SOBOLEV SPACES AND ELLIPTIC EQUATIONS

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Sobolev spaces are fundamental in the study of partial differential equations and their numerical approximations. In this chapter, we shall give brief discussions on the Sobolev spaces and the regularity theory for elliptic boundary value problems.

CONTENTS

1. ESSENTIAL FACTS FOR SOBOLEV SPACES

We shall state and explain main results (without proofs) on Sobolev spaces. We refer to [1] for comprehensive treatment of Sobolev spaces.

1.1. **Preliminaries.**

We first set up the environment of our discussion: Lipschitz domains, multi-index notation for differentiation, and some basic functional spaces.

Lipschitz domains. Our presentations here will almost exclusively be for bounded Lipschitz domains. Roughly speaking, a domain (a connected open set) $\Omega \subset \mathbb{R}^n$ is called a Lipschitz domain if its boundary $\partial\Omega$ can be locally represented by Lipschitz continuous function; namely for any $x \in \partial\Omega$, there exists a neighborhood of x, $G \subset \mathbb{R}^n$, such that $G \cap \partial\Omega$ is the graph of a Lipschitz continuous function under a proper local coordinate system.

Of course, all smooth domains are Lipschitz. A significant non-smooth example is that every polygonal domain in $\mathbb{R}^2$ or polyhedron in $\mathbb{R}^3$ is Lipschitz. A more interesting example is that every convex domain in $\mathbb{R}^n$ is Lipschitz. A simple example of non-Lipschitz domains is two polygons touching at one vertex only.

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Notation of Schwarz. Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_+^n$, where $\mathbb{Z}_+$ is the set of non-negative integers, be a vector of nonnegative integers, denote

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots x_n^{\alpha_n}} \quad \text{with } |\alpha| = \sum_{i=1}^n \alpha_i.$$

For a smooth function v and $x \in \mathbb{R}^n$, denote

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \cdots x_n^{\alpha_n}}, \text{ and } x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Some basic functional spaces. Several basic Banach spaces will often be used in this course. $C(\bar\Omega)$ is the space of continuous functions on $\bar\Omega$ with the usual maximum norm

$$\|v\|_{C(\bar\Omega)} = \max_{x\in\bar\Omega} |v(x)|.$$

$C_0^\infty(\Omega)$ denotes the space of infinitely differential functions in Ω that vanish in some neighborhood of $\partial\Omega$; namely any $v \in C_0^\infty(\Omega)$ satisfies $\operatorname{supp}(v) \subset \Omega$, where

$$\operatorname{supp}(v) = \text{closure of } \{x \in \Omega : \ v(x) \neq 0\}.$$

Let $E(\Omega)$ represent the equivalent class of Lebesgue integrable functions corresponding to the equivalence $u \sim v$ if $u = v$ almost everywhere. Given $1 \le p < \infty$, we shall denote the usual Lebesgue space of pth power integrable functions by $L^p(\Omega) = \{v \in E(\Omega) : \int_\Omega |v|^p \,\mathrm{d}x < \infty\}$ and the essentially bounded function space by $L^\infty(\Omega)$. The norm is defined by

$$\|u\|_{p,\Omega} = \Big(\int_\Omega |u|^p \,\mathrm{d}x \Big)^{1/p} \text{ for } 1 \le p < \infty,$$

and

$$\|u\|_{\infty,\Omega} = \operatorname*{ess\,sup}_\Omega |u|.$$

Exercise 1.1. *Prove that $L^p(\Omega), 1 \le p \le \infty$ defined above are Banach spaces.*

1.2. **Definition of Sobolev spaces.** We are used to understand a function from its point values. This is not adequate. It is better to understand a function as a functional through its action on a space of unproblematic test functions (convential and well-behaved functions). In this way, we can generalize the concept of functions to generalized functions or so-called distributions. "Integration by parts" is used to extend the differentiation operators from classic differentiable functions to distributions by shifting them to the test functions.

Sobolev spaces will be first defined here for integer orders using the concept of distributions and their weak derivatives. The fractional order Sobolev spaces will be introduced by looking at the pth power integrable of quotient of difference. Definitions will also be given to Sobolev spaces satisfying certain zero boundary conditions.

Distributions and weak derivatives. We begin with the nice function space $C_0^\infty(\Omega)$. Recall that it denotes the space of infinitely differentiable functions with compact support in Ω. Obviously $C_0^\infty(\Omega)$ is a real vector space and can be turned into a topological vector space by a proper topology. The space $C_0^\infty(\Omega)$ equipped with the following topology is denoted by $\mathcal{D}(\Omega)$: a sequence of functions $\{\phi_k\} \subset C_0^\infty(\Omega)$ is said to be convergent to a function $\phi \in C_0^\infty(\Omega)$ in the space $\mathcal{D}(\Omega)$ if

(1) there exists a compact set $K \subset \Omega$ such that for all k, $\operatorname{supp}(\phi_k - \phi) \subset K$, and
(2) for every $\alpha \in \mathbb{Z}_+^n$, we have $\lim_{k\to\infty} \|D^\alpha(\phi_k - \phi)\|_\infty = 0$.

The space, denoted by $\mathcal{D}'(\Omega)$, of all continuous linear functionals on $\mathcal{D}(\Omega)$ is called the (Schwarz) *distribution space*. The space $\mathcal{D}'(\Omega)$ will be equipped with the weak star topology. Namely, in $\mathcal{D}'(\Omega)$, a sequence T_n converge to T in the distribution sense if and only if $\langle T_n, \phi\rangle \to \langle T, \phi\rangle$ for all $\phi \in \mathcal{D}(\Omega)$, where $\langle \cdot, \cdot\rangle : \mathcal{D}'(\Omega) \times \mathcal{D}(\Omega) \to \mathbb{R}$ is the duality pair. A function ϕ belonging to $\mathcal{D}(\Omega)$ is called a *test function* since the action of a distribution on ϕ can be thought as a test. The property of a distribution can be extracted by clever choice of test functions.

