A SHORT INTRODUCTION TO SOBOLEV SPACES AND ELLIPTIC EQUATIONS

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1. Essential facts for Sobolev spaces

In this section, we present and explain main results (without proofs) on Sobolev spaces. We refer to [1] for comprehensive treatment of Sobolev spaces.

1.1. **Preliminaries.** To establish the context for our discussion, we introduce the basic concepts of Lipschitz domains, multi-index notation for differentiation, and several fundamental functional spaces.

Lipschitz domains. We focus on bounded Lipschitz domains. A domain $\Omega \subset \mathbb{R}^n$ is said to be Lipschitz if its boundary $\partial\Omega$ can be locally represented as the graph of a Lipschitz continuous function.

All smooth domains are Lipschitz, but many non-smooth domains also satisfy this property. For example, every polygonal domain in $\mathbb{R}^2$ and every convex domain in $\mathbb{R}^n$ is Lipschitz. Domains such as two polygons touching only at a single vertex are not Lipschitz, and a domain with a cusp on the boundary is also non-Lipschitz.

Derivatives. Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_+^n$, where $\mathbb{Z}_+$ is the set of non-negative integers, be a vector of nonnegative integers. For a smooth function v and $x \in \mathbb{R}^n$, denote $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x^\alpha} = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad |\alpha| = \sum_{i=1}^n \alpha_i.$$

Some basic functional spaces. Several basic Banach spaces will be frequently used in this course. The space $C(\bar\Omega)$ consists of continuous functions on $\bar\Omega$ equipped with the usual maximum norm

$$\|v\|_{C(\bar\Omega)} = \max_{x\in\bar\Omega} |v(x)|.$$

The space $C_0^\infty(\Omega)$ denotes infinitely differentiable functions in Ω that vanish in some neighborhood of $\partial\Omega$; namely, any $v \in C_0^\infty(\Omega)$ satisfies $\operatorname{supp}(v) \subset \Omega$, where

$$\operatorname{supp}(v) = \text{closure of } \{x \in \Omega : v(x) \neq 0\}.$$

Let $E(\Omega)$ represent the equivalence class of Lebesgue integrable functions under the relation $u \sim v$ if $u = v$ almost everywhere. For $1 \le p < \infty$, the Lebesgue space of pth-power integrable functions is defined by

$$L^p(\Omega) = \{v \in E(\Omega) : \int_\Omega |v|^p \,\mathrm{d}x < \infty\},$$

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and the space of essentially bounded functions is denoted by $L^\infty(\Omega)$. The corresponding norms are defined as

$$\|u\|_{p,\Omega} = \left(\int_\Omega |u|^p \,\mathrm{d}x \right)^{1/p}, \quad 1 \leq p < \infty, \qquad \|u\|_{\infty,\Omega} = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|.$$

1.2. **Distributions.** Traditionally, functions are defined through their pointwise values. However, this viewpoint is too restrictive for generalized functions, such as distributions. To overcome this limitation, we redefine a function as a *functional*—an object that acts on a space of smooth test functions. Within this framework, the notion of a function extends naturally to distributions, and the operation of integration by parts allows us to generalize differentiation to this broader setting.

Interaction. How do we know particles exist? Early scientists could see molecules through microscopes or chemical traces. But deeper exploration revealed particles like protons, neutrons, electrons, and photons that could never be seen directly.

Taka a classic example is X-ray crystallography. When X-rays strike a crystal, their wavelength—comparable to atomic spacing—causes a distinct interference pattern. By analyzing this pattern, scientists reconstruct the crystal's atomic structure. Thus we "see" atoms not with our eyes, but through the traces left by their interaction with light.

Mathematics follows a similar idea. We first understood a function as something we could "see": ideally, it had an explicit formula such as $f(x) = \sin x$, or at least a program that takes an input x and returns an output $y = f(x)$. This pointwise notion of a function once sufficed, but it fails when studying partial differential equations (PDEs).

Many PDEs admit solutions that are not ordinary functions—they may blow up, be discontinuous, or not even be defined at every point. Yet we still wish to understand these "invisible" functions. The approach, much like in physics, is not to observe the function directly, but to study how it interacts with well-behaved test functions.

Distributions. We begin with the space of smooth test functions $C_0^\infty(\Omega)$, consisting of infinitely differentiable functions with compact support in Ω. This space is a real vector space that can be equipped with a suitable topology, making it a topological vector space denoted by $\mathcal{D}(\Omega)$.

