Math 3150 – PDEs WS 1

Name: ____

1. Let the 1D mass flux field be given by $\phi(x) = 1 + \sin(x)$ in units of kg per second. What is the net flux out of the interval $(0, \pi/2)$? Will the mass in the region grow or shrink? Justify your answer.

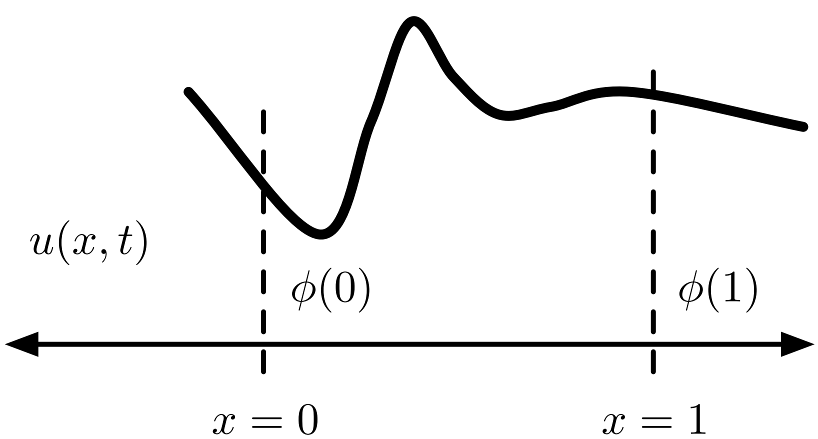
Solution: The only tricky thing here is to remember the correct signs for flux *out*. We know that positive flux $\phi > 0$ corresponds to flow to the *right*, so from the discussion in class, the net flux *out* is then simply

$$-\phi(0) + \phi(\pi/2) = +1.$$

Because the net flux *out* is positive, we could think of this as the net flux *in* as negative, so the amount of stuff is going down.

2. Suppose at time t the concentration of a chemical is given in the Figure below. Let $M(t) = \int_0^1 u(x,t)dx$ be the total mass in the unit interval.

   Given the flux rules ϕ listed below, draw arrows indicating the relative magnitudes (larger magnitude should be indicated by bigger arrows) and directions of the flux at the both $x = 0$ and $x = 1$ in the space listed below. For each case, determine the sign of dM/dt at time t.



(a) $\phi = -u_x$

$(x = 0)|$________________$|(x = 1)$ Sign of dM/dt: ____________

(b) $\phi = -u$

$(x = 0)|$________________$|(x = 1)$ Sign of dM/dt: ____________

Solution:

(a) In this case, we look and see that at $x = 0$, $\phi = -u_x$ is *positive* because the slope is negative, so the flux is to the right, $\phi(0) > 0$. Similarly, for $x = 1$, the slope is negative again, so $\phi(1) > 0$ as well. The slope at $x = 0$ is steeper, meaning the magnitude is larger on the left, and therefore more stuff is coming in than leaving, so $\mathrm{d}M/\mathrm{d}t > 0$.

(b) Here we have a different flux rule. At $x = 0$, u is positive so $-u$ is negative, meaning $\phi(0) < 0$ and stuff is flowing to the left. The same is true for the right. However, in this case, $\phi(0)$ has smaller magnitude than $\phi(1)$, so the flow in $\phi(1)$ is larger and therefore $\mathrm{d}M/\mathrm{d}t > 0$ again.

3. (Reviewing Vector Calculus/Divergence Theorem). Suppose we want to calculate the net flux through a square $R = \{(x, y) \in [0, 2] \times [0, 2]\}$, with flux field

$$\vec{\phi}(x, y) = \begin{pmatrix} x^2 \\ xy \end{pmatrix}$$

.

(a) Find the unit normal vector to each side of the square R. Call these four unit normal vectors $\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4$.

(b) The flux through one side of R is equal to $\int\limits_{\partial R} \vec{\phi} \cdot \vec{n}_i dS$, where ∂R is the boundary of R. Find the flux through all 4 sides of R.

(c) What is the net flux through R? Is it growing or shrinking?

(d) The divergence theorem says that $\int\limits_{\partial R} \vec{\phi} \cdot \vec{n} dS = \int\limits_{R}\int \nabla \cdot \vec{\phi} dA$. Can you explain in your own words what that means?

