
1. Develop a symbolic code to check the consistency and stability of the two-level difference scheme in the form of
   \[
   u^{n+1}(x) = \sum_p a_p u^n(x - ph)
   \]
   for the heat equation and the transport equation. Use the code to analyze
   (1) (HW5.3.3 in p.216) The 1-D forward Beam-Warming scheme
   \[
   u_i^{n+1} = \frac{1}{2} \left[ u_i^n + u_i^* - R(u_i^* - u_{i-1}^*) - R(2u_i^n - u_{i-1}^n - u_{i+1}^n) \right],
   \]
   for solving \( u_t + au_x = 0 \).

   (2) (HW5.8.5 in p.247) The 2-D forward alternative direction scheme
   \[
   u_{ij}^{n+1/2} = u_{ij}^n - \frac{R_x}{2} (u_{i+1,j}^n - u_{i-1,j}^n) + \frac{R_x^2}{2} (2u_{i,j}^n - u_{i-1,j}^n - u_{i+1,j}^n),
   \]
   \[
   u_{ij}^{n+1} = u_{ij}^{n+1/2} - \frac{R_y}{2} (u_{i,j+1}^{n+1/2} - u_{i,j-1}^{n+1/2}) + \frac{R_y^2}{2} (2u_{i,j}^{n+1/2} - u_{i,j-1}^{n+1/2} - u_{i,j+1}^{n+1/2}),
   \]
   for solving \( u_t + au_x + bu_y = 0 \).
   Here recall that \( R = R_x = a\Delta t/\Delta x \) and \( R_y = b\Delta t/\Delta y \).

2. Verify the upwinding scheme for solving \( u_t + au_x = 0 \) is equivalent to the centered difference scheme for solving \( u_t + au_x - \frac{ah}{2} u_{xx} = 0 \). Namely we add numerical viscosity to stabilize the scheme.

3. (HW5.9.1 in p.257) We consider upwinding scheme for convection-diffusion and convection-dominated problem
   \[
   u_t + au_x - \nu u_{xx} = 0.
   \]
   The first order derivative \( u_x \) is discretized by an unwinding scheme and the second order derivative \( u_{xx} \) by centered difference scheme.
   Let \( R = a\Delta t/\Delta x \) and \( r = \nu\Delta t/\Delta x^2 \). Show that the above difference scheme is stable if and only if
   \[
   \begin{cases}
   R \leq 2r \leq 1 + R & \text{for } a < 0 \\
   R \leq 2r \leq 1 + R & \text{for } a > 0.
   \end{cases}
   \]
   We can also apply Lax-Wendroff scheme for the hyperbolic part and centered difference for the diffusion part. Prove that the resulting scheme is stable if and only if
   \[
   r + R^2/2 \leq 1/2.
   \]
4. (HW5.6.10 in p.232) Consider the following initial-boundary-value problem
\begin{align*}
    u_t - 2u_x &= 0, \quad x \in (0, 1), t > 0 \\
    u(x, 0) &= 1 + \sin 2\pi x, \quad x \in [0, 1] \\
    u(1, t) &= 1.
\end{align*}

Code the problem using the explicit Lax-Wendroff scheme with the following numerical boundary conditions at \( x = 0 \).
\begin{enumerate}
    
    \item \( u^n_0 = 1 \);
    \item \( u^n_0 = u^n_1 \);
    \item \( u^{n+1}_0 = u^n_0 - R(u^n_1 - u^n_0) \);
    \item \( u^{n+1}_0 + u^{n+1}_1 + R(u^{n+1}_1 - u^{n+1}_0) = u^n_0 + u^n_1 - R(u^n_1 - u^n_0) \).
\end{enumerate}