The space of all continuous linear functionals on $\mathcal{D}(\Omega)$ is called the space of *distributions*, denoted by $\mathcal{D}'(\Omega)$. It is endowed with the weak-star topology: a sequence $\{T_n\}$ in $\mathcal{D}'(\Omega)$ converges to T in the sense of distributions if and only if

$$\langle T_n, \phi\rangle \to \langle T, \phi\rangle \quad \text{for all } \phi \in \mathcal{D}(\Omega),$$

where $\langle \cdot, \cdot\rangle : \mathcal{D}'(\Omega) \times \mathcal{D}(\Omega) \to \mathbb{R}$ denotes the duality pairing. A function $\phi \in \mathcal{D}(\Omega)$ is called a *test function* because the action of a distribution on ϕ can be interpreted as a test, revealing properties of the distribution through suitably chosen ϕ.

Example 1.1. By definition, an element in $\mathcal{D}'(\Omega)$ is uniquely determined by its action. The action can be abstract, as long as it is linear and continuous. For example, the Dirac delta distribution $\delta \in \mathcal{D}'(\Omega)$, defined for $0 \in \Omega \subseteq \mathbb{R}^n$, acts as

$$\langle \delta, \phi\rangle = \phi(0) \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

An essential class of distributions uses integration as the primary action. A function is locally integrable if it is Lebesgue integrable over every compact subset of Ω; denote this space by $L^1_{\mathrm{loc}}(\Omega)$. We embed $L^1_{\mathrm{loc}}(\Omega)$ into $\mathcal{D}'(\Omega)$ by the duality

$$\langle T_u, \phi\rangle = \int_\Omega u\,\phi \,\mathrm{d}x \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

For $u \in L^1_{\mathrm{loc}}(\Omega)$ we write T_u simply as u, identifying the "ordinary" function with the corresponding distribution.

Distributions are also called generalized functions, since they extend classical functions. A basic example that is not an ordinary function is the Dirac delta δ from Example 1.1. One motivation for introducing $\mathcal{D}'(\Omega)$ is to include δ within a coherent function-space framework.

Exercise 1.2. *Prove that the Dirac delta distribution cannot be represented by a locally integrable function.*

Weak Derivatives. When u is smooth, integration by parts gives, for any multi-index $\alpha \in \mathbb{Z}^n_+$ and any $\phi \in \mathcal{D}(\Omega)$,

$$\int_\Omega D^\alpha u(x)\phi(x)\,\mathrm{d}x = (-1)^{|\alpha|}\int_\Omega u(x)D^\alpha\phi(x)\,\mathrm{d}x.$$

Since ϕ has compact support in Ω, no boundary terms appear, and both sides are well defined. This identity provides the basis for defining derivatives for distributions. Specifically, for $T \in \mathcal{D}'(\Omega)$, the weak derivative $D^\alpha T$ is defined by

$$\langle D^\alpha T, \phi\rangle = (-1)^{|\alpha|}\langle T, D^\alpha\phi\rangle \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

The weak derivative of a differentiable function agrees with its classical derivative. In general, the weak derivative is less restrictive, extending differential operators from smooth functions to the larger space of distributions. For example, one can define derivatives for discontinuous functions.

Example 1.3. The Heaviside step function is defined by $S(x) = 1$ for $x > 0$ and $S(x) = 0$ for $x < 0$. By definition,

$$\int_{\mathbb{R}} S'\phi\,\mathrm{d}x = -\int_{\mathbb{R}} S\phi'\,\mathrm{d}x = -\int_0^\infty \phi'\,\mathrm{d}x = \phi(0).$$

Hence $S' = \delta$ in the distribution sense, although δ is not a function in $L^1_{\mathrm{loc}}(\Omega)$ (Exercise 1.2). Roughly speaking, every distribution is locally a derivative, possibly of higher order, of a continuous function. A precise statement can be found in Rudin [11].

The formal definition of distributions places them within a very large space. In general, no proper subset of $\mathcal{D}'(\Omega)$ contains all continuous functions and remains closed under differentiation. This shows that the distribution framework extends the notion of functions in the most economical way possible. Every distribution is infinitely differentiable in the distribution sense.

