

# FINITE VOLUME METHODS

LONG CHEN

The finite volume method (FVM) is a discretization technique for partial differential equations, especially those that arise from physical conservation laws. FVM uses a volume integral formulation of the problem with a finite partitioning set of volumes to discretize the equations. FVM is in common use for discretizing computational fluid dynamics equations.

## 1. GENERAL FORM OF FINITE VOLUME METHODS

We consider vertex-centered finite volume methods for solving diffusion type elliptic equation

$$(1) \quad -\nabla \cdot (\mathbf{K}\nabla u) = f \text{ in } \Omega,$$

with suitable Dirichlet or Neumann boundary conditions. Here  $\Omega \subset \mathbb{R}^d$  is a polyhedral domain ( $d \geq 2$ ), the diffusion coefficient  $\mathbf{K}(\mathbf{x})$  is a  $d \times d$  symmetric matrix function that is uniformly positive definite on  $\Omega$  with components in  $L^\infty(\Omega)$ , and  $f \in L^2(\Omega)$ . We have discussed finite element methods based on the discretization of the weak formulation and finite difference methods based on the classic formulation.

We shall now present finite volume methods based on the following balance equation

$$(2) \quad -\int_{\partial b} (\mathbf{K}\nabla u) \cdot \mathbf{n} \, ds = \int_b f \, dx, \quad \forall b \subset \Omega,$$

where  $\mathbf{n}$  denotes the unit outwards normal vector of  $\partial b$ . We first recall how the equation (??) is derived.

When the equilibrium is reached, numerous physical models are based on conservation and constitutive laws.

*The balance equation for the conservation law:*

$$(3) \quad \int_{\partial b} \mathbf{q} \cdot \mathbf{n} \, dS = \int_b f \, dx, \quad \text{for any domain } b \subset \Omega,$$

where  $\mathbf{q}$  denoting the flux density and  $\mathbf{n}$  the unit outward normal field of  $\partial b$ .

*The constitutive equation:*

$$(4) \quad \mathbf{q} = -\mathbf{K}\nabla u.$$

If  $u$  denotes the: chemical concentration, temperature, electrostatic potential, or pressure, then equation (??) is: Fick's law of diffusion, Fourier's law of heat conduction, Ohm's law of electrical conduction, or Darcy's law of flow in the porous medium, respectively.

Suppose  $u$  and  $\mathbf{q}$  are smooth enough. In view of Gauss theorem, (??) can be written as

$$(5) \quad \int_b \nabla \cdot \mathbf{q} \, dx = \int_b f \, dx, \quad \forall b \subset \Omega.$$

Since  $b$  is arbitrary, letting  $b \rightarrow \{\mathbf{x}\}$ , it implies

$$(6) \quad \nabla \cdot \mathbf{q}(\mathbf{x}) = f(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega.$$

Substituting  $\mathbf{q} = -\mathbf{K}(\mathbf{x})\nabla u$  into (??), we then obtain the classic formulation (??).

Finite volume methods are discretizations of the balance equation (??) so that the conservation holds in the discrete level. The discretization consists of three approximations:

- (1) approximate the function  $u$  by  $u_h$  in a  $N$ -dimensional space  $\mathbb{V}$ ;
- (2) approximate “arbitrary domain  $b \subset \Omega$ ” by a finite subset  $\mathcal{B} = \{b_i, i = 1 : M\}$ ;
- (3) approximate boundary flux  $(\mathbf{K}\nabla u) \cdot \mathbf{n}$  on  $\partial b_i$  by a discrete one  $(\mathbf{K}\nabla_h u_h) \cdot \mathbf{n}$ .

We then end with a method: to find  $u_h \in \mathbb{V}$  such that:

$$(7) \quad - \int_{\partial b_i} (\mathbf{K}\nabla_h u_h) \cdot \mathbf{n} dS = \int_{b_i} f d\mathbf{x}, \quad \forall b_i \subset \Omega, i = 1 : M.$$

We call any method in the form (??) *finite volume methods* (FVMs).