Example 1.2. By the definition, an element in $\mathcal{D}'(\Omega)$ is uniquely determined by its action. The action could be very general and abstract as long as it is linear and continuous. As an example, let us introduce the Dirac delta distribution $\delta \in \mathcal{D}'(\Omega)$ with $0 \in \Omega \subseteq \mathbb{R}^n$

$$\langle \delta, \phi\rangle = \phi(0) \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

One important class of distributions is to use the integration as the action. A function is called *locally integrable* if it is Lebesgue integrable over every compact subset of Ω. We define the space $L^1_{\text{loc}}(\Omega)$ as the space containing all locally integrable functions. We can embed $L^1_{\text{loc}}(\Omega)$ into $\mathcal{D}'(\Omega)$ using the integration as the duality action. For a function $u \in L^1_{\text{loc}}(\Omega)$, let $T_u \in \mathcal{D}'(\Omega)$ defined as

$$\langle T_u, \phi\rangle = \int_\Omega u\phi \,\mathrm{d}x \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

We shall still denoted T_u by u. The correspondence $u \mapsto T_u$ is often used to identify an "ordinary" function as a distribution.

A distribution is often also known as a *generalized function* as the concept of distribution is a more general than the concept of the classic function. One of the basic distribution which is not an "ordinary" function is the Dirac δ-distribution introduced in Example 1.2. Indeed, one motivation of the invention of the distribution space is to include Dirac delta "function".

Exercise 1.3. *Prove that it is not possible to represent the delta distribution by a locally integrable function.*

If u is a smooth function, it follows from integration by parts that, for any $\alpha \in \mathbb{Z}^n_+$

$$\int_\Omega D^\alpha u(x)\phi(x)\ \mathrm{d}x = (-1)^{|\alpha|}\int_\Omega u(x) D^\alpha\phi(x)\ \mathrm{d}x \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

There are no boundary terms, since ϕ has compact support in Ω and thus ϕ, together with its derivatives, vanishes near $\partial\Omega$. The above identity is the basis for defining derivatives for a distribution. If $T \in \mathcal{D}'(\Omega)$, then for any $\alpha \in \mathbb{Z}^n_+$, we define *weak derivative* $D^\alpha T$ as the distribution given by

$$\langle D^\alpha T, \phi\rangle = (-1)^{|\alpha|}\langle T, D^\alpha\phi\rangle \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

It is easy to see for a differentiable function, its weak derivative coincides with its classical derivative. In general, however, the weak derivative is much weaker than the classical one so that the differential operator can be extended from differential functions to a much larger space – the space of distributions. For example, we can even talk about the derivative of a discontinuous function.

Example 1.4. The Heaviside step function is defined as $S(x) = 1$ for $x > 0$ and $S(x) = 0$ for $x < 0$. By the definition

$$\int_{\mathbb{R}} S'\phi\ \mathrm{d}x = -\int_{\mathbb{R}} S\phi'\ \mathrm{d}x = -\int_0^\infty \phi'\ \mathrm{d}x = \phi(0).$$

Therefore $S' = \delta$ in the distribution sense but δ is not a function in $L^1_{\rm loc}(\Omega)$ (Exercise 1.3). Roughly speaking, any distribution is locally a (multiple) derivative of a continuous function. A precise version of this result can be found at Rudin [11].

The formal definition of distributions exhibits them as a subspace of a very large space. Generally speaking, no proper subset of the space of distributions contains all continuous functions and is closed under differentiation. This says that the distribution extension of the function concept is as economical as it possibly can be. A distribution is infinitely differentiable in the distribution sense.

Integer order Sobolev spaces. The Sobolev space of index (k, p), where k is a nonnegative integer and $p \geq 1$, is defined by

$$W^{k,p}(\Omega) \stackrel{\rm def}{=} \{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega) \quad \text{for all } \ |\alpha| \leq k\},$$

with a norm $\|\cdot\|_{k,p,\Omega}$ given by

$$\|v\|^p_{k,p,\Omega} \stackrel{\rm def}{=} \sum_{|\alpha|\leq k} \|D^\alpha v\|^p_{0,p(\Omega)}, \tag{1}$$

We will have occasions to use the seminorm $|\cdot|_{k,p,\Omega}$ given by

$$|v|^p_{k,p,\Omega} \stackrel{\rm def}{=} \sum_{|\alpha|=k} \|D^\alpha v\|^p_{0,p,\Omega}.$$

For $p = 2$, it is customary to write $H^k(\Omega) \stackrel{\rm def}{=} W^{k,2}(\Omega)$ which is a Hilbert space together with an inner product as follows

$$(u, v) = \sum_{|\alpha|\leq k} (D^\alpha u, D^\alpha v)$$

and the corresponding norm is denoted by $\|v\|_{k,\Omega} = \|v\|_{k,2,\Omega}$. The majority of our use of Sobolev space will be in this case.

We also define $W^{k,p}_{\rm loc}(\Omega) = \{u \in E(\Omega) : \text{ for any } U \subset\subset \Omega, u \in W^{k,p}(U)\}$ and $H^k_{\rm loc}(\Omega) = W^{k,2}_{\rm loc}(\Omega)$.

Exercise 1.5. *Prove that $W^{k,p}(\Omega)$ is a Banach space.*

For a function in $L^p(\Omega)$, treating it as a distribution, its weak derivatives always exists as distributions. but may not be in the space $L^p(\Omega)$. An element in $W^{k,p}(\Omega)$ possesses certain smoothness.

Example 1.6. We consider the Heaviside function restricted to $(-1, 1)$ and still denote by S. The weak derivative of S is the delta distribution which is not integrable. Therefore $S \notin H^1(-1, 1)$.

Example 1.7. Let $u(x) = |x|$ for $x \in (-1, 1)$ be an anti-derivative of $2(S - 1/2)$. Obviously $u \in L^2(-1, 1)$ and $u' \in L^2(-1, 1)$. Therefore $u \in H^1(-1, 1)$.

Example 1.6 and 1.7 explain that for a piecewise smooth function u to be in $H^1(\Omega)$ requires more global smoothness of u in Ω. See Exercise 1.18.

Fractional order Sobolev spaces. In the definition of classic derivative, it takes the pointwise limit of the quotient of difference. For functions in Sobolev space, we shall use the pth power integrability of the quotient difference to characterize the differentiability.