1.3. **Sobolev spaces.** The Sobolev space of index (k, p), where k is a nonnegative integer and $p \geq 1$, is defined by

$$W^{k,p}(\Omega) \stackrel{\text{def}}{=} \{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega) \text{ for all } |\alpha| \leq k\},$$

with the norm

$$\|v\|^p_{k,p,\Omega} \stackrel{\text{def}}{=} \sum_{|\alpha|\leq k} \|D^\alpha v\|^p_{0,p,\Omega}.$$

We also use the seminorm

$$|v|^p_{k,p,\Omega} \stackrel{\text{def}}{=} \sum_{|\alpha|=k} \|D^\alpha v\|^p_{0,p,\Omega}.$$

For $p = 2$, it is customary to write $H^k(\Omega) \stackrel{\text{def}}{=} W^{k,2}(\Omega)$, which is a Hilbert space with the inner product

$$(u, v) = \sum_{|\alpha| \leq k} (D^\alpha u, D^\alpha v),$$

and the corresponding norm $\|v\|_{k,\Omega} = \|v\|_{k,2,\Omega}$. In most cases, we will focus on Sobolev spaces with $p = 2$. It can be shown that $W^{k,p}(\Omega)$ is a Banach space.

For a function $u \in L^p(\Omega)$, viewed as a distribution, its weak derivatives always exist as distributions, but they may not belong to $L^p(\Omega)$. Hence an element in $W^{k,p}(\Omega)$ possesses a certain level of smoothness.

Example 1.4. Consider the Heaviside function S restricted to $(-1, 1)$. Its weak derivative is the Dirac delta distribution, which is not integrable. Therefore $S \notin H^1(-1, 1)$.

Example 1.5. Let $u(x) = \mathrm{ReLU}(x) := \max\{0, x\}$ for $x \in (-1, 1)$. Then $u' = S$. Clearly $u \in L^2(-1, 1)$ and $u' \in L^2(-1, 1)$. Hence $u \in H^1(-1, 1)$.

Examples 1.4 and 1.5 show that for a piecewise smooth function u to belong to $H^1(\Omega)$, it must satisfy a global smoothness condition across Ω. See Exercise 1.12.

Negative order Sobolev spaces. The space $W^{k,p}(\Omega)$ is a Banach space, i.e., it is complete under the norm $\|\cdot\|_{k,p,\Omega}$. In fact, $W^{k,p}(\Omega)$ is the closure of $C^\infty(\Omega)$ with respect to this norm. The closure of $C_0^\infty(\Omega)$ in the same topology is denoted by $W_0^{k,p}(\Omega)$. For $p = 2$, we write $H_0^k(\Omega) = W_0^{k,2}(\Omega)$.

For $k \in \mathbb{N}$, the negative-order Sobolev space $W^{-k,p}(\Omega)$ is defined as the dual of $W_0^{k,p'}(\Omega)$, where p' is the conjugate exponent of p, satisfying $1/p + 1/p' = 1$. In particular, for $p = p' = 2$,

$$H^{-k}(\Omega) = (H_0^k(\Omega))', \qquad \|f\|_{-1,\Omega} = \sup_{v \in H_0^1(\Omega)} \frac{\langle f, v\rangle}{\|v\|_{1,\Omega}}.$$

Since $C_0^\infty(\Omega)$ is dense in $W_0^{k,p}(\Omega)$, we can view $W^{-k,p}(\Omega)$ as a subspace of the distribution space $\mathcal{D}'(\Omega)$. Moreover, we have the following characterization [11], which says roughly that $D^{-k}v \in L^p(\Omega)$ if we could define D^{-k} appropriately.

Theorem 1.6. *Let $v \in \mathcal{D}'(\Omega)$. Then $v \in W^{-k,p}(\Omega)$ if and only if*

$$v = \sum_{|\alpha| \leq k} D^\alpha v_\alpha, \quad \textit{for some } v_\alpha \in L^p(\Omega).$$

Note that $C_0^\infty(\Omega)$ is not dense in $W^{k,p}(\Omega)$ for $k \in \mathbb{N}$. Hence, the dual of $W^{k,p}(\Omega)$ cannot be embedded as a subspace of $\mathcal{D}'(\Omega)$.

Example 1.7. Since the Heaviside function $S \in L^2(-1, 1)$, by Theorem 1.6, its derivative $\delta = S'$ belongs to $H^{-1}(-1, 1)$. This statement, however, holds only in one dimension.

1.4. **Embedding theorems.** Sobolev spaces $W^{k,p}(\Omega)$ are defined using weak derivatives, which describe smoothness in a weaker sense than classical derivatives. Sobolev embedding theorems bridge these two notions of smoothness, showing that sufficient weak smoothness implies a certain degree of classical regularity.

Example 1.8. In two dimensions, consider the function

$$u(x) = \begin{cases} \ln|\ln|x||, & |x| < 1/e, \\ 0, & |x| \geq 1/e. \end{cases}$$

It can be verified that $u \in H^1(\mathbb{R}^2)$, yet u is unbounded; that is, $u \notin C(\mathbb{R}^2)$.

