FINITE ELEMENT METHODS FOR MAXWELL EQUATIONS

LONG CHEN

ABSTRACT. We give a brief introduction to finite element methods for solving Maxwell equations.

Recall that the time-harmonic Maxwell equation for electric field $\boldsymbol{E}$ is

$$\nabla \times (\mu^{-1}\nabla \times \boldsymbol{E}) - \omega^2\tilde{\epsilon}\boldsymbol{E} = \tilde{\boldsymbol{J}}$$
$$\nabla \cdot (\epsilon \boldsymbol{E}) = \rho.$$

The time-harmonic Maxwell equation for magnetic field $\boldsymbol{H}$ is

$$\nabla \times (\tilde{\epsilon}^{-1}\nabla \times \boldsymbol{H}) - \omega^2\mu\boldsymbol{H} = \nabla \times \tilde{\boldsymbol{J}}$$
$$\nabla \cdot (\mu \boldsymbol{H}) = 0.$$

Those are obtained by Fourier transform in time for the original Maxwell equations. Here ω is a positive constant called the frequency. For derivation and physical meaning, we refer to *Brief Introduction to Maxwell's Equations*. In this note, we shall consider finite element methods for solving time-harmonic Maxwell equations.

1. INTRODUCTION

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. We introduce the Sobolev spaces

$$H(\operatorname{curl};\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \operatorname{curl}\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\},$$
$$H(\operatorname{div};\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \operatorname{div}\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\}$$

The intensity fields $(\boldsymbol{E}, \boldsymbol{H})$ belong to $H(\operatorname{curl};\Omega)$ while the flux field $(\boldsymbol{D}, \boldsymbol{B})$ in $H(\operatorname{div};\Omega)$. We use the unified notation $H(\mathrm{d};\Omega)$ with $\mathrm{d} = \operatorname{grad}, \operatorname{curl}$, or div. Note that $H(\operatorname{grad};\Omega)$ is the familiar $H^1(\Omega)$ space. One can verify that $H(\mathrm{d};\Omega)$ is a Hilbert space with respect to the inner product

$$(u, v) + (\mathrm{d}u, \mathrm{d}v).$$

The norm for $H(\mathrm{d};\Omega)$ is the graph norm

$$\|u\|_{\mathrm{d},\Omega} := \left(\|u\|^2 + \|\mathrm{d}u\|^2\right)^{1/2}.$$

We recall the integration by parts for vector functions below. Formally the boundary term is obtained by replacing the Hamilton operator ∇ by the unit outwards normal vector

Date: Created: Spring 2014. Last update: July 18, 2020.

$\boldsymbol{n}$. For example,

$$\int_\Omega \nabla u \cdot \boldsymbol{\phi} \,\mathrm{d}x = -\int_\Omega u \nabla \cdot \boldsymbol{\phi} \,\mathrm{d}x + \int_{\partial\Omega} \boldsymbol{n} u \cdot \boldsymbol{\phi} \,\mathrm{d}S,$$
$$\int_\Omega \nabla \times \boldsymbol{u} \cdot \boldsymbol{\phi} \,\mathrm{d}x = \int_\Omega \boldsymbol{u} \cdot \nabla \times \boldsymbol{\phi} \,\mathrm{d}x + \int_{\partial\Omega} \boldsymbol{n} \times \boldsymbol{u} \cdot \boldsymbol{\phi} \,\mathrm{d}S,$$
$$\int_\Omega \nabla \cdot \boldsymbol{u} \, \phi \,\mathrm{d}x = -\int_\Omega \boldsymbol{u} \cdot \nabla \phi \,\mathrm{d}x + \int_{\partial\Omega} \boldsymbol{n} \cdot \boldsymbol{u} \, \phi \,\mathrm{d}S.$$

The weak formulation is obtained by multiplying the original equation by a smooth test equation and applying the integration by parts. The boundary condition will be discussed later. For time-harmonic Maxwell equations, the weak formulation for $\boldsymbol{E}$ is

$$(\mu^{-1}\nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{\phi}) - \omega^2(\tilde{\epsilon}\boldsymbol{E}, \phi) = (\tilde{\boldsymbol{J}}, \phi) \quad \forall \boldsymbol{\phi} \in \mathcal{D}(\Omega). \tag{1}$$

And the equation for $\boldsymbol{H}$ is

$$(\tilde{\epsilon}^{-1}\nabla \times \boldsymbol{H}, \nabla \times \boldsymbol{\phi}) - \omega^2(\mu\boldsymbol{H}, \phi) = (\tilde{\boldsymbol{J}}, \nabla \times \boldsymbol{\phi}) \quad \forall \boldsymbol{\phi} \in \mathcal{D}(\Omega). \tag{2}$$

In (2) the source $\nabla \times \tilde{\boldsymbol{J}}$ is understood in the distribution sense and $\nabla\times$, as the adjoint of itself, is moved to the test function. The coefficient $\tilde{\epsilon}$ and the current $\tilde{\boldsymbol{J}}$ are in general complex functions and so are $\boldsymbol{E}, \boldsymbol{H}$.

The divergence constraint is build into the weak formulation when $\omega \neq 0$. For example, the condition $\operatorname{div}(\mu \boldsymbol{H}) = 0$ in the distribution sense can be obtained by applying div operator to the equation $\nabla \times (\tilde{\epsilon}^{-1}\nabla \times \boldsymbol{H}) - \omega^2\mu\boldsymbol{H} = \nabla \times \tilde{\boldsymbol{J}}$. When $\omega = 0$, we need to impose the constraint explicitly; see (4).

To simplify the discussion, we consider the following model problems:

- Symmetric and positive definite problem:

$$\nabla \times (\alpha \nabla \times \boldsymbol{u}) + \beta \boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega, \qquad \boldsymbol{u} \times \boldsymbol{n} = 0 \quad \text{on } \partial\Omega \tag{3}$$

- Saddle point system:

$$\nabla \times (\alpha \nabla \times \boldsymbol{u}) = \boldsymbol{f} \quad \text{in } \Omega, \qquad \nabla \cdot (\beta \boldsymbol{u}) = 0 \quad \text{in } \Omega, \qquad \boldsymbol{u} \times \boldsymbol{n} = 0 \quad \text{on } \partial\Omega. \tag{4}$$

where α and β are uniformly bounded and positive and real coefficients. The right hand side $\boldsymbol{f}$ is divergence free, i.e. $\operatorname{div} \boldsymbol{f} = 0$ in the distribution sense.

2. Interface and Boundary Conditions

For a vector $\boldsymbol{u} \in \mathbb{R}^3$ and a unit norm vector $\boldsymbol{n}$, we can decompose $\boldsymbol{u}$ into the normal component and the tangential component as

$$\boldsymbol{u} = (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n} + \boldsymbol{n} \times (\boldsymbol{u} \times \boldsymbol{n}) = \boldsymbol{u}_n + \boldsymbol{u}_\tau.$$

The vector $\boldsymbol{u} \times \boldsymbol{n}$ is also on the tangent plane and orthogonal to the tangential component $\boldsymbol{u}_\tau$ which is a clockwise 90° rotation of $\boldsymbol{u}_\tau$ on the tangent plane. Consequently $\{\boldsymbol{u} \times \boldsymbol{n}, \boldsymbol{u}_\tau, \boldsymbol{n}\}$ forms an orthogonal basis of $\mathbb{R}^3$; see Fig. 1.

The interface condition can be derived from the continuity requirement for piecewise smooth functions to be in $H(\mathrm{d}; \Omega)$. Let $\Omega = K_1 \cup K_2 \cup S$ with interface $S = \bar{K}_1 \cap \bar{K}_2$. Let $u_i \in H(\mathrm{d}; K_i)$. Define $u \in L^2(\Omega)$ as

$$u = \begin{cases} u_1 & x \in K_1, \\ u_2 & x \in K_2. \end{cases}$$

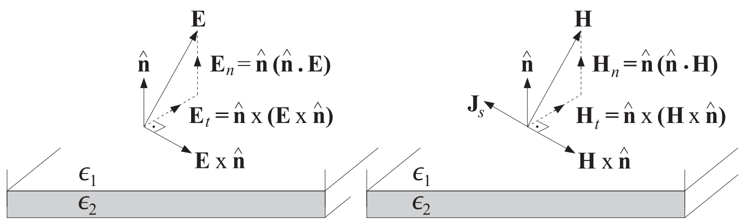

FIGURE 1. Field directions at boundary. Extract from *Electromagnetic Waves and Antennas* by Orfanidis [9].

We can always define derivative $\mathrm{d}u$ in the distribution sense. To be a weak derivative, we need to verify it coincides with the piecewise derivative, i.e.,

$$\mathrm{d}u = \begin{cases} \mathrm{d}u_1 & x \in K_1, \\ \mathrm{d}u_2 & x \in K_2. \end{cases}$$

To do so, let $\phi \in \mathcal{D}(\Omega)$, by the definition of the derivative of a distribution

$$\begin{aligned} \langle \mathrm{d}u, \phi \rangle &:= \langle u, \mathrm{d}^* \phi \rangle = (u_1, \mathrm{d}^* \phi) + (u_2, \mathrm{d}^* \phi) \\ &= (\mathrm{d}u_1, \phi) + (\mathrm{d}u_2, \phi) + \langle \gamma_S (u_1 - u_2), \phi \rangle_S, \end{aligned}$$

where d^* is the adjoint of d in L^2-inner product and γ_S is an appropriate restriction of functions on the interface depending on the differential operators. The negative sign in front of u_2 is from the fact the outwards normal direction of K_2 is opposite to that of K_1.