Since finite volume methods discretize the balance equation (??) directly, an obvious virtue of finite volume methods is the conservation property comparing with finite element methods based on the weak formulation. This property can be fundamental for the simulation of many physical models, e.g., in oil recovery simulations and in computational fluid dynamics in general.

On the other hand, the function space and the control volume can be constructed based on general unstructured triangulations for complex geometry domains. The boundary condition can be easily built into the function space or the variational form. Thus FVM is more flexible than standard finite difference methods which mainly defined on the structured grids of simple domains.

## 2. CELL-CENTERED FINITE VOLUME METHOD

Let  $\mathcal{T}$  be a triangular or Cartesian grid of  $\Omega$ . We choose the finite dimensional space  $\mathbb{V} = \{v \in L^2(\Omega) : v|_{\tau} \text{ is constant for all } \tau \in \mathcal{T}\}$ . Then  $\dim \mathbb{V} = NT$ , the number of elements of  $\mathcal{T}$ . We also choose  $\mathcal{B} = \mathcal{T}$ . See Figure ??(a). To complete the discretization, we need to assign the boundary flux of each element.

This can be done in a finite difference fashion. For example, for an interior side  $e$  (an edge in 2-D and a face in 3-D) shared by two elements  $\tau_1$  and  $\tau_2$ , we can define

$$(8) \quad \nabla_h u_h \cdot \mathbf{n}_e := \frac{u_h|_{\tau_2} - u_h|_{\tau_1}}{c_{\tau_2} - c_{\tau_1}},$$

where the normal vector  $\mathbf{n}_e$  is the outward unit normal vector of  $e$  in  $\tau_1$ , i.e. pointing from  $\tau_1$  to  $\tau_2$  and  $c_{\tau_i} \in \tau_i, i = 1, 2$  are points in each element such that the line segment connecting  $c_{\tau_2}$  and  $c_{\tau_1}$  is orthogonal to the side  $e$ . By the symmetry, for rectangles or cubes  $c_{\tau}$  is the mass center of  $\tau$ . For simplex,  $c_{\tau}$  should be the circumcenters which imposes restriction on the triangulation. When the mesh is a uniform rectangular grid, it is reduced to the cell centered finite difference method; see Section 4 of [Chapter: Finite Difference Methods](#).

The error analysis can be carried out in the finite difference fashion by considering the truncation error and stability of the resulting system. Theory and computation along this approach is summarized in the book [?].

Another approach to discretize the boundary flux is through mixed finite element methods. The gradient operator is understood as  $\nabla : L^2 \rightarrow H^{-1}$ . Optimal error estimate can be easily obtained by using that of mixed finite element methods [?].

Since the control volume is the element (also called cell) of the mesh and the unknown is associated to each element/cell, it is often called cell-centered finite volume methods and the difference scheme (??) is also known as cell centered finite difference methods.



FIGURE 1. Meshes and control volumes of two FVMs. The unknowns are associated to black nodes.

### 3. VERTEX-CENTERED FINITE VOLUME METHOD

We now discuss another popular choice of  $\mathbb{V}$  and  $\mathcal{B}$ . To simplify the notation, we consider two dimensional triangular grids and homogenous Dirichlet boundary condition. We refer to [?] for the general treatment in high dimensional simplicial grid and [?] for rectangle grids.

Let  $\Omega \subset \mathbb{R}^2$  be a polygon and let  $\mathcal{T}$  be a triangular grid of  $\Omega$ . Denoted by  $\mathbb{V}_{\mathcal{T}}$  be the linear finite element spaces of  $H_0^1(\Omega)$  based on  $\mathcal{T}$ :

$$\mathbb{V}_{\mathcal{T}} = \{v \in H_0^1(\Omega) : v|_{\tau} \in \mathcal{P}_1(\tau), \forall \tau \in \mathcal{T}\},$$

where  $\mathcal{P}_1(\tau)$  is the linear polynomial space on  $\tau$ . We shall choose  $\mathbb{V} = \mathbb{V}_{\mathcal{T}}$ . The dimension  $N$  is the number of interior vertices of  $\mathcal{T}$ .