Exercise 1.9. *Prove that if* $u \in W^{1,1}(0,1)$*, then* $u \in L^\infty(0,1)$.

We now present the general embedding theorem. For two Banach spaces B_1 and B_0, we say that B_1 is *continuously embedded* into B_0, written $B_1 \hookrightarrow B_0$, if every $u \in B_1$ also belongs to B_0 and the embedding map is continuous; that is, for all $u \in B_1$,

$$\|u\|_{B_0} \lesssim \|u\|_{B_1}.$$

If, in addition, the embedding map is compact—meaning it maps every bounded set in B_1 into a precompact set in B_0—then B_1 is said to be *compactly embedded* into B_0, denoted by $B_1 \subset\subset B_0$.

Theorem 1.10 (General Sobolev embedding)**.** *Let* $1 \le p \le \infty, k \in \mathbb{Z}_+$ *and* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^n$.

Case 1. $kp > n$

$$W^{k,p}(\Omega) \hookrightarrow C(\bar{\Omega}).$$

Case 2. $kp = n$

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \textit{for all } q \in [1, \infty).$$

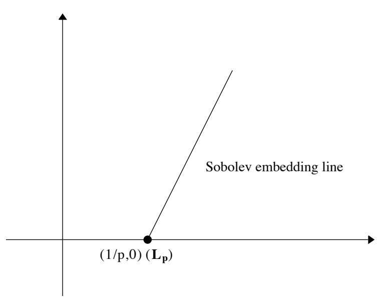
Furthermore

$$W^{n,1}(\Omega) \hookrightarrow C(\bar{\Omega}).$$

Case 3. $kp < n$

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \textit{with } \frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$$

We can visualize Sobolev spaces in the upper-right quadrant of a plane, where the x-axis represents L^p spaces, with each L^p identified by $x = 1/p$ rather than $x = p$. The y-axis corresponds to the order of smoothness. Thus, the point $(1/p, k)$ represents the Sobolev space $W^{k,p}(\Omega)$.



The line passing through $(1/p, 0)$ with slope n (the spatial dimension) serves as the boundary for embeddings of Sobolev spaces into $L^p(\Omega)$. Any Sobolev space whose indices correspond to a point above this line is embedded into $L^p(\Omega)$.
This visualization, adapted from DeVore [6] (p. 93), provides a simple and intuitive way to organize and interpret various Sobolev embedding results.

To quickly determine whether a point lies above or below the demarcation line, we introduce the *Sobolev number*:

$$\mathrm{sob}_n(k,p) = k - \frac{n}{p},$$

which measures the deviation from the line $k = n/p$ in the $(1/p, k)$ plane. If $\mathrm{sob}_n(k,p) > 0$, the point lies above the embedding line, and functions in $W^{k,p}(\Omega)$ are continuous (or, more precisely, can be represented by continuous functions in their equivalence classes).

In general, we have the embedding result:

$$W^{k,p}(\Omega) \hookrightarrow W^{l,q}(\Omega) \quad \text{if } k > l \text{ and } \mathrm{sob}_n(k,p) > \mathrm{sob}_n(l,q).$$

This means that if the point representing $W^{k,p}(\Omega)$ lies above that of $W^{l,q}(\Omega)$ and $k > l$, then $W^{k,p}(\Omega)$ is continuously embedded in $W^{l,q}(\Omega)$.

Sobolev spaces corresponding to points on the demarcation line may or may not embed into $L^p(\Omega)$. For example, for $n \geq 1$,

$$W^{n,1}(\Omega) \hookrightarrow L^\infty(\Omega), \qquad W^{1,n}(\Omega) \not\hookrightarrow L^\infty(\Omega).$$

However, we have

$$W^{1,n}(\Omega) \hookrightarrow L^p(\Omega) \quad \text{for all } 1 \leq p < \infty.$$

Moreover, there exists a constant $C(n, \Omega)$ depending only on the dimension n and the Lipschitz domain Ω such that

$$\|v\|_{0,p,\Omega} \leq C(n, \Omega)\, p^{1-1/n} \|v\|_{1,n,\Omega} \quad \text{for all } 1 \leq p < \infty. \tag{1}$$

The inequality (1) provides a quantitative estimate comparing the L^p and Sobolev norms.