Then $u \in H(\mathrm{d}; \Omega)$ if and only if

$$\begin{cases} u_1|_S = u_2|_S & \text{for } \mathrm{d} = \mathrm{grad}\,, \\ \boldsymbol{n} \times \boldsymbol{u}_1|_S = \boldsymbol{n} \times \boldsymbol{u}_2|_S & \text{for } \mathrm{d} = \mathrm{curl}\,, \\ \boldsymbol{n} \cdot \boldsymbol{u}_1|_S = \boldsymbol{n} \cdot \boldsymbol{u}_2|_S & \text{for } \mathrm{d} = \mathrm{div}\,. \end{cases}$$

Here strictly speaking, the restriction operator $(\cdot)|_S$ should be replaced by appropriate trace operators which will be discussed in the next section.

So for a function in $H(\mathrm{curl}\,; \Omega)$, its tangential component should be continuous across the interface and for a function in $H(\mathrm{div}; \Omega)$, its normal component should be continuous. This will be the key of constructing finite element spaces for these Sobolev spaces.

When the interface S contains surface charge ρ_S and surface current J_S, the interface condition for $\boldsymbol{H}$ and $\boldsymbol{D}$ is changed to

$$(\boldsymbol{H}_1 - \boldsymbol{H}_2) \times \boldsymbol{n} = \boldsymbol{J}_S, \quad (\boldsymbol{D}_1 - \boldsymbol{D}_2) \cdot \boldsymbol{n} = \rho_S.$$

The interface condition for $\boldsymbol{H}$ can be build into the right hand side of the weak formulation (2) using a surface integral on S.

The boundary condition can be thought of as an interface condition when one side of the interface is the free space. The following are popular boundary conditions for Maxwell-type equations.

- If one side is a perfect conductor, then $\sigma = \infty$. By Ohm's law, to have a finite current, the electric field $\boldsymbol{E}$ should be zero. So we obtain the boundary condition $\boldsymbol{E} \times \boldsymbol{n} = 0$ for a perfect conductor.
- Impedance boundary condition •1

$$n \times \boldsymbol{H} - \lambda \boldsymbol{E}_t = g.$$

•1 *more on this*

3. TRACES

The trace of functions in $H(\mathrm{d};\Omega)$ is not simply the restriction of the function values since the differential operator div or curl controls only partial component of the vector function. The best way to look at the trace is, again, through integration by parts.

Recall that $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is the trace operator for H^1 functions. It is continuous and surjective. When u is also continuous on $\bar{\Omega}$, $\gamma u = u|_{\partial\Omega}$.

3.1. $H(\operatorname{div};\Omega)$ **space.** For functions $\boldsymbol{v} \in \boldsymbol{C}^1(\Omega), \phi \in C^1(\Omega)$ and Ω is a domain with smooth boundary, we have the following integration by parts

$$\int_\Omega \operatorname{div} \boldsymbol{v}\phi \,\mathrm{d}x = -\int_\Omega \boldsymbol{v} \cdot \operatorname{grad} \phi \,\mathrm{d}x + \int_{\partial\Omega} \boldsymbol{n}\cdot\boldsymbol{v}\,\phi \,\mathrm{dS}. \tag{5}$$

Then we relax the smoothness of functions and the domain such that (5) still holds. First since for Lipschitz domains, the normal vector $\boldsymbol{n}$ of $\partial\Omega$ is well defined almost everywhere, we can relax the smoothness of domain Ω to be a bounded Lipschitz domain only. Second we only need $\boldsymbol{v} \in H(\operatorname{div};\Omega)$ and $\phi \in H^1(\Omega)$ so that the volume integral is finite. Then (5) can be used to define the trace of $\boldsymbol{v} \in H(\operatorname{div};\Omega)$:

$$\langle \boldsymbol{n}\cdot\boldsymbol{v}, \gamma\phi\rangle_{\partial\Omega} := \int_\Omega \operatorname{div} \boldsymbol{v}\phi \,\mathrm{d}x + \int_\Omega \boldsymbol{v} \cdot \operatorname{grad} \phi \,\mathrm{d}x, \text{ for all } \phi \in H^1(\Omega). \tag{6}$$

In the left hand side of (6) we change from a boundary integral to an abstract duality action and $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is the trace operator for H^1 functions. Since γ is an onto, $\gamma\phi$ will run over all $H^{1/2}(\partial\Omega)$ when ϕ runs over $H^1(\Omega)$. That is $\boldsymbol{n}\cdot\boldsymbol{v}$ is a dual of $H^{1/2}(\partial\Omega)$. Note that $\partial(\partial\Omega) = 0$. So the right space for $\boldsymbol{n}\cdot\boldsymbol{v}$ is $H^{-1/2}(\partial\Omega)$. We summarize as the following theorem.

Theorem 3.1 (Trace of $H(\operatorname{div};\Omega)$)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain in $\mathbb{R}^3$ with unit outward normal $\boldsymbol{n}$. Then the mapping $\gamma_n : C^\infty(\bar{\Omega}) \to C^\infty(\partial\Omega)$ with $\gamma_n\boldsymbol{v} = \boldsymbol{n}\cdot\boldsymbol{v}|_{\partial\Omega}$ can be extended to a continuous linear map γ_n from $H(\operatorname{div};\Omega)$ onto $H^{-1/2}(\partial\Omega)$, namely*

$$\|\gamma_n\boldsymbol{v}\|_{-1/2,\partial\Omega} \lesssim \|\boldsymbol{v}\|_{\operatorname{div},\Omega}. \tag{7}$$

and the following Green's identity holds for functions $\boldsymbol{v} \in H(\operatorname{div};\Omega)$ and $\phi \in H^1(\Omega)$

$$\langle \gamma_n\boldsymbol{v}, \gamma\phi\rangle_{\partial\Omega} = \int_\Omega \operatorname{div} \boldsymbol{v}\phi \,\mathrm{d}x + \int_\Omega \boldsymbol{v} \cdot \operatorname{grad} \phi \,\mathrm{d}x. \tag{8}$$

The space $H_0(\operatorname{div};\Omega)$ can be defined as

$$H_0(\operatorname{div};\Omega) = \{\boldsymbol{v} \in H(\operatorname{div};\Omega) : \gamma_n\boldsymbol{v} = 0\}.$$

Proposition 3.2. *The trace operator γ_n from $H(\operatorname{div};\Omega)$ onto $H^{-1/2}(\partial\Omega)$ is surjective and there exists a continuous right inverse. Namely for any $g \in H^{-1/2}(\partial\Omega)$, there exists a function $\boldsymbol{v} \in H(\operatorname{div};\Omega)$ such that $\gamma_n\boldsymbol{v} = g$ in $H^{-1/2}(\partial\Omega)$ and $\|\boldsymbol{v}\|_{\operatorname{div},\Omega} \lesssim \|g\|_{-1/2,\partial\Omega}$.*

Proof. For a given $g \in H^{-1/2}(\partial\Omega)$, let $f = -|\Omega|^{-1}\langle g, 1\rangle$. We solve the Poisson equation $-\Delta p = f$ with Neumann boundary condition $\partial_n p = g$:

$$(\nabla p, \nabla\phi) = (f, \phi) + \langle g, \gamma\phi\rangle_{\partial\Omega} \quad \text{ for all } \phi \in H^1(\Omega).$$

The existence and uniqueness of the solution $p \in H^1(\Omega) \cap L_0^2(\Omega)$ is ensured by the choice of f which satisfies the compatable condition with the boundary data g. By choosing $v \in H_0^1(\Omega)$, we conclude $-\Delta p = f$ in $L^2(\Omega)$, i.e. $\boldsymbol{v} = \nabla p$ is in $H(\operatorname{div};\Omega)$.

Note that $\langle \gamma_n \boldsymbol{v}, \gamma\phi\rangle = (\operatorname{div} \boldsymbol{v}, \phi) + (\boldsymbol{v}, \nabla\phi) = -(f, \phi) + (\nabla p, \nabla\phi) = \langle g, \gamma\phi\rangle$. Since $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is surjective, we conclude $\gamma_n \boldsymbol{v} = g$ in $H^{-1/2}(\partial\Omega)$. That is we found a function $\boldsymbol{v} \in H(\mathrm{div}; \Omega)$ such that $\gamma_n \boldsymbol{v} = g$.

