HOMEWORK 3: ITERATIVE METHODS FOR FEM

DUE NOV 16

We consider the finite element method for solving the Poisson equation. Let $\mathbb{V}^h \subset \mathbb{V} = H_0^1(\Omega)$ be the linear finite element space based on a quasi-uniform mesh $\mathcal{T}_h$. We look for a $u_h \in \mathbb{V}^h$ such that

$$a(u_h, v_h) = \langle f, v_h \rangle, \quad \forall v_h \in \mathbb{V}^h. \tag{1}$$

Here $a(u, v) = (\nabla u, \nabla v) = \int_\Omega \nabla u \cdot \nabla v$, and $\langle \cdot, \cdot \rangle$ is the dual pair. In particular when $f \in L^2(\Omega)$, $\langle f, v \rangle = (f, v) := \int_\Omega f v$.

The bilinear form $a(\cdot, \cdot)$ will introduce a mapping

$$A : \mathbb{V} \mapsto \mathbb{V}' \cong \mathbb{V}, \quad u \mapsto Au := a(u, \cdot).$$

The restriction of A to $\mathbb{V}^h$ introduce a map $A_h : \mathbb{V}^h \mapsto (\mathbb{V}^h)' \cong \mathbb{V}^h$

Let $\{\phi_i\}_{i=1}^N$ be the nodal basis of $\mathbb{V}^h$. Every function in $\mathbb{V}^h$ has a representation $v_h = \sum_{i=1}^N v_i \phi_i$. We define an isomorphism $\mathbb{V}^h \longleftrightarrow \mathbb{R}^N$ by

$$v_h = \sum_{i=1}^N v_i \phi_i \longleftrightarrow \boldsymbol{v} = (v_1, \cdots, v_N)^t.$$

Here and after, we shall use boldface letters to denote vectors or matrixes. The *stiffness matrix* is defined as

$$\boldsymbol{A} = (a_{ij}), \text{ where } a_{ij} = a(\phi_j, \phi_i).$$

The *mass matrix* is defined as

$$\boldsymbol{M} = (m_{ij})_{N \times N} \quad \text{with} \quad m_{ij} = (\phi_j, \phi_i).$$

For any $v \in \mathbb{V}^h$, the H^1 norm and L^2 norm can be realized by

$$\|v\|^2 = (v, v) = (\boldsymbol{M}\boldsymbol{v}, \boldsymbol{v}) = \boldsymbol{v}^t \boldsymbol{M} \boldsymbol{v},$$
$$|v|_1^2 = (Av, v) = (\boldsymbol{A}\boldsymbol{v}, \boldsymbol{v}) = \boldsymbol{v}^t \boldsymbol{A} \boldsymbol{v}.$$

Let $u = \sum_j u_j \phi_j$. Taking $v_h = \phi_i, i = 1, \cdots N$ in (1), we get an algebraic equation

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}. \tag{2}$$

Here $\boldsymbol{f} = (f_1, ..., f_N)^t \in \mathbb{R}^N$ denotes the vector representation of f with $f_i = (f, \phi_i)$. By the definition, both $\boldsymbol{A}$ and $\boldsymbol{M}$ are $N \times N$ symmetric and positive definite matrix.

(1) Prove the mass matrix is well conditioned by showing: for any $v \in \mathbb{V}^h$

$$\|v\|^2 \simeq h^d \sum_{i=1}^N v_i^2 = h^d \|\boldsymbol{v}\|_{l_2}^2. \tag{3}$$

Note that (3) implies that

$$h^d \lesssim \lambda_{\min}(\boldsymbol{M}) \le \lambda_{\max}(\boldsymbol{M}) \lesssim h^d.$$

And thus $\kappa(\boldsymbol{M}) \simeq 1$ i.e. $\boldsymbol{M}$ is well conditioned.

Date: November 5, 2009.

(2) Prove that
$$1 \lesssim \lambda_{\min}(A_h) \leq \lambda_{\max}(A_h) \lesssim h^{-2}.$$
(3) Prove that $\lambda_{\max}(\boldsymbol{A}) \eqsim h^d \lambda_{\max}(A_h)$ and $\lambda_{\min}(\boldsymbol{A}) \eqsim h^d \lambda_{\min}(A_h)$. Therefore
$$\kappa(A_h) = \kappa(\boldsymbol{A}) \simeq h^{-2}.$$
(4) Estimate the convergence rate of Richardson's method (with appropriate choice of relaxation parameter), damped Jacobi (choosing $B = \alpha D^{-1}$ with appropriate choice of α), Gauss-Seidal method, and Conjugate Gradient method for solving the linear algebraic equation (2) from the linear finite element discretization.
(5) Hilbert matrix $H = (h_{ij})$ of n-th order is defined as $h_{ij} = \frac{1}{i+j-1}(\,i, j = 1, 2, ..., n)$. Use any numerical algorithm to solve the linear system: $Hx = b$, where b is generated by setting $x = (1, 1, ..., 1)^T$ for both $n = 3$ and $n = 30$. Compare your results with exact solution and try to explain why.