For $0 < \sigma < 1$ and $1 \le p < \infty$, we define

$$W^{\sigma,p}(\Omega) = \left\{ v \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^p}{|x-y|^{n+\sigma p}} \, \mathrm{d}x \, \mathrm{d}y < \infty \right\} \tag{2}$$

and

$$H^\sigma(\Omega) = W^{\sigma,2}(\Omega).$$

In $W^{\sigma,p}(\Omega)$, we define the following semi-norm

$$|v|^p_{\sigma,p,\Omega} \stackrel{\text{def}}{=} \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^p}{|x-y|^{n+\sigma p}} \, \mathrm{d}x \, \mathrm{d}y, \tag{3}$$

and norm

$$\|v\|^p_{\sigma,p,\Omega} \stackrel{\text{def}}{=} \|v\|^p_{0,p,\Omega} + |v|^p_{\sigma,p,\Omega}. \tag{4}$$

In the definition of the fractional Sobolev space, the index of the dominator in the quotient of difference depends on the dimension of the space. This can be seen more clearly if we use polar coordinate in the integration. The fractional derivative seems weird. We now give a concrete example.

Example 1.8. For the Heavisde function S restricted to $(-1,1)$, we look at the integral (3) in the skewed coordinate $\tilde{x} = x - y, \tilde{y} = x + y$. To be in the space $W^{\sigma,p}(\Omega)$, it is essential to have the integral

$$\int_0^1 \frac{1}{\tilde{x}^{\sigma p}} \, \mathrm{d}\tilde{x} < \infty.$$

That is to require $\sigma < 1/p$. In particular, we conclude $S \in H^{1/2-\epsilon}(-1,1)$ for any $0 < \epsilon \le 1/2$ but $S \notin H^{1/2}(-1,1)$.

Given $s = k + \sigma$ with a real number $\sigma \in (0,1)$ and an integer $k \ge 0$, define

$$W^{s,p}(\Omega) \stackrel{\text{def}}{=} \{v \in W^{k,p}(\Omega) : \ D^\alpha v \in W^{\sigma,p}(\Omega), |\alpha| \le k\}.$$

In $W^{s,p}(\Omega)$, we define the following semi-norm and norm

$$|v|_{s,p,\Omega} = \left(\sum_{|\alpha|=k} |D^\alpha v|^p_{\sigma,p,\Omega} \right)^{1/p}, \quad \|v\|_{s,p,\Omega} = \left(\sum_{|\alpha|\le k} \|D^\alpha v\|^p_{\sigma,p,\Omega} \right)^{1/p}.$$

Negative order Sobolev spaces. $W^{k,p}(\Omega)$ is a Banach space, i.e., it is complete in the topology induced by the norm $\|\cdot\|_{k,p,\Omega}$. Indeed $W^{k,p}(\Omega)$ is the closure of $C^\infty(\Omega)$ with respect to $\|\cdot\|_{k,p,\Omega}$. The closure of $C_0^\infty(\Omega)$ with respect to the same topology is denoted by $W_0^{k,p}(\Omega)$. For $p = 2$, we usually write $H_0^k(\Omega) = W_0^{k,2}(\Omega)$. Roughly speaking for $u \in H_0^1(\Omega)$, $u|_{\partial\Omega} = 0$ in an appropriate sense. Except $k = 0$ or $\Omega = \mathbb{R}^n$, $W_0^{k,p}(\Omega)$ is a proper subspace of $W^{k,p}(\Omega)$.

For $k \in \mathbb{N}$, $W^{-k,p}(\Omega)$ is defined as the dual space of $W_0^{k,p'}(\Omega)$, where p' is the conjugate of p, i.e., $1/p + 1/p' = 1$. In particular $H^{-k}(\Omega) = (H_0^k(\Omega))'$, and for $f \in H^{-1}(\Omega)$

$$\|f\|_{-1,\Omega} = \sup_{v \in H_0^1(\Omega)} \frac{\langle f, v \rangle}{\|v\|_{1,\Omega}}.$$

Since $C_0^\infty(\Omega)$ is dense in $W_0^{k,p}(\Omega)$, $W^{-k,p}(\Omega)$ can be thought as a subspace of $\mathcal{D}'(\Omega)$. Furthermore, we have the following characterization of $W^{-k,p}(\Omega)$ [11] which says, roughly speaking, $D^{-k}v \in L^p$ suppose we can define D^{-k} in an appropriate way.

Theorem 1.9. *Let $v \in \mathcal{D}'(\Omega)$. Then $v \in W^{-k,p}(\Omega)$ if and only if*

$$v = \sum_{|\alpha|\leq k} D^\alpha v_\alpha, \quad \text{for some } v_\alpha \in L^p(\Omega).$$

Note that $C_0^\infty(\Omega)$ is not dense in $W^{k,p}(\Omega)$ for $k \in \mathbb{N}$. Therefore the dual space of $W^{k,p}(\Omega)$ cannot be embedded as a subspace of the distribution space.

Example 1.10. Since the Heavside function $S \in L^2(-1,1)$, by Theorem 1.9, we have $\delta = S' \in H^{-1}(-1,1)$. But this is only true in one dimension; see Example 1.11.

Chacterization of Sobolev spaces using Fourier transform. When $\Omega = \mathbb{R}^n$, we can characterize the Sobolev space $H^k(\mathbb{R}^n)$ using the Fourier transform. Denoted by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz functions which consists of, roughly speaking, smooth functions and all derivatives decay faster than polynomials approaching to the infinity. Although the Fourier transform

$$\hat{f}(\xi) = \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,\mathrm{d}x$$

is well defined for $f \in L^1(\mathbb{R}^n)$, restricted to $\mathcal{S}(\mathbb{R}^n)$, the transform is isomorphism. And more importantly, restricted to $\mathcal{S}(\mathbb{R}^n)$, the differentiation changes to a multiplication, e.g. $\widehat{D^\alpha v} = (2\pi i\xi)^\alpha \hat{v}$ which can be proved using integration by parts.