From the stability of $-\Delta$ operator, we have

$$\|\boldsymbol{v}\| = \|\nabla p\| \lesssim \|f\| + \|g\|_{-1/2} \lesssim \|g\|_{-1/2}.$$

Together with the identity $\|\operatorname{div} \boldsymbol{v}\| = \|f\|$, we obtain $\|\boldsymbol{v}\|_{\mathrm{div},\Omega} \lesssim \|g\|_{-1/2,\partial\Omega}$. □

3.2. $H(\mathrm{curl}\,; \Omega)$ **space.** Similarly we can use the integration by parts

$$\int_\Omega \operatorname{curl} \boldsymbol{v} \cdot \phi \,\mathrm{d}x = \int_\Omega \boldsymbol{v} \cdot \operatorname{curl} \boldsymbol{\phi} \,\mathrm{d}x - \int_{\partial\Omega} (\boldsymbol{v} \times \boldsymbol{n}) \cdot \boldsymbol{\phi} \,\mathrm{d}S$$

to define the trace of $H(\mathrm{curl}\,; \Omega)$. The trace only controls the tangential part of $\boldsymbol{v}|_{\partial\Omega}$.

Theorem 3.3 (Trace of $H(\mathrm{curl}\,; \Omega)$)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain in $\mathbb{R}^3$ with unit outward normal $\boldsymbol{n}$. Then the mapping $\gamma_\tau : C^\infty(\bar{\Omega}) \to C^\infty(\partial\Omega)$ with $\gamma_\tau \boldsymbol{v} = \boldsymbol{v}|_{\partial\Omega} \times \boldsymbol{n}$ can be extended by continunity to a continuous linear map γ_τ from $H(\mathrm{curl}\,; \Omega)$ to $H^{-1/2}(\partial\Omega)$, namely*

$$\|\gamma_\tau \boldsymbol{v}\|_{-1/2,\partial\Omega} \lesssim \|\boldsymbol{v}\|_{\mathrm{curl}\,,\Omega}. \tag{9}$$

and the following Green's identity holds for functions $\boldsymbol{v} \in H(\mathrm{curl}\,; \Omega)$ and $\boldsymbol{\phi} \in \boldsymbol{H}^1(\Omega)$

$$\langle \gamma_\tau \boldsymbol{v}, \gamma\phi\rangle_{\partial\Omega} = \int_\Omega \boldsymbol{v} \cdot \operatorname{curl} \boldsymbol{\phi} \,\mathrm{d}x - \int_\Omega \operatorname{curl} \boldsymbol{v} \cdot \boldsymbol{\phi} \,\mathrm{d}x. \tag{10}$$

The trace γ_τ from $H(\mathrm{curl}\,; \Omega)$ to $H^{-1/2}(\partial\Omega)$, however, is not surjective since in (10) the test function ϕ can be further extend from $H^1(\Omega)$ to $H(\mathrm{curl}\,; \Omega)$.

Let us denote by $\Gamma = \partial\Omega$ and introduce the tangential component trace π_τ as $\pi_\tau \boldsymbol{v} = \boldsymbol{v}_\tau = \boldsymbol{n} \times (\boldsymbol{v} \times \boldsymbol{n})$. The boundary pair can be written as

$$\langle \boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{\phi}\rangle_\Gamma = \langle \boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{\phi}_\tau\rangle_\Gamma = \langle \gamma_\tau \boldsymbol{v}, \pi_\tau \boldsymbol{\phi}\rangle_\Gamma. \tag{11}$$

Let $\mathrm{curl}_\Gamma, \mathrm{div}_\Gamma$ be the $\mathrm{curl}\,, \mathrm{div}$ operators on the boundary surface Γ, which can be defined intrinsically using metrics on the tangent planes. It is, however, advantageous to define through the operator ∇ in space and operations with the normal vector

$$\boldsymbol{n} \cdot (\nabla \times \boldsymbol{v}) = \mathrm{curl}_\Gamma(\pi_\tau \boldsymbol{v}) = \mathrm{div}_\Gamma(\gamma_\tau \boldsymbol{v}).$$

For a function $\boldsymbol{v} \in H(\mathrm{curl}\,; \Omega)$, $\operatorname{curl} \boldsymbol{v} \in H(\mathrm{div}; \Omega)$ as $\operatorname{div} \operatorname{curl} \boldsymbol{v} = 0$. Therefore $\gamma_n(\nabla \times \boldsymbol{v}) = \boldsymbol{n} \cdot (\nabla \times \boldsymbol{v}) = \mathrm{curl}_\Gamma(\pi_\tau \boldsymbol{v}) \in H^{-1/2}(\Gamma)$, i.e., $\pi_\tau \boldsymbol{v} \in H^{-1/2}(\mathrm{curl}_\Gamma; \Gamma)$. As its rotation, $\gamma_\tau \boldsymbol{v} \in H^{-1/2}(\mathrm{div}_\Gamma; \Gamma)$. The duality pair in (11) is $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) - H^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$. The precise characterization of the trace operator is

$$\gamma_\tau : H(\mathrm{curl}\,; \Omega) \to H^{-1/2}(\mathrm{div}_\Gamma; \Gamma)$$

and this mapping is onto. Details can be found in the book [6] (page 58–60) and [2]. Especially to verify the mapping is surjective, one has to construct a lifting operator (analogy of Proposition 3.2) for a given trace in $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. Construction of such lifting operator is technical and firstly by Tartar [11] which can be also found in [2].

The space $H_0(\mathrm{curl}\,; \Omega)$ can be defined as

$$H_0(\mathrm{curl}\,; \Omega) = \{\boldsymbol{v} \in H(\mathrm{curl}\,; \Omega) : \gamma_\tau \boldsymbol{v} = 0\}.$$

4. Well-posedness of Weak Formulations

Let $V = H_0(\mathrm{curl}\,;\Omega)$. The weak formulation of (3) is: given an $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, find $\boldsymbol{u} \in V$ such that

$$(\alpha\nabla\times\boldsymbol{u}, \nabla\times\boldsymbol{v}) + (\beta\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in V. \tag{12}$$

In (12), the first term is obtained by integration by part (moving $\nabla\times$ in front of $\boldsymbol{\phi}$ to $\boldsymbol{u}$)

$$(\alpha\nabla\times\boldsymbol{u}, \nabla\times\boldsymbol{\phi}) = (\nabla\times(\alpha\times\nabla\times\boldsymbol{u}), \boldsymbol{\phi}) + (\alpha\nabla\times\boldsymbol{u}, \boldsymbol{n}\times\boldsymbol{\phi})_{\partial\Omega}$$

and chose the test function $\boldsymbol{\phi} \in V$ to remove the boundary term. The boundary condition for $\boldsymbol{u}$ is of Dirichlet type: $\boldsymbol{u}\times\boldsymbol{n} = 0$ on $\partial\Omega$ or more rigorously $\gamma_\tau \boldsymbol{u} = 0$.

Assume the positive coefficients α and β are uniformly bounded below and above. The well-posedness of (12) is then trivial since the bilinear form is equivalent to the inner product of $H(\mathrm{curl}\,;\Omega)$. The existence and uniqueness of the solution to (12) can be obtained by the Riesz representation theorem. The stability constant, however, will be proportional to $1/\beta$ and thus will below up as $\beta \to 0$. Unlike the Poisson equation, where $(\nabla u, \nabla v)$ will define an inner product on $H_0^1(\Omega)$, for space $H_0(\mathrm{curl}\,;\Omega)$, the zero trace cannot take care of the much larger kernel space of curl operator which consists of the image of grad for simply connected domain Ω. We will revisit this issue (robustness as $\beta \to 0^+$) after we have discussed the saddle point formulation.

For the saddle point formulation of Maxwell equation (4), the natural Sobolev space for $\boldsymbol{u}$ is again $V = H_0(\mathrm{curl}\,;\Omega)$ and the bilinear form

$$a(\boldsymbol{u}, \boldsymbol{v}) := (\alpha\nabla\times\boldsymbol{u}, \nabla\times\boldsymbol{v}), \quad \text{for } \boldsymbol{u}, \boldsymbol{v} \in H_0(\mathrm{curl}\,;\Omega),$$

which induces an operator $A : V \to V'$, $\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = a(\boldsymbol{u}, \boldsymbol{v})$.

As a function in $H(\mathrm{curl}\,;\Omega)$ space, however, the divergence operator cannot be applied directly. It should be understood in the weak sense, i.e.,

$$-\langle \mathrm{div}^w(\beta\boldsymbol{u}), q\rangle := (\beta\boldsymbol{u}, \mathrm{grad}\, q) \quad \forall q \in Q := H_0^1(\Omega).$$

We define the bilinear form

$$b(\boldsymbol{v}, q) = (\beta\boldsymbol{v}, \mathrm{grad}\, q) = -(\mathrm{div}^w(\beta\boldsymbol{v}), q), \quad \text{for } \boldsymbol{v} \in H_0(\mathrm{curl}\,;\Omega), q \in H_0^1(\Omega)$$

which induces operator $B : V \to Q'$ as $\langle B\boldsymbol{u}, q\rangle = b(\boldsymbol{u}, q)$ for all $q \in H_0^1(\Omega)$ and $B' : Q \to V'$ as the dual of B. A Lagrangian multiplier $p \in H_0^1(\Omega)$ can be introduced to impose the constraint $\mathrm{div}^w(\beta\boldsymbol{u}) = 0$. That is we consider the inf-sup problem

$$\inf_{\boldsymbol{u}\in V} \sup_{p\in Q} \frac{1}{2}(\alpha\nabla\times\boldsymbol{u}, \nabla\times\boldsymbol{u}) - (\boldsymbol{f}, \boldsymbol{u}) + (\beta\boldsymbol{u}, \nabla p).$$

The Euler's equation is the following saddle point formulation of (4): given $\boldsymbol{f} \in V'$, find $\boldsymbol{u} \in V, p \in Q$ s.t.

$$\begin{pmatrix} A & B' \\ B & O \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ 0 \end{pmatrix}, \tag{13}$$

which is the operator form of the mixed formulation

$$(\alpha\nabla\times\boldsymbol{u}, \nabla\times\boldsymbol{v}) + (\beta\boldsymbol{v}, \nabla p) = (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in V, \tag{14}$$

$$(\beta\boldsymbol{u}, \nabla q) = 0 \qquad \forall\, q \in Q. \tag{15}$$

The well-posedness of the saddle point system (13) is a consequence of the inf-sup condition of B and the coercivity of A in the null space $X = \ker(B) = H_0(\mathrm{curl}\,;\Omega) \cap \ker(\mathrm{div}^w)$; see *Inf-sup conditions for operator equations*.