(e) Use the divergence theorem to find the net flux through R.

Solution:

(a) This is fairly easy if we think geometrically. Label the sides starting with the bottom (and going counter clockwise) S_1, S_2, S_3, S_4. Therefore, the normal S_1 is the side facing down, so the unit normal is just $\vec{n}_1 = (0, -1)$. For the right side, the unit normal is $\vec{n}_2 = (1, 0)$, the top is $\vec{n}_3 = (0, 1)$ and the left is $\vec{n}_4 = (-1, 0)$.

(b) This is a little tedious, but not too bad.

$$\begin{aligned} \text{bottom: } & \int_{S_1} \vec{\phi} \cdot \vec{n}_1 \, \mathrm{d}S \\ & = \int_0^2 (x^2, xy) \cdot (0, -1)\Big|_{y=0} \mathrm{d}x \\ & = 0. \end{aligned}$$

$$\text{right: } \int_{S_2} \vec{\phi} \cdot \vec{n}_2 \, \mathrm{d}S$$

$$= \int_0^2 (x^2, xy) \cdot (1, 0)\big|_{x=2} \, \mathrm{d}y$$

$$= \int_0^2 4 \, \mathrm{d}y = 8.$$

$$\text{top: } \int_{S_3} \vec{\phi} \cdot \vec{n}_3 \, \mathrm{d}S$$

$$= \int_0^2 (x^2, xy) \cdot (0, 1)\big|_{y=2} \, \mathrm{d}y$$

$$= \int_0^2 2x \, \mathrm{d}x = 4.$$

$$\text{left: } \int_{S_4} \vec{\phi} \cdot \vec{n}_4 \, \mathrm{d}S$$

$$= \int_0^2 (x^2, xy) \cdot (-1, 0)\big|_{x=0} \, \mathrm{d}y$$

$$= 0.$$

(c) Here, we can see the net flux is $8+4+0+0 = 12$, so the amount of stuff is going up.

(d) Basically the divergence theorem says: instead of adding up all the fluxes on the surface, we can add up the divergence of flux on the entire interior. Intuitively, everything cancels out on the inside except on the surface, which gives us the net flux through the boundary.

(e) Hopefully we get the same answer if the divergence theorem is true. We quickly can compute the divergence of our flux

$$\nabla \cdot \vec{\phi} = (\partial_x, \partial_y) \cdot (x^2, xy) = 3x,$$

and then use it in the divergence theorem

$$\iint_R \nabla \cdot \vec{\phi} \, \mathrm{d}A = \int_0^2 \int_0^2 3x \, \mathrm{d}x \, \mathrm{d}y$$

$$= 2 \int_0^2 3x \, \mathrm{d}x$$

$$= 2 \cdot 6 = 12.$$

4. Match the linear algebra term to its definition.
   (a) A vector space S contained inside a vector space V.
   (b) A set of vectors $W = \{u_i\}_{i=1}^n$ where there is no linear combination of the other $n-1$ vectors in W that equals any other vector already in W can be described this way.
   (c) A vector space formed by all linear combinations of a set of vectors $\{u_i\}_{i=1}^n$.
   (d) A linearly independent set of vectors $B = \{u_i\}_{i=1}^n$ (where $u_i \in V$) for which every vector in V can be represented by a unique linear combination of B.
   (e) A set that is closed under addition and scalar multiplication, which has an additive inverse and additive identity in that set and commutativity, associativity, and distributivity holds.

   1. vector space: ________________________________
   2. subspace: ________________________________
   3. linearly independent: ________________________________
   4. basis: ________________________________
   5. span: ________________________________

Solution: While these terms may not appear explicitly in the entirety of the PDEs course, they're lurking in the shadows at all times. Basically the study of linear anything (ODEs, PDEs) is the study of linear algebra (vector spaces).

Intuitively, the way we solve these equations is by finding a *basis* for the *subspace* formed by solutions to our PDE/ODE, which we can think of as the minimal building blocks for our solution because they *span* (hit all possible solutions), and are *linearly independent* (we have no redundancy in our build blocks).

(a): subspace, (b): linearly independent, (c): span, (d): basis, (e): vector space.