Given a $v \in H^k(\mathbb{R}^n)$, by the Plancherel's theorem, we deduce that

$$\|D^\alpha v\|_{0,\mathbb{R}^n} = \|\xi^\alpha \hat{v}\|_{0,\mathbb{R}^n}.$$

Thus, by definition

$$\|v\|_{k,\mathbb{R}^n}^2 = \int_{\mathbb{R}^n}\Big(\sum_{|\alpha|\leq k}\xi^{2\alpha}\Big)|\hat{v}(\xi)|^2\,d\xi.$$

Using the elementary inequalities

$$(1+|\xi|^2)^k \lesssim \sum_{|\alpha|\leq k}(|\xi|^2)^\alpha \lesssim (1+|\xi|^2)^k, \tag{5}$$

we conclude that

$$\|v\|_{k,\mathbb{R}^n} \eqsim \|(1+|\cdot|^2)^{k/2}\hat{v}\|_{0,\mathbb{R}^n}.$$

This relation shows that the Sobolev space $H^k(\mathbb{R}^n)$ may be equivalently defined by

$$H^k(\mathbb{R}^n) = \{v \in L^2(\mathbb{R}^n), (1+|\xi|^2)^{k/2}\hat{v} \in L^2(\mathbb{R}^n)\}.$$

This alternative definition can be used to characterize Sobolev spaces with real index. Namely, for any given $s \in [0,\infty)$, the Sobolev space $H^s(\mathbb{R}^n)$ can be defined as follows:

$$H^s(\mathbb{R}^n) \stackrel{\text{def}}{=} \{v \in L^2(\mathbb{R}^n), (1+|\xi|^2)^{s/2}\hat{v} \in L^2(\mathbb{R}^n)\} \quad \text{for all } s \in [0,\infty) \tag{6}$$

with a norm defined by

$$\|v\|_{s,\mathbb{R}^n} \stackrel{\text{def}}{=} \|(1+|\cdot|^2)^{s/2}\hat{v}\|_{0,\mathbb{R}^n}. \tag{7}$$

Example 1.11. In $\mathbb{R}^n$, the Fourier transform of δ-distribution is one. By looking at the integration in the polar coordinate, we see $\delta \in H^{-s}(\mathbb{R}^n)$ for $s > n/2$ only. Therefore for $n \geq 2$, the delta distribution is not in $H^{-1}(\Omega)$.

1.3. **Extension theorems.** The extension theorem presented below is a fundamental result for Sobolev spaces. Fourier transform is a powerful tool. But unfortunately it only works for functions defined in the entire space $\mathbb{R}^n$. To extend results proved on the whole $\mathbb{R}^n$ to a bounded domain Ω, we can try to extend the function defined in $W^{k,p}(\Omega)$ to $W^{k,p}(\mathbb{R}^n)$. The extension of a function $u \in L^p(\Omega)$ is trivial. For example, we can simply set $u(x) = 0$ when $x \notin \Omega$ which is called zero extension. But such trivial extension will create a bad discontinuity along the boundary and thus cannot control the norm of derivatives especially the boundary is non-smooth. The extension of Sobolev space $W^{k,p}(\Omega)$ is subtle. We only present the result here.

Theorem 1.12. *For any bounded Lipschitz domain Ω, for any $s \geq 0$ and $1 \leq p \leq \infty$, there exits a linear operator $E : W^{s,p}(\Omega) \to W^{s,p}(\mathbb{R}^n)$ such that*

(1) $Eu|_\Omega = u$, and

(2) E is continuous. More precisely there exists a constant $C(s,\Omega)$ which is increasing with respect to $s \geq 0$ such that, for all $1 \leq p \leq \infty$,

$$\|Ev\|_{s,p,\mathbb{R}^n} \leq C(s,\Omega)\|v\|_{s,p,\Omega} \quad \textit{for all } v \in W^{s,p}(\Omega).$$

Theorem 1.12 is well-known for integer order Sobolev spaces defined on smooth domains and the corresonding proof can be found in most text books on Sobolev spaces. But for Lipschitz domains and especially for fractional order spaces Theorem 1.12 is less well-known and the the proof of the theorem for these cases is quite complicated. For integer order Sobolev spaces on Lipschitz domains, we refer to Calderón [4] or Stein [12]. For factional order Sobolev spaces on Lipschitz domains, we refer to the book by McLean [10].

1.4. **Embedding theorems.** Embedding theorems of Sobolev spaces are what make the Sobolev spaces interesting and important. The Sobolev spaces $W^{k,p}(\Omega)$ are defined using weak derivatives. The smoothness using weak derivatives is weaker than that using classic derivatives.

Example 1.13. In two dimensions, consider the function $u(x) = \ln|\ln|x||$ when $|x| < 1/e$ and $u(x) = 0$ when $|x| \geq 1/e$. It is easy to verify that $u \in H^1(\mathbb{R}^2)$. But u is unbounded, i.e., $u \notin C(\mathbb{R}^2)$.

Sobolev embedding theorem connects ideas of smoothness using "weak" and "classic" derivatives. Roughly speaking, it says that if a function is weakly smooth enough, then it implies certain classic smoothness.

Exercise 1.14. *Prove that if $u \in W^{1,1}(0,1)$, then $u \in L^\infty(0,1)$.*

We now present the general embedding theorem. For two Banach spaces B_1, B_0, we say B_1 is continuously embedded into B_0, denoted by $B_1 \hookrightarrow B_0$, if for any $u \in B_1$, it is also in B_0 and the embedding map is continuous, i.e., for all $u \in B_1$

$$\|u\|_{B_0} \lesssim \|u\|_{B_1}.$$

Theorem 1.15 (General Sobolev embedding)**.** *Let $1 \leq p \leq \infty, k \in \mathbb{Z}_+$ and Ω be a bounded Lipschitz domain in $\mathbb{R}^n$.*

Case 1. $kp > n$

$$W^{k,p}(\Omega) \hookrightarrow C(\bar{\Omega}).$$

Case 2. $kp = n$

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \textit{for all } q \in [1, \infty).$$

Furthermore

$$W^{n,1}(\Omega) \hookrightarrow C(\bar{\Omega}).$$

Case 3. $kp < n$

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \textit{with } \frac{1}{q} = \frac{1}{p} - \frac{k}{p}.$$

Exercise 1.16. *Use the Fourier transform to prove that if $u \in H^s(\mathbb{R}^n)$ for $s > n/2$, then $u \in L^\infty(\mathbb{R}^n)$. Then use the extension theorem to prove similar results for Lipschitz domains Ω.*

We take the following visualization of Sobolev spaces from DeVore [6] (page 93). This will give us a simple way to keep track of various results and also add to our understanding. We shall do this by using points in the upper right quadrant of the plane. The x-axis will correspond to the L^p spaces except that L^p is identified with $x = 1/p$ not with $x = p$. The y-axis will correspond to the order of smoothness. For example, the point $(1/p, k)$ represents the Sobolve space $W^{k,p}(\Omega)$. The line with slope n (the dimension of Euclidean spaces) passing through $(1/p, 0)$ is the demarcation line for embeddings of Sobolev spaces into $L^p(\Omega)$ (see Figure 1). Any Sobolev space with indices corresponding to a point above that line is embedded into $L^p(\Omega)$.