Lemma 4.1. *For $\beta = 1$, we have the inf-sup condition*

$$\inf_{p\in Q}\sup_{\boldsymbol{v}\in V}\frac{\langle B\boldsymbol{v}, p\rangle}{\|\boldsymbol{v}\|_{\rm curl}\,|p|_1} = 1. \tag{16}$$

Proof. Here we follow the convention in the Stokes equation to write out the formulation in term of the (negative) divergence operator B. It is more natural to show the adjoint $B' = \operatorname{grad} : H_0^1(\Omega) \to H_0'(\operatorname{curl};\Omega)$ is injective. We can interpret

$$\|\nabla p\|_{V'} = \sup_{\boldsymbol{v}\in V}\frac{\langle B\boldsymbol{v}, p\rangle}{\|\boldsymbol{v}\|_{\rm curl}} = \sup_{\boldsymbol{v}\in V}\frac{(\boldsymbol{v}, \nabla p)}{\|\boldsymbol{v}\|_{\rm curl}},$$

and it suffices to prove

$$\|\nabla p\|_{V'} = \|\nabla p\|. \tag{17}$$

First by the Cauchy-Schwarz inequality and the definition of the curl norm, we have $\|\nabla p\|_{V'} \leq \|\nabla p\|$. To prove the inequality in the opposite direction, we simply chose $\boldsymbol{v} = \nabla p$. Then $\langle B\boldsymbol{v}, p\rangle = |p|_1^2$ and $\|\boldsymbol{v}\|_{\rm curl} = \|\boldsymbol{v}\| = |p|_1$. Therefore $\|\nabla p\|_{V'} \geq \|\nabla p\|$ by the definition of sup. □

Exercise 4.2. *For coefficients $\beta_{\min} \leq \beta \leq \beta_{\max}$, prove that*

$$\beta_{\min}|p|_1 \leq \sup_{\boldsymbol{v}\in V}\frac{\langle B\boldsymbol{v}, p\rangle}{\|\boldsymbol{v}\|_{\rm curl}} \leq \beta_{\max}|p|_1.$$

The coercivity in the null space

$$X = \ker(B) = H_0(\operatorname{curl};\Omega) \cap \ker(\operatorname{div}^w)$$

can be derived from the following Poincaré-type inequality.

Lemma 4.3 (Poincaré inequality. Lemma 3.4 and Theorem 3.6 in [5])**.** *When Ω is simply connected and $\partial\Omega$ consists of only one component, we have*

$$\|\boldsymbol{v}\| \lesssim \|\operatorname{curl}\boldsymbol{v}\| \quad \textit{for } \boldsymbol{v} \in X. \tag{18}$$

A heuristic argument for the above Poincaré inequality is: using identity $-\Delta\boldsymbol{u} = \operatorname{grad}\operatorname{div}\boldsymbol{u} + \operatorname{curl}\operatorname{curl}\boldsymbol{u}$, we get $\|\boldsymbol{u}\|_1 \eqsim \|\operatorname{curl}\boldsymbol{u}\|$ for $\boldsymbol{u} \in X$. Together with the Poincaré inequality $\|\boldsymbol{u}\| \lesssim \|\boldsymbol{u}\|_1$, we get the desired result. The subtlety to make this argument rigorous is the boundary condition. For $\boldsymbol{u} \in H_0(\operatorname{curl};\Omega)$, only the tangential component is zero while to apply Poincaré inequality for H^1 vector function, both tangential and normal component trace should be zero.

A sketch of a proof of (18) is: show that $\operatorname{curl} : X \to H := H_0(\operatorname{div};\Omega) \cap \ker(\operatorname{div})$ is one-to-one and continuous. Then by the open mapping theorem, the inverse is also continuous which leads to (18). For each $\boldsymbol{\psi} \in H$, i.e., $\operatorname{div}\boldsymbol{\psi} = 0$, with the assumption of the domain Ω, there exists a vector potential $\boldsymbol{v}$ such that $\boldsymbol{\psi} = \operatorname{curl}\boldsymbol{v}$, which is not unique. But if we further require $\operatorname{div}\boldsymbol{v} = 0$ and impose boundary condition $\boldsymbol{v}\times\boldsymbol{n} = 0$, then the potential is unique. Details can be found in [5, Chapter 1 Theorem 3.6]. The condition: Ω is simply connected and $\partial\Omega$ consists of only one component is to remove the non-trivial harmonic form; see §5.4. We will name it as trivial topology.

Another approach is through the compact embedding. By modifying the proof in [5, Chapter 1, Section 3.4], i.e. using H^s instead of H^2-regularity of Poisson equation, we can prove the following result.

Lemma 4.4. *For a Lipschitz polyhedron domain Ω, there exists a constant $s \in (1/2, 1]$ depending only on Ω such that $X \hookrightarrow \boldsymbol{H}^s(\Omega)$ and*

$$\|\boldsymbol{v}\|_s \lesssim \|\boldsymbol{v}\|_{\operatorname{curl};\Omega}.$$

Consequently X is compactly imbedded in $\boldsymbol{L}^2(\Omega)$. When Ω is convex, $s = 1$.

With the compact embedding, we can mimic the proof for H^1-type Poincaré inequality to get (18). Here is a sketch.

Proof of Lemma 4.3 using Lemma 4.4. Assume (18) does not hold. Then we can find a sequence $\{\boldsymbol{v}_n\} \subset X$ s.t. $\|\boldsymbol{v}_n\| = 1$ and $\|\operatorname{curl} \boldsymbol{v}_n\| \leq 1/n \to 0$ as $n \to +\infty$. As $X \hookrightarrow \boldsymbol{L}^2$ is compact, we can find an L^2-convergent subsequence $\{\boldsymbol{v}_{n_k}\}$ which converges to an element $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$. Then by the definition of weak derivatives and convergence in L^2, we can show $\operatorname{curl} \boldsymbol{v} = 0, (\boldsymbol{v}, \nabla\phi) = 0$ for all $\phi \in H_0^1(\Omega)$, and $\|\boldsymbol{v}\| = 1$. As γ_τ is continuous, we also have $\gamma_\tau \boldsymbol{v} = 0$, i.e. $\boldsymbol{v} \in X$. Then there exists a scalar potential $p \in H_0^1(\Omega)$ s.t. $\boldsymbol{v} = \nabla p$. Taking $\phi = p$ in $(\boldsymbol{v}, \nabla\phi) = 0$, we obtain $\|\nabla p\| = 0$ and thus $p = 0, \boldsymbol{v} = 0$. Contradicts with the condition $\|\boldsymbol{v}\| = 1$. □

We summarize the well-posedness as the following theorem.

Theorem 4.5. *Let Ω be a Lipschitz polyhedron domain and Ω is topologically trivial. Then there exists a unique solution $(\boldsymbol{u}, p)$ to the saddle point system (13) and*

$$\|\boldsymbol{u}\| + \|\alpha^{1/2}\, \nabla \times \boldsymbol{u}\| + \|\beta^{1/2}\, \nabla p\| \lesssim \|\boldsymbol{f}\|_{V'}.$$

Furthermore if $\operatorname{div} \boldsymbol{f} = 0$, then the Lagrange multiplier $p = 0$.

Proof. The well-posedness is from Brezzi theory. When $\operatorname{div} \boldsymbol{f} = 0$, chose the test function $\boldsymbol{v} = \nabla p$ in (14), we get $\|\beta^{1/2}\nabla p\| = 0$ which implies $p = 0$ as $p \in H_0^1(\Omega)$. □

Now we revisit the stability of weak formulation (12). We further require $\operatorname{div} \boldsymbol{f} = 0$. We consider the stability in the space X for which we can apply Poincaé inequality (18) to obtain a coercivity even β is near 0.

Theorem 4.6. *Let Ω be a Lipschitz polyhedron domain and β be a positive constant. For a given $\boldsymbol{f} \in V'$ and $\operatorname{div} \boldsymbol{f} = 0$, there exists a unique solution $\boldsymbol{u}$ to the symmetric and positive definite problem (12) and*

$$\|\boldsymbol{u}\|_{\operatorname{curl}} \lesssim \frac{1}{\alpha_{\min}} \|\boldsymbol{f}\|_{V'},$$

with stability constant independent of β.