Example 1.11. By the embedding theorem, we have:

(1) $H^1(\Omega) \hookrightarrow C(\bar{\Omega})$ in one dimension (Exercise 1.9).
(2) $H^1(\Omega) \not\hookrightarrow C(\bar{\Omega})$ for $n \geq 2$ (Example 1.8); namely, functions in $H^1(\Omega)$ for $n \geq 2$ do not admit a continuous representative, so their pointwise values are not well defined.
(3) In two dimensions, $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for all $1 \leq p < \infty$; in three dimensions, $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for $1 \leq p \leq 6$.

For a piecewise smooth function which is also in $H^1(\Omega)$, then it is globally continuous.

Exercise 1.12. *Let Ω, Ω_1 and Ω_2 are three open and bounded domains in $\mathbb{R}^n$. Suppose $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ and $\Omega_1 \cap \Omega_2 = \varnothing$. Let u be a function defined in Ω such that $u|_{\Omega_1} \in C^1(\Omega_1) \cap H^1(\Omega_1)$ and $u|_{\Omega_2} \in C^1(\Omega_2) \cap H^1(\Omega_2)$. Prove that $u \in H^1(\Omega)$ if and only if $u \in C(\Omega)$.*

1.5. **Trace Theorems.** The trace theorem provides a rigorous way to define function values on the boundary. If $u \in C(\overline{\Omega})$, then $u(x)$ is well-defined for all $x \in \partial\Omega$. However, for $u \in W^{k,p}(\Omega)$, the function is defined only as an equivalence class of Lebesgue integrable functions, where $u \sim v$ if $u = v$ almost everywhere. Since $\partial\Omega$ has measure zero in the n-dimensional Lebesgue sense, the pointwise value $u|_{\partial\Omega}$ is not well-defined for Sobolev functions.

Theorem 1.13 (Trace theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth or Lipschitz boundary $\Gamma = \partial\Omega$. Then the trace operator $\gamma : C^1(\overline{\Omega}) \to C(\Gamma)$ extends uniquely and continuously to a linear map*

$$\gamma : H^1(\Omega) \to H^{1/2}(\Gamma),$$

and the trace inequality holds:

$$\|\gamma(u)\|_{1/2,\Gamma} \lesssim \|u\|_{1,\Omega}, \qquad \forall\, u \in H^1(\Omega). \tag{2}$$

Moreover, γ is surjective and admits a continuous right inverse.

More generally, if Ω is Lipschitz, the trace operator $\gamma : H^s(\Omega) \to H^{s-1/2}(\Gamma)$ is bounded for $1/2 < s < 3/2$; see McLean [10, pp. 100–106].

1.6. **Norm Equivalence Theorems.** The norm $\|\cdot\|_{k+1,p}$ involves all derivatives up to order $k+1$, among which the highest-order derivative plays the dominant role. In fact, we have the equivalence

$$\|u\|_{k+1,p} \eqsim \|u\|_{0,p} + |u|_{k+1,p}, \qquad \forall\, u \in W^{k+1,p}(\Omega).$$

Consider the decomposition

$$W^{k+1,p}(\Omega) = \mathcal{P}_k(\Omega) \oplus \big(W^{k+1,p}(\Omega)/\mathcal{P}_k(\Omega)\big),$$

where $\mathcal{P}_k(\Omega)$ denotes the space of polynomials of degree at most k. On the quotient space $W^{k+1,p}(\Omega)/\mathcal{P}_k(\Omega)$, the seminorm $|\cdot|_{k+1,p}$ defines a genuine norm.

Lemma 1.14. *For any $u \in W^{k+1,p}(\Omega)$, there holds*

$$\inf_{v\in\mathcal{P}_k(\Omega)} \|u+v\|_{k+1,p,\Omega} \eqsim |u|_{k+1,p,\Omega}.$$

In the polynomial space $\mathcal{P}_k(\Omega)$, the norm $\|\cdot\|_{0,p}$ can be replaced by any other norm, since this space is finite-dimensional. Consider a seminorm $F(\cdot)$ on $W^{k+1,p}(\Omega)$ satisfying

$$F(u+v) \le F(u) + F(v), \qquad F(\alpha u) \le |\alpha|\, F(u).$$

The key difference between a seminorm and a norm lies in its null space $\ker(F)$. If $\ker(F) = \{0\}$, then F is a norm. When $\ker(F) \cap \mathcal{P}_k(\Omega) = \{0\}$, the functional F defines a norm on $\mathcal{P}_k(\Omega)$.