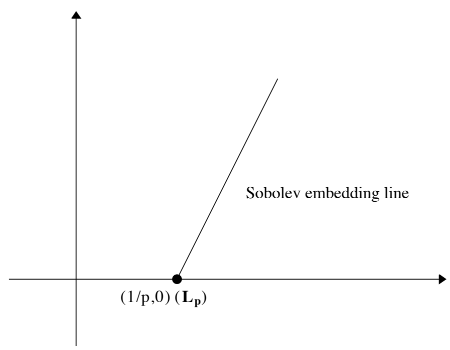

FIGURE 1. Sobolev embedding

To be quickly determine if a point lies above the demarcation line or not, we introduce the *Sobolev number*

$$\mathrm{sob}_n(k, p) = k - \frac{n}{p}.$$

If $\mathrm{sob}_n(k, p) > 0$, functions from $W^{k,p}(\Omega)$ are continuous (or more precisely can find a continuous representative in its equivalent class). In general

$$W^{k,p}(\Omega) \hookrightarrow W^{l,q}(\Omega) \quad \text{if } k > l \text{ and } \mathrm{sob}_n(k, p) > \mathrm{sob}_n(l, q).$$

Sobolev spaces corresponding to points on the demarcation line may or may not be embedded in $L^p(\Omega)$. For example, for $n \geq 1$

$$W^{n,1}(\Omega) \hookrightarrow L^\infty(\Omega) \qquad \text{but } W^{1,n}(\Omega) \not\hookrightarrow L^\infty(\Omega)$$

Note that $W^{1,n}(\Omega) \hookrightarrow L^p(\Omega)$, for all $1 \leq p < \infty$. Furthermore there exists a constant $C(n, \Omega)$ depending on the dimension of Euclidean space and the Lipschitz domain such that

$$\|v\|_{0,p,\Omega} \leq C(n,\Omega)\, p^{1-1/n} \|v\|_{1,n,\Omega} \quad \text{for all } 1 \leq p < \infty. \tag{8}$$

Example 1.17. By the emebedding theorem, we have

(1) $H^1(\Omega) \hookrightarrow C(\bar{\Omega})$ in one dimension (Exercise 1.14)
(2) $H^1(\Omega) \not\hookrightarrow C(\bar{\Omega})$ for $n \geq 2$ (Example 1.13). Namely there is no continuous representation of functions in $H^1(\Omega)$ for $n \geq 2$ and thus the point value is not well defined.
(3) In 2-D, $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for any $1 \leq p < \infty$; in 3-D, $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for $1 \leq p \leq 6$.

For a piecewise smooth function which is also in $H^1(\Omega)$, then it is globally continuous.

Exercise 1.18. *Let Ω, Ω_1 and Ω_2 are three open and bounded domains in $\mathbb{R}^n$. Suppose $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ and $\Omega_1 \cap \Omega_2 = \varnothing$. Let u be a function defined in Ω such that $u|_{\Omega_1} \in C^1(\Omega_1) \cap H^1(\Omega_1)$ and $u|_{\Omega_2} \in C^1(\Omega_2) \cap H^1(\Omega_2)$. Prove that $u \in H^1(\Omega)$ if and only if $u \in C(\Omega)$.*

1.5. **Interpolation theory.** Let H_0 and H_1 be two Hilbert spaces with H_1 continuously embedded and dense in H_0. An *intermediate space* H is a space satisfying

$$H_1 \subset H \subset H_0.$$

Hilbert scale is a technique to define a family of intermediate spaces between H_1 and H_0.

Assume that $(\cdot,\cdot)$ is the inner product on H_0. By a classical result, the space H_1 may be defined as the domain of an (unbounded) positive selfadjoint operator $\Lambda : H_1 \to H_0$ connecting the norms as follows:

$$\|u\|_{H_1} = \|\Lambda u\|_{H_0}.$$

Given $s \in (0,1)$, we define the intermediate spaces H_s to be the domain of Λ^s with a norm given by

$$\|u\|_{H_s} = \|\Lambda^s u\|_{H_0}.$$

Example 1.19. Given $-\infty < s_0 < s_1 < \infty$, let $H_0 = H^{s_0}(\mathbb{R}^n)$ and $H_1 = H^{s_1}(\mathbb{R}^n)$. The operator that connects H_0 and H_1 is as follows

$$\Lambda = \mathcal{F}^{-1}(1+|\xi|^2)^{(s_1-s_0)/2}\mathcal{F}.$$

In fact, by definition

$$\|v\|_{s_1,\mathbb{R}^n} = \|\Lambda v\|_{s_0,\mathbb{R}^n}.$$

By Hilbert scale, we have

$$[H^{s_1}(\mathbb{R}^n), H^{s_0}(\mathbb{R}^n)]_\theta = D(\Lambda^\theta) = H^{s_\theta}(\mathbb{R}^n) \text{ with } s_\theta = (1-\theta)s_0 + \theta s_1.$$

We now discuss a special case that the spectrum of Λ is discrete (λ_i) and the eigenvectors (ϕ_i) form a complete orthonormal basis for H_0. Then we may expand any element of H_0 as

$$u = \sum_{i=1}^{\infty} (u, \varphi_i)\varphi_i.$$

If $u \in H_1$, then

$$\Lambda u = \sum_{i=1}^{\infty} \lambda_i (u,\varphi_i)\varphi_i \ \text{ and } \ \|u\|_{H_1}^2 = \sum_{i=1}^{\infty} \lambda_i^2 (u,\varphi_i)^2.$$

In this case, the intermediate spaces H_s consists of those elements of H_0 for which the norm

$$\|u\|_{H_s} = \left(\sum_{i=1}^{\infty} \lambda_i^{2s}(u, \varphi_i)^2\right)^{1/2} \tag{9}$$

is finite.

Example 1.20. Assume Ω is a bounded Lipschitz domain. Let $H_1 = H_0^1(\Omega)$ and $H_0 = L^2(\Omega)$. Then H_1 can be viewed as the domain of $\Lambda = (-\Delta)^{1/2}$ with a norm given by

$$\|\nabla v\| = \|\Lambda v\| \quad \text{for all for } v \in H_0^1(\Omega).$$

The interpolated spaces $[H_0^1(\Omega), L^2(\Omega)]_s = H_0^s(\Omega)$ for $s \in (1/2, 1]$.