Proof. As the bilinear form is equivalent to the inner product of $H(\operatorname{curl}; \Omega)$, the existence and uniqueness of the solution $\boldsymbol{u}$ to (12) can be obtained by the Riesz representation theorem. As $\beta > 0$ and $\operatorname{div} \boldsymbol{f} = 0$, we take $\boldsymbol{v} = \nabla p$ in (12) to conclude $\operatorname{div}^w \boldsymbol{u} = 0$, i.e. $\boldsymbol{u} \in X$. Then we can apply Poincaé inequality (18) to obtain a coercivity

$$\alpha_{\min}(\|\boldsymbol{u}\|^2 + \|\nabla \times \boldsymbol{u}\|^2) \lesssim a(\boldsymbol{u}, \boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{u}) \lesssim \|\boldsymbol{f}\|_{V'} \|\boldsymbol{u}\|_{\operatorname{curl}}$$

from which the desired stability follows. □

5. Finite Element Methods for Maxwell Equations

In this section we first present two finite element spaces for Maxwell equations, discuss the interpolation error, and give convergence analysis of finite element methods for Maxwell equations using these spaces.

5.1. **Edge Elements.** We describe two types of edge elements developed by Nédélec [7, 8] in 1980s. We also briefly mention the implementation of these elements in MATLAB and recommend the readers to do the project *Project: Edge Finite Element Method for Maxwell-type Equations.*

5.1.1. *First family: lowest order.* For the k-th edge $\boldsymbol{e}_k$ with vertices (i, j) and the direction from i to j, the basis $\boldsymbol{\phi}_k$ and corresponding degree of freedom $l_k(\cdot)$ are

$$\boldsymbol{\phi}_k = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i,$$
$$l_k(\boldsymbol{v}) = \int_{e_k} \boldsymbol{v} \cdot \boldsymbol{t} \, ds \approx \frac{1}{2}[\boldsymbol{v}(i) + \boldsymbol{v}(j)] \cdot \boldsymbol{e}_k,$$

where the quadrature is exact when $\boldsymbol{v} \cdot \boldsymbol{t}$ is linear.

We verify the duality $l_k(\boldsymbol{\phi}_k) = 1$ as follows

$$\boldsymbol{\phi}_k(i) \cdot \boldsymbol{e}_k = \nabla \lambda_j \cdot \boldsymbol{e}_k = \int_{e_k} \nabla \lambda_j \cdot \boldsymbol{t} \, ds = \lambda_j(j) - \lambda_j(i) = 1$$
$$\boldsymbol{\phi}_k(j) \cdot \boldsymbol{e}_k = \nabla \lambda_i \cdot \boldsymbol{e}_k = \int_{e_k} \nabla \lambda_i \cdot \boldsymbol{t} \, ds = \lambda_i(j) - \lambda_i(i) = -1,$$

and consequently $l_k(\boldsymbol{\phi}_k) = 1$.

If we change the integral to another edge (m, n). If $(m, n) \cap (i, j) = \varnothing$, then $\lambda_i|_{e_{mn}} = \lambda_j|_{e_{mn}} = 0$. Without loss of generality, consider $m = i$ and $n \notin \{i, j\}$. Then in the basis $\boldsymbol{\phi}_k$ either $\nabla \lambda_j \cdot \boldsymbol{t}_{mn} = 0$ or $\lambda_j|_{e_{mn}} = 0$ and therefore $\boldsymbol{\phi}_k \cdot \boldsymbol{t}_{mn} = 0$. This verifies $l_i(\boldsymbol{\phi}_k) = 0$ for $i \neq k$.

The lowest order edge element is

$$\mathrm{NE}^0 = \mathrm{span}\{\boldsymbol{\phi}_k, k = 1, 2, \cdots, 6\}$$

which is a linear polynomial. For a 2D triangle, the formulae for the basis is the same and three basis functions on a triangle is shown below We also show three basis function

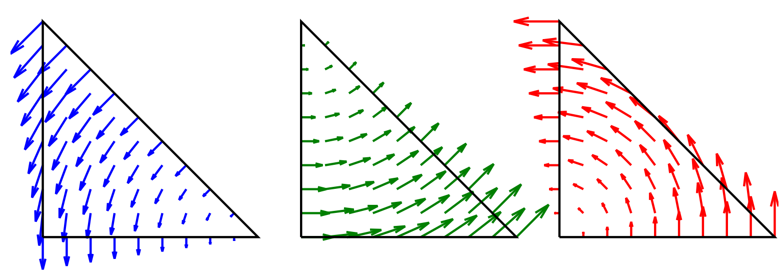

FIGURE 2. Basis of NE^0 in a triangle.

associated to three edges on one face in a tetrahedron in Fig. 3. Notice that the vector field $\boldsymbol{\phi}_k$ of edge k is orthogonal to other edges.

The lowest order element NE^0 is not $\mathcal{P}_1^3$ whose dimension is $4 \times 3 = 12$. In other words, the lowest order edge element is an incomplete linear polynomial space and can only reproduce constant vector. From the approximation point of view, the L^2 error can be only first order. The $H(\mathrm{curl}\,)$ norm of error is also first order.

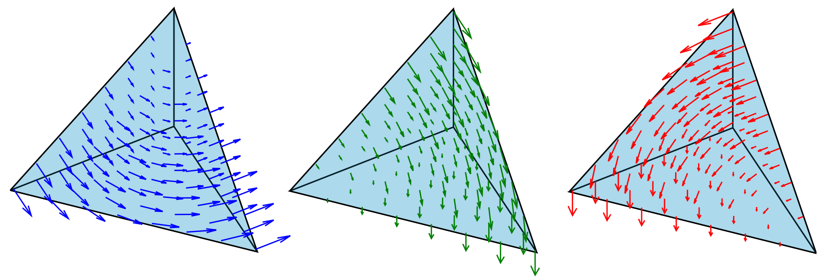

FIGURE 3. Three basis of NE^0 associated to three edges on one face in a tetrahedron.

5.1.2. *Second family: linear polynomial.* In addition to ϕ_k, for each edge, we add one more basis

$$\psi_k = \lambda_i \nabla \lambda_j + \lambda_j \nabla \lambda_i,$$
$$l_k^1(\boldsymbol{v}) = 3\int_{e_k} \boldsymbol{v}\cdot\boldsymbol{t}(\lambda_i - \lambda_j)\, ds \approx \frac{1}{2}[\boldsymbol{v}(i) - \boldsymbol{v}(j)]\cdot \boldsymbol{e}_k.$$

The quadrature is obtained by the Simpson's rule with the fact $\lambda_i - \lambda_j = 0$ at the middle point, which is exact when $\boldsymbol{v}\cdot\boldsymbol{t}$ is linear. Obviously $\{l_k(\cdot), l_k^1(\cdot), k = 1, 2, \cdots, 6\}$ are linear independent. We then show it is dual to $\{\phi_k, \psi_k\}$

The Simpson's rule is exact for $l_k^1(\psi_k)$ and thus

$$l_k^1(\psi_k) = \frac{1}{2}\left[\psi_k\cdot \boldsymbol{e}_{ij}(i) - \psi_k \cdot \boldsymbol{e}_{ij}(j)\right] = \frac{1}{2}\left[(\lambda_i - \lambda_j)(i) - (\lambda_i - \lambda_j)(j)\right] = 1.$$

The verification of $\psi_k\cdot \boldsymbol{e}_l = 0$, for $l \neq k$, is similar as before. Therefore $\{l_k^1\}$ is a dual basis of $\{\psi_k\}$.

We need to verify one more duality

$$l_k(\psi_l) = 0, \quad l_k^1(\phi_l) = 0, \quad \forall l = 1, 2, \cdots, 6.$$

We only need to worry about $l = k$ since $\psi_k\cdot\boldsymbol{t}_l = \phi_k\cdot\boldsymbol{t}_l = 0$ if $k \neq l$. Notice that $\psi_k\cdot\boldsymbol{t}_k$ is odd (respect to the middle point) and thus the integral is zero. Similarly $\phi_k\cdot\boldsymbol{t}_k = 1$ and thus $l_k^1(\phi_k) = 0$.

