac{4x}{5}.$$

Here, we can see the two functions plotted and see we get a pretty good approximation.



rubric: 3 points for right approach: plugging into orthogonal projection formula. 1 pt for right c_j values.

1 3. In the previous question, in what sense are these the "best" choices for c_1, c_2?

Hint: what do they minimize?

Solution: The big thing here is that we are approximating f with $\hat{f}$ and measuring the error using the *squared* error

$$\text{squared error} = \|f - \hat{f}\|^2 = \int_0^1 (f - \hat{f})^2\,\mathrm{d}x.$$

This is *one* way to measure the error between f and $\hat{f}$, and if this is the error we care about, the orthogonal projection formula provides the best choice by minimizing this type of error. If we cared about a different type of error, say

$$\text{other type of error} = \max_{x\in[0,1]} |f(x) - \hat{f}(x)|,$$

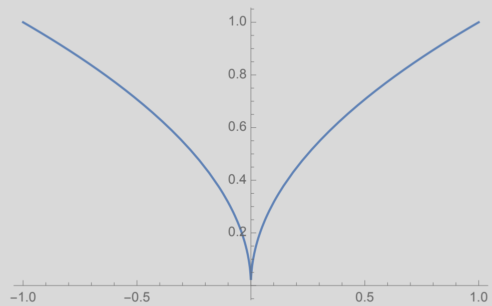
then these choices of c_j are not optimized. **rubric:** 1/2 pt for mentioning squared error, 1/2 a point for any coherent thought.

3 4. Suppose instead we wanted to construct the Fourier *cosine* series of $f(x)$. Explain the steps to do so, but don't calculate the coefficients.

Hint: you should start with some sort of extension of $f(x)$.

Solution:

1. If we want the *cosine* series, we remember that $\cos(x)$ is an *even* function, so we construct f_{even} which is the even extension of $f(x)$, shown below



2. From there, we just take $f = f_{even}$ and construct a typical Fourier series on $[-1, 1]$, consisting of

$$\hat{f} = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x).$$

3. Because this is a cosine series, we know $b_n = 0$ and then

$$a_n = \frac{\langle f_{even}, \cos(n\pi x)\rangle}{\langle \cos(n\pi x), \cos(n\pi x)\rangle} = \frac{\int_{-1}^{1} f_{even} \cos(n\pi x)\,\mathrm{d}x}{\int_{-1}^{1} \cos(n\pi x)^2\,\mathrm{d}x},$$

but by symmetry, this becomes

$$a_n = \frac{2\int_0^1 f \cos(n\pi x)\,\mathrm{d}x}{2\int_0^1 \cos(n\pi x)^2\,\mathrm{d}x}.$$

4. We can find these coefficients, and then only consider $\hat{f}$ on $[0, 1]$ and we've constructed a Fourier series with only cosine terms as desired.

rubric: 1 pt for mentioning even extension, 1 pt for writing Fourier series, 1 pt for writing how to get coefficients