MATH 226 HW4: MULTIGRID METHODS

(1) Although Jacobi method will not damp the high frequency, due to the shift, the rate of ρ_k is $\mathcal{O}(h)$ for k near $N/2$. Recall that for $\omega = 1/4$, the rate is $1/2$ for k near $N/2$. One can then apply two Richardson iterations consecutively with different parameters to have a better smoothing effect. In general, one can use Chebyshev acceleration technique

$$(I - \omega_l A) \cdots (I - \omega_1 A)$$

and chose l-parameters by the following optimization problem:

$$\min_{\omega_i \in \mathbb{R}, i=1,\ldots,l} \left\{ \max_{\lambda \in [\lambda_{N/2}(A), \lambda_N(A)]} |(I - \omega_l \lambda) \cdots (I - \omega_1 \lambda)| \right\}.$$

Figure out an approximated solution for $l = 1, 2, 3$.

(2) Prove the mass matrix $M = (m_{ij})$ with $m_{ij} = \int_\Omega \phi_i \phi_j \, \mathrm{d}x$ is well conditioned for a quasi-uniform triangulation. For a shape regular but non quasi-uniform grids, prove $D^{-1}M$ is well conditioned, where D is the diagonal matrix of M. Therefore the diagonal preconditioner is always a good preconditioner for the mass matrix.

(3) Smoothing assumption $(\bar{S}_P)$. The smoother $\overline{R}_k$ will smooth the frequency $(P_k - P_{k-1})\mathbb{V}$, i.e., there exists $\alpha \geq 0$

$$(\overline{R}_k^{-1} u_k, u_k) \leq \alpha (A u_k, u_k), \quad \text{for all } u_k \in (P_k - P_{k-1})\mathbb{V},\ k = 0, \cdots, J.$$

Prove if $(\bar{S}_P)$ holds, then the constant $\alpha \geq 1$.