Theorem 1.15 (Sobolev norm equivalence)**.** *Let F be a seminorm on $W^{k+1,p}(\Omega)$ such that $\ker(F) \cap \mathcal{P}_k(\Omega) = \{0\}$. Then*

$$\|v\|_{k+1,p} \eqsim F(v) + |v|_{k+1,p}, \qquad \forall\, v \in W^{k+1,p}(\Omega). \tag{3}$$

As a special case of the Sobolev norm equivalence theorem, we derive several classical inequalities of Poincaré and Friedrichs type. Let $\Gamma \subset \partial\Omega$ be a measurable subset with positive $(n-1)$-dimensional Lebesgue measure. By choosing $F(v) = \big|\int_\Gamma v\, ds\big|$ in (3), we obtain the *Friedrichs inequality*:

$$\|v\|_{1,p,\Omega} \lesssim |v|_{1,p,\Omega} + \Big|\int_\Gamma v\, ds\Big|, \qquad \forall\, v \in W^{1,p}(\Omega).$$

This implies the *Poincaré inequality*:

$$\|v\|_{0,p,\Omega} \lesssim |v|_{1,p,\Omega}, \qquad \forall\, v \in W^{1,p}_0(\Omega).$$

Choosing instead $F(v) = \big|\int_\Omega v\, dx\big|$ yields the *Poincaré–Friedrichs inequality*:

$$\|v\|_{1,p,\Omega} \lesssim |v|_{1,p,\Omega} + \Big|\int_\Omega v\, dx\Big|, \qquad \forall\, v \in W^{1,p}(\Omega).$$

Letting $\bar v = |\Omega|^{-1}\int_\Omega v\, dx$ denote the average of v, we further obtain the *mean-zero Poincaré inequality*:

$$\|v - \bar v\|_{0,p,\Omega} \lesssim |v|_{1,p,\Omega}, \qquad \forall\, v \in W^{1,p}(\Omega).$$

The hidden constants in the notation $\lesssim$ depend on the size and shape of Ω.

2. Elliptic boundary value problems and regularity

In this section, we present the weak formulation of the Poisson equation, as well as its H^2-regularity result on convex or smooth domains.

2.1. **Weak formulation of the Poisson equation.**

Consider the Poisson equation with homogeneous Dirichlet boundary conditions:

$$-\Delta u = f \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0, \tag{4}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. A function $u \in C^2(\Omega) \cap C_0(\Omega)$ that satisfies (4) is called a *classical solution*. However, this smoothness requirement may exclude many physically relevant solutions. To broaden the admissible space, we instead seek a *weak solution* in a Sobolev space. Here "weak" refers to differentiability defined in the sense of distributions.

We now reinterpret (4) distributionally. We seek $u \in \mathcal{D}'(\Omega)$ such that for all $\phi \in C_0^\infty(\Omega)$,

$$\langle -\nabla\cdot\nabla u, \phi\rangle := \langle \nabla u, \nabla\phi\rangle = \langle f, \phi\rangle.$$

This leads to the *weak formulation*: find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla\phi = \int_\Omega f\,\phi, \quad \forall\,\phi \in C_0^\infty(\Omega). \tag{5}$$

In practice, however, we cannot discretize (5) directly, since constructing finite-dimensional subspaces of $C_0^\infty(\Omega)$ is not feasible.

To extend the action of ∇u from $C_0^\infty(\Omega)$ to a broader space, we define the bilinear form

$$a(u,\phi) = \int_\Omega \nabla u \cdot \nabla\phi\,dx, \qquad u \in H^1(\Omega),\ \phi \in C_0^\infty(\Omega).$$

By the Cauchy–Schwarz inequality,

$$|a(u,\phi)| \le \|\nabla u\|_{0,\Omega}\,\|\nabla\phi\|_{0,\Omega},$$

so $a(u,\cdot)$ is continuous in the H^1 topology. Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, the bilinear form $a(\cdot,\cdot)$ can be extended continuously to $U \times V := H^1(\Omega) \times H_0^1(\Omega)$. Here U is the *trial space*, where the solution is sought, and V is the *test space*.

In (5), the right-hand side f belongs to $L^2(\Omega) \cong L^2(\Omega)'$. After extending the test space to $H_0^1(\Omega)$, we can allow $f \in H^{-1}(\Omega) := (H_0^1(\Omega))' = V'$.

Boundary conditions can be imposed by choosing appropriate trial and test spaces:

- Homogeneous Dirichlet boundary condition:

$$U = H_0^1(\Omega), \qquad V = H_0^1(\Omega); \tag{6}$$

- Homogeneous Neumann boundary condition:

$$U = H^1(\Omega)/\mathbb{R}, \qquad V = H^1(\Omega). \tag{7}$$

We are now ready to state the abstract variational (weak) formulation of the Poisson problem: given $f \in V'$, find $u \in U$ such that

$$a(u,v) = \langle f, v\rangle \qquad \forall\, v \in V. \tag{8}$$

Exercise 2.1. Prove existence and uniqueness of the weak solution for the Poisson equation with homogeneous Dirichlet or Neumann boundary conditions.