The control volume will be given by another mesh  $\bar{\mathcal{B}} = \{b_i, i = 1, \dots, M\}$  satisfying

$$\bar{\Omega} = \cup_{i=1}^M b_i, \text{ and } b_i \cap b_j = \emptyset, i \neq j,$$

and to reflect to the Dirichlet boundary condition, we set

$$\mathcal{B} = \{b_i \in \bar{\mathcal{B}}, b_i \subset \overset{\circ}{\Omega}\}.$$

The element  $b_i$  of  $\mathcal{B}$  is not necessary to be polygons. But for practical reasons,  $b_i$  are chosen as polygons such that the boundary integral is easy to evaluate.

Given a triangulation  $\mathcal{T}$ , one construction of  $\bar{\mathcal{B}}$  is given as follows: for each triangle  $\tau \in \mathcal{T}$ , select a point  $c_{\tau} \in \tau$ . The point  $c_{\tau}$  can coincides with middle points of edges, but not the vertices of triangles (to avoid the degeneracy of the control volume). In each triangle, we connect  $c_{\tau}$  to three middle points on the boundary edges. This will divide each triangle in  $\mathcal{T}$  into three regions. For each vertex  $x_i$  of  $\mathcal{T}$ , we collect all regions containing this vertex and define it as  $b_i$ . In Figure ?? we only draw the control volume for interior vertices since the Dirichlet boundary condition is build into the space  $\mathbb{V}_{\mathcal{T}}$  and the unknown is only associated to interior vertices. Obviously for Neumann boundary condition, we should use  $\bar{\mathcal{B}}$ .

There are three common choices of  $c_{\tau}$ :

- Type A:  $c_{\tau}$  is the barycenter of  $\tau$ .
- Type B:  $c_{\tau}$  is the middle point of the longest edge.
- Type C:  $c_{\tau}$  is the circumcenter of  $\tau$ .

Type A is preferable for triangulations composed by equilateral triangles. In this case  $\tau$  will be divided into three parts with equal area. This symmetric property is important to get optimal rate of convergence in  $L^2$  norm. Type B is better for right triangles, and can

be easily obtained by the longest edge bisection method. Type  $C$  is suitable for Delaunay triangulations. The edges of the control volumes will be orthogonal to the intersected edges of triangles, and if the grid  $\mathcal{T}$  is a Delaunay triangulation,  $\mathcal{B}$  will be a Voronoi diagram.

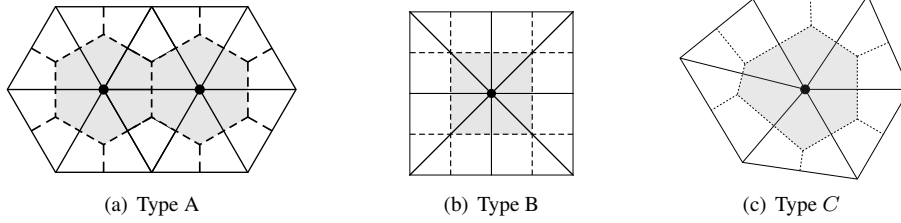


FIGURE 2. Three types of grids and dual grids. The gray areas are the control volumes of interior nodes. Type  $A$ : The point  $c_\tau$  is the barycenter of  $\tau$ . Type  $B$ : The points  $c_\tau$  is the middle point of the longest edge. Type  $C$ : The point  $c_\tau$  is the circumcenter of  $\tau$ .

Since we associate control volumes and unknowns to vertices, it is called vertex-centered finite volume methods. It is also known as box method [?, ?] (since the control volume is called box in these work), finite volume element methods [?, ?, ?] (to emphasis the approximation of  $u$  is from finite element space), and generalized finite difference methods [?, ?]. High order finite volume methods can be found in [?, ?].