A simple application of Hölder inequality gives that

Theorem 1.21. *For $u \in H_1$*

$$\|u\|_{H_s} \leq \|u\|_{H_0}^{1-s} \|u\|_{H_1}^{s}$$

This definition use an embedding operator Λ. It can be shown that the definition of H_θ does not depend on the choice of Λ using the K-method or J-method; See the book by Bergh and Lofstrom [3].

Exercise 1.22. *Integrate by parts to prove the interpolation inequality*

$$\int_\Omega |Du|^2 \, \mathrm{d}x \leq C \left(\int_\Omega u^2 \, \mathrm{d}x\right)^{1/2} \left(\int_\Omega |D^2 u|^2 \, \mathrm{d}x\right)^{1/2}$$

for $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Now we consider the interpolation of operators between interpolation spaces. Let $\widetilde{H}_1 \hookrightarrow \widetilde{H}_0$ and $H_1 \hookrightarrow H_0$. Further let L be a linear operator such that

$$L : H_0 \mapsto \widetilde{H}_0, \ \text{ with } \|Lu\|_{\widetilde{H}_0} \leq C_0 \|u\|_{H_0} \quad \text{for all } u \in H_0,$$

and

$$L : H_1 \mapsto \widetilde{H}_1, \ \text{ with } \|Lu\|_{\widetilde{H}_1} \leq C_1 \|u\|_{H_1} \quad \text{for all } u \in H_1.$$

It is reasonable to expect $L : H_\theta \mapsto \widetilde{H}_\theta$ is also bounded.

Theorem 1.23. *Suppose that $H_i, \widetilde{H}_i, i = 0, 1$ and L are given above. Then $L : H_\theta \mapsto \widetilde{H}_\theta$ is also bounded and*

$$\|Lu\|_{\widetilde{H}_\theta} \leq C_0^{1-\theta} C_1^{\theta} \|u\|_{H_\theta} \quad \textit{for all } u \in H_1.$$

1.6. **Trace Theorems.** The trace theorem is to define function values on the boundary. If $u \in C(\overline{\Omega})$, then $u(x)$ is well defined for $x \in \partial\Omega$. For $u \in W^{k,p}(\Omega)$, however, the function is indeed defined as an equivalent class of Lebesgue integrable functions, i.e., $u \sim v$ if and only if $u = v$ almost everywhere. The boundary $\partial\Omega$ is a measure zero set (in nth dimensional Lebesgue measure) and thus the point wise value of $u|_{\partial\Omega}$ is not well defined for functions in Sobolev spaces.

Theorem 1.24. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth or Lipschitz boundary $\Gamma = \partial\Omega$. Then the trace operator $\gamma : C^1(\overline{\Omega}) \mapsto C(\Gamma)$ can be continuously extended to $\gamma : H^1(\Omega) \mapsto H^{1/2}(\Gamma)$, i.e., the trace inequality holds:*

$$\|\gamma(u)\|_{1/2,\Gamma} \lesssim \|u\|_{1,\Omega}, \quad \textit{for all } u \in H^1(\Omega). \tag{10}$$

Furthermore the trace operator is surjective and has a continuous right inverse.

More general, if Ω is a Lipschitz domain, then the trace operator $\gamma : H^s(\Omega) \to H^{s-1/2}(\Gamma)$ is bounded for $1/2 < s < 3/2$. See McLean [10] (pages 100–106).

1.7. **Norm Equivalence Theorems.** The definition of norm $\|\cdot\|_{k+1,p}$ involves all the ith derivatives for $i \leq k+1$. Among them the highest one is the critical one. That is we can prove

$$\|u\|_{k+1,p} \eqsim \|u\|_{0,p} + |u|_{k+1,p} \quad \text{for all } u \in W^{k+1,p}(\Omega).$$

Let $\mathcal{P}_k(\Omega)$ denote the space of polynomials with degree $\leq k$. Considering this fact for the decomposition

$$W^{k+1,p}(\Omega) = \mathcal{P}_k(\Omega) \oplus (W^{k+1,p}(\Omega)/\mathcal{P}_k(\Omega)),$$

on the quotient space $W^{k+1,p}(\Omega)/\mathcal{P}_k(\Omega)$, the seminorm $|u|_{k+1,p}$ is indeed a norm.

Lemma 1.25. *For any $u \in W^{k+1,p}(\Omega)$, one has*

$$\inf_{p \in \mathcal{P}_k(\Omega)} \|u+p\|_{k+1,p,\Omega} \eqsim |u|_{k+1,p,\Omega}.$$

On the polynomial space $\mathcal{P}_k(\Omega)$, since it is finite dimensional, the norm $\|\cdot\|_{0,p}$ can be replaced by any other norms defined on $\mathcal{P}_k(\Omega)$. In general, we can consider a semi-norm $F(\cdot)$ on $W^{k+1,p}(\Omega)$, i.e., $F: W^{k+1,p}(\Omega) \to \mathbb{R}^+$ and

$$F(u+v) \leq F(u) + F(v), \text{ and } F(\alpha\, u) \leq |\alpha| F(u).$$

The difference of a norm and a semi-norm is the null space $\ker(F)$. If it is trivial, i.e., containing only zero, then F is a norm. For a semi-norm F of $W^{k+1,p}(\Omega)$, with the condition: for $p \in \mathcal{P}_k(\Omega)$, $F(p) = 0$ if and only if $p = 0$, it will define a norm on the subspace $\mathcal{P}_k(\Omega)$.