2.2. **Regularity theory.** The regularity theory for elliptic equations studies the smoothness of weak solutions and their relation to classical solutions. It serves as a bridge between these two notions and is fundamental in the analysis of finite element methods and the convergence of multigrid algorithms. However, this theory is highly nontrivial and involves deep analytic tools. Here we give a brief overview.

One approach is to consider the space

$$H(\Delta;\Omega) = \{\, u \in H^1(\Omega) : \Delta u \in L^2(\Omega) \,\}.$$

Clearly, $H^2(\Omega) \cap H_0^1(\Omega) \subset H(\Delta;\Omega)$, but the reverse inclusion does not hold in general.

We first give a formal derivation for the model problem $-\Delta u = f$ in $\mathbb{R}^n$, assuming that u is smooth and decays sufficiently fast as $|x| \to \infty$ to justify integration by parts:

$$\begin{aligned}\int_{\mathbb{R}^n} f^2\,dx &= \int_{\mathbb{R}^n} (\Delta u)^2\,dx = \sum_{i,j=1}^n \int_{\mathbb{R}^n} \partial_{ii}u\,\partial_{jj}u\,dx \\ &= -\sum_{i,j=1}^n \int_{\mathbb{R}^n} \partial_{iij}u\,\partial_j u\,dx = \sum_{i,j=1}^n \int_{\mathbb{R}^n} \partial_{ij}u\,\partial_{ij}u\,dx = \int_{\mathbb{R}^n} |D^2u|^2\,dx.\end{aligned}$$

The same result can be obtained using the Fourier transform of the Laplacian on $\mathbb{R}^n$. Let v be a distribution on $\mathbb{R}^n$ such that $\Delta v \in L^2(\mathbb{R}^n)$. By the Fourier transform properties,

$$\widehat{D^\alpha v}(\xi) = (i\xi)^\alpha \widehat{v}(\xi) = -(i\xi)^\alpha |\xi|^{-2} \widehat{\Delta v}(\xi).$$

Since $|(i\xi)^\alpha| \cdot |\xi|^{-2} \le 1$ for $|\alpha| = 2$, it follows that

$$\|\widehat{D^\alpha v}\|_{0,\mathbb{R}^n} \le \|\widehat{\Delta v}\|_{0,\mathbb{R}^n}.$$

By Parseval's identity,

$$\|D^\alpha v\|_{0,\mathbb{R}^n} \le \|\Delta v\|_{0,\mathbb{R}^n}, \qquad |\alpha| = 2.$$

This inequality shows an important fact: if $\Delta v \in L^2$, then all second derivatives of v also belong to L^2. In other words, the Frobenius norm of the Hessian matrix D^2v can be controlled by its trace Δv. This is a remarkable property, since Δv represents only a special linear combination of the second-order derivatives of v.

Smooth and bounded domains. The proofs given above can be extended to bounded domains with smooth boundaries, although this extension is nontrivial. The following classical result, found in most textbooks on elliptic boundary value problems, e.g. [7], establishes the H^2-regularity of the Poisson equation.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain. Then, for each $f \in L^2(\Omega)$, there exists a unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ to the problem $-\Delta u = f$ such that*

$$\|u\|_{2,\Omega} \le C\,\|f\|_{0,\Omega},$$

where $C > 0$ depends only on Ω.

Here is an outline of the proof. For general domains, if $f \in L^2(\Omega)$, then the weak solution u satisfies $u \in H^2_{\mathrm{loc}}(\Omega)$; that is, for every subdomain $U \subset\subset \Omega$, we have $u \in H^2(U)$. This can be established by finite difference approximation of D^2u. The assumption $U \subset\subset \Omega$ allows the use of a smooth cut-off function supported away from the boundary.

Near the boundary, a suitable change of coordinates is introduced so that all tangential derivatives can be handled in the same way as the interior case. For the normal derivative, the equation itself provides an estimate in terms of f and the tangential derivatives. Since the boundary of Ω is smooth, a partition of unity can be employed to combine the local estimates into a global one. A detailed proof can be found, for example, in [7].

Convex domains. Theorem 2.2 can be extended to non-smooth but convex domains.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a convex and bounded domain. Then, for each $f \in L^2(\Omega)$, there exists a unique solution $u \in H^2(\Omega)$ to (8) satisfying*

$$\|u\|_{2,\Omega} \leq C \, \|f\|_{0,\Omega},$$

where $C > 0$ depends only on the diameter of Ω.