The lowest order second family of edge element is

$$\mathrm{NE}^1 = \mathrm{span}\{\phi_k, \psi_k, k = 1, 2, \cdots, 6\},$$

which is a full linear polynomial and will reproduce linear polynomials. Therefore the L^2-norm of error will be second order. The $H(\mathrm{curl})$ norm, however, is still first order since $\psi_k = \nabla(\lambda_i\lambda_j)$ and $\nabla\times\psi_k = 0$ has no contribution to the approximation of curl. Plot of ψ_k in a triangle is given below

The global finite element space is obtained by gluing piecewise one. Using the barycentric coordinate in each tetrahedron, for an edge, the basis ϕ_k, ψ_k can be extend to all tetrahedron surrounding this edge. Given a triangulation $\mathcal{T}$, let $\mathcal{E}$ be the edge set of $\mathcal{T}$. Define

$$\mathrm{NE}^0(\mathcal{T}) = \mathrm{span}\{\phi_e, e\in\mathcal{E}\},$$
$$\mathrm{NE}^1(\mathcal{T}) = \mathrm{span}\{\phi_e, \psi_e, e\in\mathcal{E}\}.$$

To show the obtained spaces are indeed in $H(\mathrm{curl}\,;\Omega)$, it suffices to verify the tangential continuity of the piecewise polynomials. Given a triangular face f, in one tetrahedron, we label the vertex opposite to f as x_f and the corresponding barycentric coordinate will be

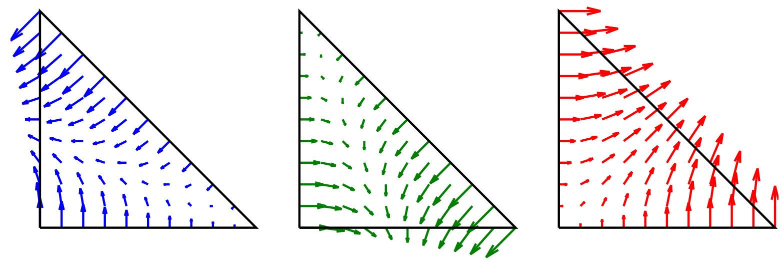

FIGURE 4. Basis vectors $\boldsymbol{\psi}_k$ of NE^1 in a triangle.

denoted by λ_f. For an edge e using x_f as an vertex, the corresponding basis $\boldsymbol{\phi}_e$ or $\boldsymbol{\psi}_e$ is a linear combination of $\lambda_i \nabla \lambda_f$ and $\lambda_f \nabla \lambda_i$. Restrict to f, $\lambda_f|_f = 0$ and $\nabla \lambda_f \times n_f = 0$ since $\nabla \lambda_f$ is a norm vector of f. Therefore we showed that $\boldsymbol{\phi}_e|_f \times n_f = \boldsymbol{\psi}_e|_f \times n_f = 0$ for edges e containing n_f. Therefore for $\boldsymbol{v} \in \mathrm{NE}^0(\mathcal{T})$ or $\mathrm{NE}^1(\mathcal{T})$, the trace $\boldsymbol{v}|_f \times n_f$ depends only on the basis function of edges of f which is the ideal continuity of a $H(\mathrm{curl}\,;\Omega)$ function.

We introduce the canonical interpolation to the edge element space. Given a triangulation $\mathcal{T}_h$ with mesh size h. Define $I_h^{\mathrm{curl}} : V \cap \mathrm{dom}(I_h^{\mathrm{curl}}) \to \mathrm{NE}^0(\mathcal{T}_h)$ as follows: given a function $\boldsymbol{u} \in V$, define $\boldsymbol{u}_I = I_h^{\mathrm{curl}} \boldsymbol{u} \in \mathrm{NE}^0(\mathcal{T}_h)$ by matching the d.o.f.

$$l_e(I_h^{\mathrm{curl}} \boldsymbol{u}) = l_e(\boldsymbol{u}) \quad \forall e \in \mathcal{E}_h(\mathcal{T}_h).$$

Namely

$$\boldsymbol{u}_I = \sum_{e \in \mathcal{E}_h} \left(\int_e \boldsymbol{u} \cdot \boldsymbol{t} \,\mathrm{d}s \right) \boldsymbol{\phi}_e$$

For the second family edge element space, add $l_e^1(\cdot)$ and $\boldsymbol{\psi}_e$.

Exercise 5.1. In one tetrahedron τ, verify I_h^{curl} to $\mathrm{NE}^0(\tau)$ will preserve constant vector and linear vectors for space $\mathrm{NE}^1(\tau)$.

For the error $\nabla \times (\boldsymbol{u} - \boldsymbol{u}_I)$, if we want to use Bramble-Hilbert lemma, we need to introduce the Piola transformation to connecting the curl operators $\nabla\times$ and $\hat{\nabla}\times$ in the reference element. Instead we introduce the lowest order face element for $H(\mathrm{div};\Omega)$ and use the commuting diagram to change to the estimate of L^2-error.

5.2. **Face Element.** Give a face f formed by vertices $[i, j, k]$, we introduce a basis vector

$$\boldsymbol{\phi}_l = 2(\lambda_i \nabla \lambda_j \times \nabla \lambda_k + \lambda_j \nabla \lambda_k \times \nabla \lambda_i + \lambda_k \nabla \lambda_i \times \nabla \lambda_j), \tag{19}$$

and the corresponding degree of freedom

$$l_f(\boldsymbol{v}) = \int_{f_l} \boldsymbol{v} \cdot \boldsymbol{n} \,\mathrm{d}S \approx \boldsymbol{v}(\boldsymbol{c}) \cdot \boldsymbol{n}_f |f|,$$

where the quadrature is exact for linear polynomial $\boldsymbol{v}$.

Exercise 5.2. [Face element]

(1) Verify $\{l_{f_i}, i = 1, 2, 3, 4\}$ is a dual basis of $\{\boldsymbol{\phi}_{f_j}, j = 1, 2, 3, 4\}$.
(2) For a triangle in 2D, the degree of freedom remains the same. Write out the basis functions.

We define the lowest order face element space, known as Raviart-Thomas element [10]

$$\mathrm{RT}^0(\tau) = \mathrm{span}\{\boldsymbol{\phi}_{f_j}, j = 1,2,3,4\}$$

and the global version

$$\mathrm{RT}^0(\mathcal{T}) = \mathrm{span}\{\boldsymbol{\phi}_f, f \in \mathcal{F}(\mathcal{T})\},$$

where $\mathcal{F}(\mathcal{T})$ is the set of all faces of a triangulation $\mathcal{T}$. A plot of basis for 2D RT element can be found in Fig. 5.

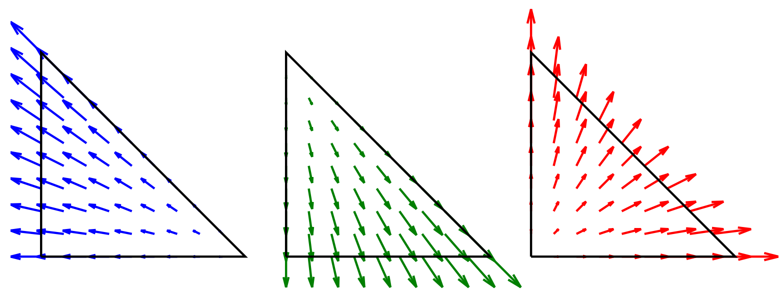

FIGURE 5. Basis vectors $\boldsymbol{\phi}_k$ of RT^0 in a triangle.

Given a triangulation $\mathcal{T}_h$ with mesh size h. Define $I_h^{\mathrm{div}} : V \to \mathrm{RT}^0(\mathcal{T}_h)$ as follows: given a function $\boldsymbol{u} \in V$, define $\boldsymbol{u}_I = I_h^{\mathrm{div}}\boldsymbol{u} \in \mathrm{RT}^0(\mathcal{T}_h)$ by matching the d.o.f.

$$l_f(I_h^{\mathrm{div}}\boldsymbol{u}) = l_f(\boldsymbol{u}) \quad \forall f \in \mathcal{F}_h(\mathcal{T}_h).$$

Namely

$$\boldsymbol{u}_I = \sum_{f\in\mathcal{F}_h} \left(\int_f \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S\right)\boldsymbol{\phi}_f$$

We verify the crucial commuting property.

Lemma 5.3. *For function $\boldsymbol{u}$ smooth enough so that $I_h^{\mathrm{curl}}\boldsymbol{u}$ and $I_h^{\mathrm{div}}\nabla\times\boldsymbol{u}$ are well defined, then*

$$\nabla\times I_h^{\mathrm{curl}}\boldsymbol{u} = I_h^{\mathrm{div}}\nabla\times\boldsymbol{u}.$$

Proof. By the Stokes' theorem and the definition of interpolation operators:

$$\begin{aligned}\int_f I_h^{\mathrm{div}}(\nabla\times\boldsymbol{u})\cdot\boldsymbol{n}_f\,\mathrm{d}S &= \int_f(\nabla\times\boldsymbol{u})\cdot\boldsymbol{n}_f\,\mathrm{d}S = \int_{\partial f}\boldsymbol{u}\cdot\boldsymbol{t}\,\mathrm{d}s\\ &= \int_{\partial f} I_h^{\mathrm{curl}}\boldsymbol{u}\cdot\boldsymbol{t}\,\mathrm{d}s = \int_f(\nabla\times I_h^{\mathrm{curl}}\boldsymbol{u})\cdot\boldsymbol{n}_f\,\mathrm{d}S.\end{aligned}$$

□

The commuting diagram can be extended to the whole sequence and summarized in Fig. 6.

Exercise 5.4. Prove the commuting diagram shown in Fig. 6.

Remark 5.5. The domain of the canonical interpolation $I_h^{\mathrm{curl}}, I_h^{\mathrm{div}}$ are smooth subspace of $H(\mathrm{curl};\Omega)$ or $H(\mathrm{div};\Omega)$, respectively. For example, even for a H^1 function $\boldsymbol{u}$, the trace $\boldsymbol{u}$ restricted on an edge is not well defined. The arguments above require the function smooth enough. Quasi-interpolation, which relaxes the smoothness of the function and preserves the nice commuting diagram, have been constructed recently.