Theorem 1.26 (Sobolev norm equivalence)**.** *If F is a seminorm on $W^{k+1,p}(\Omega)$ such that for $p \in \mathcal{P}_k(\Omega)$, $F(p) = 0$ if and only if $p = 0$, then*

$$\|v\|_{k+1,p} \eqsim F(v) + |v|_{k+1,p} \quad \textit{for all } v \in W^{k+1,p}(\Omega). \tag{11}$$

As a special case of the Sobolev norm equivalence theorem, we present the following variants of Poincaré or Friedriches inequalities. Assume that Γ is a measurable subset of $\partial\Omega$ with positive $n-1$ dimensional Lebesgue measure. Choosing $F(v) = |\int_\gamma v\, ds|$ in (11), we obtain the Friedrichs inequality

$$\|v\|_{1,p,\Omega} \lesssim |v|_{1,p,\Omega} + \left|\int_\Gamma v\, ds\right| \quad \text{for all } v \in W^{1,p}(\Omega).$$

Consequently, we have the Poincaré inequality

$$\|v\|_{0,p} \lesssim |v|_{1,p,\Omega} \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

Choosing $F(v) = \int_\Omega v\, dx$, we obtain the Poincaré-Friedrichs inequality

$$\|v\|_{1,p,\Omega} \lesssim |v|_{1,p,\Omega} + \left|\int_\Omega v\, dx\right| \quad \text{for all } v \in W^{1,p}(\Omega).$$

Consequently, let $\bar{v} = \int_\Omega v dx/|\Omega|$ denote the average of v over Ω, we get the average-type Poincaré inequality

$$\|v - \bar{v}\|_{0,p,\Omega} \lesssim |v|_{1,p,\Omega} \quad \text{for all } v \in W^{1,p}(\Omega).$$

All constants hidden in the notation $\lesssim$ depends on the size and shape of the domain Ω, which is important when apply them to one simplex with diameter h as in finite element methods.

2. ELLIPTIC BOUNDARY VALUE PROBLEMS AND REGULARITY

2.1. **Weak formulation.** Let us take the Poisson equation with homogenous Dirichlet boundary condition

$$-\Delta u = f \text{ in } \Omega, \quad \text{and } u|_{\partial\Omega} = 0, \tag{12}$$

as an example to illustrate the main idea. If there exists a function $u \in C^2(\Omega) \cap C_0(\Omega)$ satisfying the Poisson equation, we call u a *classic solution*. The smoothness of u excludes many interesting solutions for physical problems. We need to seek a weak solution in more broader spaces – Sobolev spaces. Here "weak" means the smoothness is imposed by weak derivative which is weaker than the classic one.

Recall the basic idea of Sobolev space is to treat function as functional. Let us try to understand the equation (12) in the distribution sense. We seek a solution $u \in \mathcal{D}'(\Omega)$ such that for any $\phi \in C_0^\infty(\Omega)$,

$$\langle -\nabla \cdot \nabla u, \phi\rangle := \langle \nabla u, \nabla \phi\rangle = \langle f, \phi\rangle,$$

which suggests a weak formulation: find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla \phi = \int_\Omega f\phi, \quad \forall \phi \in C_0^\infty(\Omega). \tag{13}$$

We will not discretize (13) directly since it is difficult to construct a finite dimensional subspace of $C_0^\infty(\Omega)$.

We first extend the action of ∇u on $C_0^\infty(\Omega)$ to a broader space. Let us define a bilinear form on $H^1(\Omega) \times C_0^\infty(\Omega)$:

$$a(u, \phi) = \int_\Omega \nabla u \cdot \nabla \phi \, dx.$$

By Cauchy-Schwarz inequality,

$$a(u, \phi) \leq a(u, u)a(\phi, \phi) \lesssim \|\nabla u\|\|\nabla \phi\|.$$

Thus $a(u, \cdot)$ is continuous in the H^1 topology. Thanks to the fact $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, the bilinear form $a(\cdot, \cdot)$ can be continuously extend to $U \times V := H^1(\Omega) \times H_0^1(\Omega)$. Here the space U is the one we seek a solution and thus called *trial space* and V is still called *test space*.

In (13) the right-hand side $f \in L^2(\Omega) \cong L^2(\Omega)'$. After $C_0^\infty(\Omega)$ is extend to $H_0^1(\Omega)$, we can take $f \in H^{-1}(\Omega) := (H_0^1(\Omega))' = V'$.

Boundary conditions can be imposed by choosing different trial and test spaces. For example:

- homogenous Dirichlet boundary condition:

$$U = H_0^1(\Omega), \ V = H_0^1(\Omega) \tag{14}$$

- homogenous Neumann boundary condition:

$$U = H^1(\Omega)/\mathbb{R}, \ V = H^1(\Omega). \tag{15}$$

We are in the position to present an abstract version of the variational (or so-called weak) formulation of the Poisson equation: given an $f \in V'$, find a solution $u \in U$ such that

$$a(u, v) = \langle f, v\rangle \quad \forall v \in V. \tag{16}$$

2.2. **Regularity theory.** The weak solution u can be proved to be a solution of (12) in a more classic sense if u is smooth enough such that we can integration by parts back. The theory for proving the smoothness of the weak solution is called *regularity* theory, which is a bridge to connect classical and weak solutions.

The regularity theory on elliptic equations is very important for the theory of finite element approximation and convergence of multigrid methods. However it is often not very straightforward. We only give a brief account of this theory.

We first give a formal derivation for the model problem $-\Delta u = f$ in $\mathbb{R}^n$ and assume u is smooth and vanishes sufficiently rapidly as $|x| \to \infty$ to justify the following integration by parts:

$$\int_{\mathbb{R}^n} f^2 dx = \int_{\mathbb{R}^n} (\Delta u)^2 \, \mathrm{d}x = \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \partial_{ii} u \partial_{jj} u \, \mathrm{d}x$$

$$= - \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \partial_{iij} u \partial_j u \, \mathrm{d}x = \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \partial_{ij} u \partial_{ij} u \, \mathrm{d}x = \int_{\mathbb{R}^n} |D^2 u|^2 \, \mathrm{d}x.$$

We can also see it from Fourier transform for the Laplacian operator on the whole space. Given a distribution v defined on $\mathbb{R}^n$ such that $\Delta v \in L^2$, by the properties of Fourier transform, we have

$$\widehat{D^\alpha v}(\xi) = (i\xi)^\alpha \hat{v}(\xi) = -(i\xi)^\alpha |\xi|^{-2} \widehat{\Delta v}(\xi).$$

The function $(i\xi)^\alpha |\xi|^{-2}$ is bounded by 1 if $|\alpha| = 2$, hence

$$\|\widehat{D^\alpha v}\|_{0,\mathbb{R}^n} \leq \|\widehat{\Delta v}\|_{0,\mathbb{R}^n}.$$

By Parseval identity, we have

$$\|D^\alpha v\|_{0,\mathbb{R}^n} \leq \|\Delta v\|_{0,\mathbb{R}^n} \quad \text{for all } |\alpha| = 2.$$

The above inequality illustrates an important fact that if v is a function such that $\Delta v \in L^2$, then all its second order derivatives are also in L^2. That is the Frobenius norm of the Hessian matrix can be bounded by its trace which is true if looking at the Fourier transform. This is a rather significant fact since Δv is a very special combination of the second order derivatives of v.