We outline a proof for a convex polygon and refer to [13] for details. First, approximate the convex polygon from the interior by a sequence of smooth convex domains $\{\Omega_k\}$. On each Ω_k, establish the H^2-regularity estimate $|u_k|_{2,\Omega_k} \leq C(\Omega_k)\|f\|_{0,\Omega_k}$ for the solution of $-\Delta u_k = f$ in Ω_k with $u_k = 0$ on $\partial\Omega_k$. Then, pass to the limit in an appropriate topology to obtain the desired result on Ω. To ensure convergence, the constants $C(\Omega_k)$ must remain uniformly bounded, which requires a careful analysis of the regularity estimate on Ω_k.

A key identity used in this analysis is the refined integration-by-parts formula (cf. [8]):

$$\int_U (\Delta u)^2 \, dx = \sum_{i,j=1}^{n} \int_U |\partial_{ij} u|^2 \, dx + \int_{\partial U} \operatorname{div} n \left| \frac{\partial u}{\partial n} \right|^2 \, dS, \tag{9}$$

where n denotes the outward unit normal on ∂U. If U is C^2 and convex, then $\operatorname{div} n = \operatorname{trace}(Dn) \geq 0$, so the boundary term is nonnegative, giving

$$|D^2 u|_{0,U} \leq |\Delta u|_{0,U}.$$

The lower-order terms in the H^2-norm can be controlled by the Poincaré inequality, and the associated constant depends only on the domain diameter, $\operatorname{diam}(\Omega_k) \leq \operatorname{diam}(\Omega)$, uniformly in k.

Exercise 2.4. Prove identity (9) using the identity

$$-\Delta \boldsymbol{f} = -\operatorname{grad} \, \operatorname{div} \boldsymbol{f} + \operatorname{curl} \operatorname{curl} \boldsymbol{f},$$

and then integration by parts by multiplying $\boldsymbol{f} = \nabla u$ to both sides and integrate over U.

Concave polygons. For concave polygons, full H^2-regularity generally fails. Two main approaches can be used to obtain weaker regularity results.

The first approach, developed in [9, 5, 2], is based on fractional Sobolev spaces. In this setting,

$$-\Delta : H_0^1(\Omega) \cap H^{1+s}(\Omega) \to H^{-1+s}(\Omega)$$

is continuous for any $s \in [0, s_0)$, where $s_0 \in (0, 1]$ depends on the geometry of Ω.

The second approach employs L^p-based Sobolev spaces instead of L^2 spaces. In particular, it can be shown (cf. [8]) that

$$-\Delta : H_0^1(\Omega) \cap W^{2,p}(\Omega) \to L^p(\Omega) \tag{10}$$

is an isomorphism for all $p \in (1, p_0)$, where $p_0 > 1$ depends on the domain Ω. This perspective is especially useful in the analysis of adaptive finite element methods.

Loss of regularity. Theorem 2.2 does not hold for general Lipschitz domains. The smoothness of $\partial\Omega$ or convexity is essential for establishing regularity. On Lipschitz domains, the regularity of u depends strongly on the geometry of Ω. In particular, corner singularities and the way boundary segments meet can significantly affect the smoothness of the solution. The following example illustrates how the shape of Ω can lead to a loss of regularity.

Example 2.5. Let $\beta \in (0, 1)$ and consider the non-convex domain

$$\Omega = \{(r, \theta) : 0 < r < 1,\ 0 < \theta < \pi/\beta\}.$$

Define

$$v = r^\beta \sin(\beta\theta),$$

which is the imaginary part of the analytic function z^β and hence harmonic in Ω. Set $u = (1 - r^2)v$. A direct computation shows that

$$-\Delta u = 4(1 + \beta)v \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0.$$

Note that $4(1 + \beta)v \in L^\infty(\Omega) \subset L^2(\Omega)$, but $u \notin H^2(\Omega)$.

Nevertheless, a slightly weaker result holds for general Lipschitz domains.

Theorem 2.6. *Let Ω be a bounded Lipschitz domain. Then there exists a constant $\alpha \in (0, 1]$ such that*

$$\|u\|_{1+\alpha,\Omega} \le C\, \|f\|_{\alpha-1,\Omega}, \tag{11}$$

for the solution u of (8), where $C > 0$ depends only on Ω.

A complete proof of this regularity result is highly technical; see [9, 5, 2] for details.

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