HOMEWORK 2 OF MATH 226C: CONSERVATION LAWS

The following are selected homework problems from the text book: J.W. Thomas. *Numerical Partial Differential Equations: Finite Difference Methods*.

1. Consider Burgers' equation

$$u_t + (\frac{1}{2}u^2)_x = 0, \ u(\cdot, 0) = u_0.$$

An entropy function and entropy flux function for the Burgers' equation is

$$S(v) = |v - c|, \ \Phi(v) = \frac{v - c}{|v - c|}\left[\frac{1}{2}v^2 - \frac{1}{2}c^2\right].$$

For the following FTBS numerical scheme

$$u_k^{n+1} = u_k^n - \frac{R}{2}(u_k^n)^2 + \frac{R}{2}(u_{k-1}^n)^2, \tag{1}$$

find a numerical entropy flux function Ψ which is consistent with Φ and proof the scheme (1) satisfies the discrete entropy condition.

2. We consider different formulations of a numerical scheme and transformation among these forms.

- Conservative form: $u_k^{n+1} = u_k^n - R(h_{k+1/2} - h_{k-1/2})$;
- Incremental form or I-form: $u_k^{n+1} = u_k^n + C_{k+1/2}^n \delta_+ u_k^n - D_{k-1/2}^n \delta_- u_k^n$;
- Q-form: $u_k^{n+1} = u_k^n - \frac{R}{2}\delta_0 F_k^n + \frac{1}{2}\delta_+(Q_{k-1/2}^n \delta_- u_k^n)$.

(1) Transform a difference scheme from conservative form to I-form;
(2) Transform a difference scheme from Q-form to conservative form and I-form;
(3) Transform a difference scheme from conservative form and I-form to Q-from.

3. Let $S(u) = |u - c|$ and $\Phi(u) = \text{sign}(u - c)[F(u) - F(c)]$. For a conservative and consistent scheme, let

$$\Psi_{k+1/2}^n = h_{k+1/2}^n(\tilde{u}_{k-p}^n, \ldots, \tilde{u}_{k+q}^n) - h_{k+1/2}^n(\tilde{\tilde{u}}_{k-p}^n, \ldots, \tilde{\tilde{u}}_{k+q}^n),$$

where $\tilde{u} = \max\{u, c\}$ and $\tilde{\tilde{u}} = \min\{u, c\}$. Prove $\Psi_{k+1/2}^n$ is consistent with Φ.

4. We study Godunov scheme in this exercise.

(1) Write the Godunov scheme in Q-form;
(2) Write the Godunov scheme in conservative form;
(3) Show that the numerical flux functions associated with the Godunov scheme can be written as

$$h_{k+1/2}^n = \begin{cases} \min_{u_k^n < u < u_{k+1}^n} F(u) & \text{if } u_k^n < u_{k+1}^n \\ \max_{u_{k+1}^n < u < u_k^n} F(u) & \text{if } u_k^n > u_{k+1}^n. \end{cases}$$

(4) Show that the Godunov scheme is a monotone scheme.

5. We consider slope-limiter schemes for the one way wave equation $u_t + au_x = 0$ with speed $a > 0$. Show that if we choose $\sigma_k^n = \delta_+ u_k^n / \Delta x$, the difference scheme

$$u_k^{n+1} = u_k^n - aR\delta_- u_k^n - \frac{aR}{2}(1 - aR)\Delta x \delta_- \sigma_k^n$$

reduces to the Lax-Wendroff scheme.