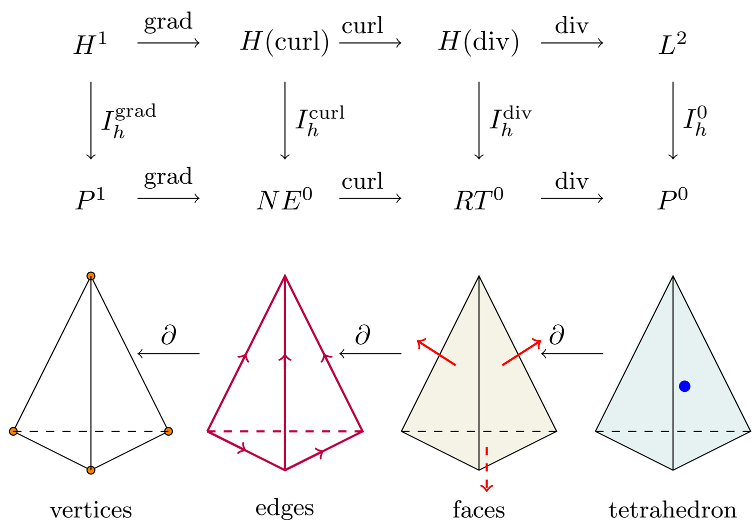

FIGURE 6. Commuting diagram of finite element spaces.

5.3. **Interpolation Error Estimate.** We first prove a stability result for I_h^{div} operator.

Lemma 5.6. *For $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, we have*

$$\|I_h^{\text{div}}\boldsymbol{v}\| \lesssim \|\boldsymbol{v}\| + h\|\nabla \boldsymbol{v}\|.$$

Proof. It suffices to prove the inequality restricted to one element K. By definition and Minkowski inequality,

$$\|I_h^{\text{div}}\boldsymbol{v}\|_{0,K} = \|\sum_{f\in\mathcal{F}_h(K)} l_f(\boldsymbol{v})\boldsymbol{\phi}_f\| \leq \sum_{f\in\mathcal{F}_h(K)} |l_f(\boldsymbol{v})|\|\boldsymbol{\phi}_f\|_{0,K}.$$

We use the scaled trace theorem for $w \in H^1(K)$

$$\|w\|_{0,f} \lesssim h^{-1/2}\|w\|_{0,K} + h^{1/2}\|\nabla w\|_{0,K}.$$

and the scaling of $\boldsymbol{\phi}_f$ to get the desired result. □

Lemma 5.7. *Assume $\operatorname{curl}\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and $\boldsymbol{u} \in \operatorname{Dom}(I_h^{\text{curl}})$. Let $\boldsymbol{u}_I = I_h^{\text{curl}}\boldsymbol{u}$. Then we have the first order interpolation error estimate*

$$\|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_I)\| \lesssim h|\operatorname{curl}\boldsymbol{u}|_1.$$

Proof. We use the commutative property and the fact that I_h^{div} preserves the constant vector to get

$$\|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_I)\| = \|(I - I_h^{\text{div}})\nabla \times \boldsymbol{u}\| = \|(I - I_h^{\text{div}})(\nabla \times \boldsymbol{u} - \boldsymbol{c})\|.$$

Then by the stability of I_h^{div} operator, we get

$$\|(I - I_h^{\text{div}})(\nabla \times \boldsymbol{u} - \boldsymbol{c})\| \lesssim \|\operatorname{curl}\boldsymbol{u} - \boldsymbol{c}\| + h|\operatorname{curl}\boldsymbol{u}|_1.$$

As it holds for arbitrary constant vector $\boldsymbol{c}$, choosing $\boldsymbol{c}$ as the average of $\operatorname{curl}\boldsymbol{u}$ and using Poincaré inequality to get the desired error estimate. □

Notice that L^2-error estimate of the interpolation error for $I_h^{\rm curl}$ is not easy and can be proved using an inequality for $l_e(\boldsymbol{v})$ on the reference element; see [4, Theorem 3.14].

5.4. **de Rham complex and finite element de Rham complex.** We revisit the diagram

$$
\begin{array}{ccccccccccc}
\mathbb{R} & \xrightarrow{\subset} & H^1(\Omega) & \xrightarrow{\rm grad} & H(\mathrm{curl}\,;\Omega) & \xrightarrow{\rm curl} & H(\mathrm{div};\Omega) & \xrightarrow{\rm div} & L^2(\Omega) & \longrightarrow & 0 \\
 & & \downarrow \mathcal{I}_h^{\rm grad} & & \downarrow \mathcal{I}_h^{\rm curl} & & \downarrow \mathcal{I}_h^{\rm div} & & \downarrow \mathcal{I}_h^{0} & & \\
\mathbb{R} & \xrightarrow{\subset} & S_h & \xrightarrow{\rm grad} & V_h & \xrightarrow{\rm curl} & U_h & \xrightarrow{\rm div} & Q_h & \longrightarrow & 0
\end{array},
$$

where S_h is the standard linear finite element space, V_h is the lowest order edge element space, U_h the lowest face element space, and Q_h is piecewise constant space.

The above sequence is called a Hilbert complex, known as de Rham complex, satisfying: $\mathrm{curl}\,\mathrm{grad}\ = 0, \mathrm{div}\,\mathrm{curl}\ = 0$ and each operator has a closed range (which implies a corresponding Poincaré inequality). The sequence is exact if $\ker(\,\mathrm{d}) = \mathrm{img}(\,\mathrm{d}^-)$ where we use d^- to denote the operator before d in the sequence. When the domain Ω has trivial topology in the sense it is simply connected and its boundary is also connected, one can verify the top sequence is exact.

In general $\ker(\,\mathrm{d}_{k+1})/\mathrm{img}(\,\mathrm{d}_k)$ is defined as the k-th *cohomology space* of Ω. They are finite dimensional vector spaces whose dimensions are the so-called Betti numbers β_k of the manifold Ω. For a bounded connected region in $\mathbb{R}^3$, $\beta_0 = 1$ (number of connected component), $\beta_1 =$ genus (number of handles), $\beta_2 =$ connected components of the boundary (number of holes), and $\beta_3 = 0$.

We give an example on the exactness. Consider a vector function u and $\mathrm{curl}\, u = 0$ in a connected domain Ω. Try to find a scalar function p s.t. $u = \mathrm{grad}\, p$, where p is called the potential function of u. The idea is to use the line integral along a curve C connecting a fixed point x_0 and x, i.e, $p(x) = \int_{C[x_0,x]} u(s)\cdot \mathrm{d}s$. Then one can easily verify $u = \mathrm{grad}\, p$. The famous example is the gravitation field.

To be a well defined function, the line integral should be independent of the choice of the curve. For any two curves C_1 and C_2 with the same ending vertices forms the boundary of a 2-D surface S. Then from $\mathrm{curl}\, u = 0$, we can apply the Stokes theorem to conclude

$$\int_S \mathrm{curl}\, u\cdot\,\mathrm{d}S = \int_{\partial S} u\cdot\,\mathrm{d}s = \int_{C_1} u\cdot\,\mathrm{d}s - \int_{C_2} u\cdot\,\mathrm{d}s = 0.$$

Here the negative sign is due to the orientation. The topological constraint comes from the fact not every close curve is a boundary of a close surface, e.g., a tours.

For a divergence free vector function u, find its vector potential ϕ is much harder. Namely if $\mathrm{div}\, u = 0$, there exists ϕ s.t. $u = \mathrm{curl}\,\phi$. We will skip it here but to mention that the existence of such potential depends on the topology of the domain. A physical example is the electric field generated by a charge at the origin

$$u(x,y,z) = \frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}.$$

Then $\mathrm{div}\, u = 0$ except the origin. But u cannot be written as $\mathrm{curl}\,\phi$ as $\int_S u\cdot n\,\mathrm{d}S = 4\pi$ for any close and positive orientated surface S enclosing the origin.

In the diagram, the interpolation operators are not canonical interpolation operators which are not well defined for $H(\mathrm{d};\Omega)$ space. We assume there exists quasi-interpolation operators $\mathcal{I}_h^{\rm d}$ with the following properties

- $\mathrm{dom}(\mathcal{I}_h^{\rm d}) \subseteq H(\,\mathrm{d};\Omega)$;

- L^2-stable: $\|\mathcal{I}_h^{\rm d} v\| \lesssim \|v\|$;
- projection: $\mathcal{I}_h^{\rm d} v = v$ if v is in finite element space;
- commutative with differential operators $\mathrm{d}\, \mathcal{I}_h^{\rm d} = \mathcal{I}_h^{\rm d^+}\, \mathrm{d}$.

Such operators can be found in [1, page 65-67] and [3].

We will use the commuting property to verify that the bottom sequence is also exact, which is called finite element de Rham complex. Given a function $v_h \in V_h$ and $\operatorname{curl} v_h = 0$, as $V_h \subset H(\operatorname{curl};\Omega)$, we can find a potential $p \in H^1(\Omega)$ s.t. $v_h = \operatorname{grad} p$. The scalar function p may not be in the finite element space. We let $p_h = \mathcal{I}_h^{\rm grad} p$ and use the commuting diagram to conclude

$$\operatorname{grad} \mathcal{I}_h^{\rm grad} p = \mathcal{I}_h^{\rm curl} \operatorname{grad} p = \mathcal{I}_h^{\rm curl} v_h = v_h.$$

Other blocks can be verified similarly. Modification to space with zero trace is also straightforward.