Exercise 2.1. Prove the existence and uniqueness of the weak solution for the Poisson equation with homogenous Dirichlet or Neumann boundary conditions.

Smooth and bounded domains. The properties of elliptic operators, described above can be extended to bounded domains with smooth boundary, but such an extension is not trivial. The following theorem is well-known and it can be found in most of the text books on elliptic boundary value problems.

Theorem 2.2. *Let Ω be a smooth and bounded domain of $\mathbb{R}^n$. Then for each $f \in L^2(\Omega,$ there exists a unique $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying $-\Delta u = f$ and*

$$\|u\|_{2,\Omega} \leq C \|f\|_{0,\Omega}$$

where C is a positive constant depending on Ω.

For general domains, when $f \in L^2(\Omega)$, the solution $u \in H^2_{\text{loc}}(\Omega)$. Namely for each domain $U \subset\subset \Omega$, $u \in H^2(U)$. The local result can be proved by using the difference quotient to define a discrete approximation of $D^2 u$. The condition $U \subset\subset \Omega$ enables us to choose a smooth cut-off function away from the boundary. Along the boundary, by means of a suitable change of coordinates, locally all tangential derivatives can be proved similarly. For the one involving normal derivative, use the equation to bound by f and combination of others. The boundary of Ω is smooth enough so that a partition of unit can be found and glue the local chat together. For a detailed proof, see e.g. [7].

Lipschitz domains. Theorem 2.2, however, does not hold on general Lipschitz domains. The requirements of Ω is necessary. When $\partial\Omega$ is only Lipschitz continuous, we may not be able to glue boundary pieces on the corner points. This is not only a technical difficulty. We shall give an example in which the regularity of u depends on the shape of Ω.

Example 2.3. Given $\beta \in (0,1)$, consider the following non-convex domain

$$\Omega = \{(r,\theta) : 0 < r < 1, 0 < \theta < \pi/\beta\}.$$

Let $v = r^\beta \sin(\beta\theta)$. Being the imaginary part of the complex analytic functin z^β, v is harmonic in Ω. Define $u = (1-r^2)v$. A direct calculation shows that

$$-\Delta u = 4(1+\beta)v \text{ in } \Omega, \quad \text{and } u|_{\partial\Omega} = 0.$$

Note that $4(1+\beta)v \in L^\infty(\Omega) \subset L^2(\Omega)$, but $u \notin H^2(\Omega)$.

Nevertheless, a slightly weaker result does hold for general Lipschitz domains.

Theorem 2.4. *Assume that Ω is a bounded Lipschitz domain. Then there exists a constant $\alpha \in (0,1]$ such that*

$$\|u\|_{1+\alpha} \le C\|f\|_{\alpha-1}, \tag{17}$$

for the solution u of (16), where C is a constant depending on the domain Ω.

Again a proof for the above regularity result is quite complicated, we refer to [9, 5, 2].

Convex domains. A remarkable fact is that we can take $\alpha = 1$ in the above theorem for convex domains. This means that Theorem 2.2 can be extended to convex domains.

Theorem 2.5. *Let Ω be a convex and bounded domain of $\mathbb{R}^n$. Then for each $f \in L^2(\Omega)$, there exists a unique $u \in H^2(\Omega)$ to the solution of (16) that satisfies*

$$\|u\|_{2,\Omega} \le C\|f\|_{0,\Omega}$$

where C is a positive constant depending only on the diameter of Ω.

We outline a proof for a convex polygon and refer to [13] for details. First of all, the convex polygon can be approximated from interior by a sequence of smooth and convex domains $\{\Omega_k\}$. On each Ω_k, we establish a H^2-regularity result $\|u_k\|_2 \le C(\Omega_k)\|f\|$ for the solution $-\Delta u_k = f$ in Ω_k and $u_k = 0$ on $\partial\Omega_k$. Then taking limit in an appropriate topology to get the desired result on Ω. To be able to take the limit, the constant $C(\Omega_k)$ should be uniformly bounded which requires a careful study on the regularity result on Ω_k. One useful identity, cf. [8], is the refined integration by parts

$$\int_U (\Delta u)^2 \,\mathrm{d}x = \sum_{i,j=1}^n \int_U |\partial_{ij} u|^2 \,\mathrm{d}x + \int_{\partial U} \operatorname{div} \boldsymbol{\nu} \left|\frac{\partial u}{\partial \boldsymbol{\nu}}\right|^2 \,\mathrm{d}S, \tag{18}$$

where $\boldsymbol{\nu}$ is the outwards unit normal vector of ∂U. When U is C^2 and convex, $\operatorname{div}\boldsymbol{\nu} = \operatorname{trace}(D\boldsymbol{\nu}) \ge 0$ and thus the last term is positive which leads to $\|D^2 u\|_U \le \|\Delta u\|_U$. Lower order terms in H^2-norm can be bounded by Poincaré inequality and the constant depends only on the diameter $\operatorname{diam}(\Omega_k) \le \operatorname{diam}(\Omega)$ uniformly.

Exercise 2.6. Prove identity (18) using the identity

$$-\Delta \boldsymbol{f} = -\mathrm{grad}\ \mathrm{div}\ \boldsymbol{f} + \mathrm{curl}\,\mathrm{curl}\,\boldsymbol{f},$$

and then integration by parts by multiplying $\boldsymbol{f} = \nabla u$ to both sides and integrate over U.

Concave polygons. When Ω is concave polygon, we do not have full regularity. There are at least two ways to obtain analogous (but weaker) results for concave domain. The first one (see [9, 5, 2]) is to use fractional Sobolev spaces: namely $-\Delta : H_0^1(\Omega) \cap H^{1+s}(\Omega) \mapsto H^{-1+s}(\Omega)$ which holds for any $s \in [0, s_0)$ for some $s_0 \in (0, 1]$ depending on Ω.

Another approach is to use L^p space, instead of L^2. It can be proved that (cf. [8])

$$-\Delta : H_0^1(\Omega) \cap W^{2,p}(\Omega) \mapsto L^p(\Omega) \text{ is an isomorphism} \tag{19}$$

which holds for any $p \in (1, p_0)$ for some $p_0 > 1$ that depends on the domain Ω. We shall discuss more in the discussion of adaptive finite element methods.

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