A systematic way of studying finite element methods using differential forms is known as FEEC (finite element exterior calculus). Here we only give a glimpse and refer to the survey by Arnold, Falk and Winther [1] for the general framework. For more specific application to electromagnetism, we refer to Hiptmair [4].

5.5. **Finite element approximation.** For finite element approximation, we chose edge element space $V_h \subset H_0(\operatorname{curl};\Omega)$ and define the subspace

$$X_h = V_h \cap \ker(\operatorname{div}_h),$$

where $\operatorname{div}_h : V_h \to S_h \subset H_0^1(\Omega)$ is the discrete weak divergence operator defined as the adjoint of grad, i.e.,

$$(\operatorname{div}_h v_h, p_h) := (v_h, \operatorname{grad} p_h) \quad \forall p_h \in S_h.$$

Note that $X_h \not\subset X$. We can lift $v_h \in X_h$ to X through L^2-projection Q_X. That is: $v = Q_X v_h \in X$ satisfies

$$(v, \xi) = (v_h, \xi) \quad \forall \xi \in X.$$

As X is a subspace of $L^2(\Omega)$, such L^2-projection exists and unique.

Lemma 5.8. *Given $v_h \in X_h$, let $v = Q_X v_h$ be its L^2-projection to X. Then*

(1) $\operatorname{curl} v = \operatorname{curl} v_h$*;*
(2) $\|v\| \eqsim \|v_h\|$*;*
(3) $\|v - v_h\| \lesssim h^s \|\operatorname{curl} v_h\|$.

Proof. We first solve a Poisson equation: find $p \in H_0^1(\Omega)$ s.t.

$$(\nabla p, \nabla \phi) = -(v_h, \nabla \phi) \quad \forall \phi \in H_0^1(\Omega).$$

Then $v = v_h + \nabla p$ is orthogonal to $\nabla H_0^1(\Omega)$. (1) is then trivial as $\operatorname{curl} \operatorname{grad} = 0$.

By the property of L^2-projection, $\|v\| \leq \|v_h\|$. To control $v - v_h$, we use the partial orthogonality

$$(v - v_h, \nabla \phi_h) = (\nabla p, \nabla \phi_h) = -(v_h, \nabla \phi_h) = 0 \quad \forall \phi_h \in S_h.$$

We claim $\nabla \times (\mathcal{I}_h^{\rm curl} v - v_h) = 0$ as

$$\nabla \times \mathcal{I}_h^{\rm curl} v = \mathcal{I}_h^{\rm curl} (\nabla \times v) = \mathcal{I}_h^{\rm curl} (\nabla \times v_h) = \nabla \times v_h.$$

Then by the exactness of the finite element de Rham complex, there exists $\phi_h \in S_h$ s.t. $\nabla \times (\mathcal{I}_h^{\rm curl} v - v_h) = \nabla \phi_h$. Then by the partial orthogonality

$$(v - v_h, v - v_h) = (v - v_h, v - \mathcal{I}_h^{\rm curl} v) + (v - v_h, \mathcal{I}_h^{\rm curl} v - v_h) = (v - v_h, v - \mathcal{I}_h^{\rm curl} v),$$

which implies

$$\|v - v_h\| \le \|v - \mathcal{I}_h^{\rm curl} v\|. \tag{20}$$

Then $\|v_h\| \lesssim \|v\|$ is from the triangle inequality and the L^2-stability of $\mathcal{I}_h^{\rm curl}$.

To prove (3), we use the embedding result Lemma 4.4 and the interpolation error estimate

$$\|v - v_h\| \le \|v - \mathcal{I}_h^{\rm curl} v\| \lesssim h^s \|v\|_s \lesssim h^s \|\mathrm{curl}\, v\| = h^s \|\mathrm{curl}\, v_h\|.$$

□

By the exact sequence, $V_h = X_h \oplus \mathrm{grad}\,(S_h)$. The bilinear form $a(\cdot,\cdot)$ is not coercive on V_h but on the subspace X_h due to the following discrete Poincaré inequality.

Lemma 5.9 (Discrete Poincaré inequality)**.** *When Ω is topologically trivial and $\mathcal{T}_h$ is shape regular. Then*

$$\|v_h\| \lesssim \|\mathrm{curl}\, v_h\| \quad \textit{for } v_h \in X_h.$$

Proof. It is not a simple consequence of the Poincaré inequality in Lemma 4.3 as $X_h \not\subset X$. We lift v_h to X, i.e. $v = Q_X v_h$ and apply Poincaré inequality to v. The desired discrete version is from the properties of v in Lemma 5.8

$$\|v_h\| \lesssim \|v\| \lesssim \|\mathrm{curl}\, v\| = \|\mathrm{curl}\, v_h\|.$$

□

Now we can consider finite element discretization of the saddle point formulation: find $\boldsymbol{u}_h \in V_h, p_h \in S_h$ s.t.

$$(\alpha \nabla \times \boldsymbol{u}_h, \nabla \times \boldsymbol{v}_h) + (\beta \boldsymbol{v}_h, \nabla p_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \qquad \forall\, \boldsymbol{v}_h \in V_h, \tag{21}$$

$$(\beta \boldsymbol{u}_h, \nabla q_h) = 0 \qquad \forall\, q_h \in Q_h. \tag{22}$$

The discrete inf-sup condition for div_h is easy as its adjoint $\mathrm{grad}\,: S_h \to V_h$ is injective. The coercivity in $\ker(\mathrm{div}_h)$ is ensured by the discrete Poincaré inequality. Therefore the well-posedness of the (21)-(22) is from Brezzi theory and we obtain the first order error estimate. We summarize as the following theorem.

Theorem 5.10. *There exists a unique solution $(\boldsymbol{u}_h, p_h)$ to (21)-(22). When $\mathrm{curl}\, u \in H^1(\Omega)$ and $p \in H^2(\Omega)$, we have*

$$\|\alpha^{1/2} \nabla \times (\boldsymbol{u} - \boldsymbol{u}_h)\| + \|\beta^{1/2} \nabla (p - p_h)\| \lesssim h\left(|\nabla \times \boldsymbol{u}|_1 + |p|_2\right).$$

When $\mathrm{div}\, \boldsymbol{f} = 0$, we have both $p = p_h = 0$ and

$$\|\alpha^{1/2} \nabla \times (\boldsymbol{u} - \boldsymbol{u}_h)\| \lesssim h |\nabla \times \boldsymbol{u}|_1.$$

Proof. By Brezzi theory and interpolation error estimate, we have

$$\begin{aligned}\|\alpha^{1/2} \nabla \times (\boldsymbol{u} - \boldsymbol{u}_h)\| + \|\beta^{1/2} \nabla (p - p_h)\| &\lesssim \|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_I)\| + \|\nabla (p - p_I)\| \\ &\lesssim h\left(|\nabla \times \boldsymbol{u}|_1 + |p|_2\right).\end{aligned}$$

When $\mathrm{div}\, \boldsymbol{f} = 0$, we have both $p = p_h = 0$. □

The symmetric formulation (12) is simpler and leave as an exercise.

Exercise 5.11. Present the finite element discretization of (12) and its corresponding error estimate.

ACKNOWLEDGMENT

Thank Dr. Shuhao Cao for the discussion and help on the nice figures of basis functions. Thank Dr. Yongke Wu on the proof of discrete Poincaré inequality.

REFERENCES

[1] D. N. Arnold, R. S. Falk, and R. Winther. Finite element exterior calculus, homological techniques, and applications. *Acta Numer.*, pages 1–155, 2006. 15

[2] A. Buffa, M. Costabel, and D. Sheen. On traces for H(curl,Omega) in Lipschitz domains. *Journal of Mathematical Analysis and Applications*, 276(2):845–867, 2002. 5

[3] S. H. Christiansen and R. Winther. Smoothed projections in finite element exterior calculus. *Math. Comp.*, 77(262):813, 2008. 15

[4] R. Hiptmair. Finite elements in computational electromagnetism. *Acta Numer.*, pages 237–339, 2002. 14, 15

[5] V. Girault and P. A. Raviart. *Finite element methods for Navier–Stokes equations: Theory and algorithms*. Springer-Verlag, Berlin, 1986. 7

[6] P. Monk. *Finite Element Methods for Maxwell's Equations*. Oxford University Press, 2003. 5

[7] J. Nédélec. Mixed finite elements in R^3. *Numer. Math.*, 35:315–341, 1980. 9

[8] J. Nédélec. A new family of mixed finite elements in R^3. *Numer. Math.*, 50:57–81, 1986. 9

[9] S. J. Orfanidis. *Electromagnetic Waves and Antennas*. Online, 2011. 3

[10] P. A. Raviart and J. Thomas. A mixed finite element method fo 2-nd order elliptic problems. In I. Galligani and E. Magenes, editors, *Mathematical aspects of the Finite Elements Method*, Lectures Notes in Math. 606, pages 292–315. Springer, Berlin, 1977. 12

[11] L. Tartar. On the characterization of traces of a sobolev space used for maxwell's equation. In *Proceedings of a Meeting*. Bordeaux, France, 1997. 5