#### 4. PETROV-GALERKIN FORMULATION

We shall follow Bank and Rose [?] to formulate the vertex-centered linear finite volume method as a Petrov-Galerkin method by choosing different trial space and test space for an appropriate bilinear form.

We first introduce a function space for the dual mesh. Let  $\mathcal{B}$  be the dual mesh of a triangulation  $\mathcal{T}$  constructed in the previous subsection. We define a piecewise constant function space on  $\mathcal{B}$  by:

$$(9) \quad \mathbb{V}_{\mathcal{B}} = \{v \in L^2(\Omega) : v|_{b_i} = \text{constant}, \forall b_i \in \mathcal{B}\}.$$

The set of interior sides of the mesh  $\mathcal{B}$  is denoted by  $\mathcal{E}(\mathcal{B})$ . For each  $e \in \mathcal{E}(\mathcal{B})$ , we shall fix a unit normal direction  $\mathbf{n}_e$  of  $e$ . That is  $\mathbf{n}_e$  is independent of the element containing  $e$ . Suppose  $e$  is shared by two control volumes  $b_i$  and  $b_j$ . Without loss of generality, we suppose the outward normal direction of  $e$  in  $b_i$  coincides with  $\mathbf{n}_e$ . For any function  $v \in \mathbb{V}_{\mathcal{B}}$ , the jump of  $v$  across  $e$  is denoted by  $[v] = v|_{b_i} - v|_{b_j}$ .

We define a bilinear form on  $\mathbb{V}_{\mathcal{T}}$  and  $\mathbb{V}_{\mathcal{B}}$

$$(10) \quad \bar{a}(u, v) = - \sum_{e \in \mathcal{E}(\mathcal{B})} \int_e (\mathbf{K} \nabla u) \cdot \mathbf{n}_e [v] dS, \quad \forall u \in \mathbb{V}_{\mathcal{T}}, v \in \mathbb{V}_{\mathcal{B}},$$

and formulate the linear finite volume method as: find  $\bar{u} \in \mathbb{V}_{\mathcal{T}}$  such that

$$(11) \quad \bar{a}(\bar{u}, v) = (f, v) \quad \text{for all } v \in \mathbb{V}_{\mathcal{B}}.$$

**Remark 4.1.** For Neumann boundary condition, we shall choose

$$\begin{aligned} \mathbb{V}_{\mathcal{T}} &= \{v \in H^1(\Omega) : v|_{\tau} \in \mathcal{P}_1(\tau), \forall \tau \in \mathcal{T}\}, \quad \text{and} \\ \mathbb{V}_{\mathcal{B}} &= \{v \in L^2(\Omega) : v|_{b_i} = \text{constant}, \forall b_i \in \bar{\mathcal{B}}\}. \end{aligned}$$

For  $e \in \partial b_i \cap \partial\Omega$ , the flux  $(\mathbf{K}\nabla u) \cdot \mathbf{n}_e$  will be given by the boundary condition and can be moved to the right hand side. All algorithms and analysis in this notes can be applied to Neumann boundary condition in a straightforward way.  $\square$

Since the trial space  $\mathbb{V}_{\mathcal{T}}$  and the test space  $\mathbb{V}_{\mathcal{B}}$  are different, the weak formulation (??) is known as Petrov-Galerkin method.

## 5. RELATION TO FINITE ELEMENT METHODS

We shall recall the finite element method and show the close relation between them. Let  $a(u, v)$  be the bilinear form

$$(12) \quad a(u, v) = \int_{\Omega} (\mathbf{K}\nabla u) \cdot \nabla v \, d\mathbf{x}.$$

The linear finite element method is: find  $u_L \in \mathbb{V}_{\mathcal{T}}$  such that

$$(13) \quad a(u_L, v) = (f, v) \quad \text{for all } v \in \mathbb{V}_{\mathcal{T}}.$$

For FEM, the trial space and the test space are the same, which is known as the Galerkin method.

To see the close relation, we now formulate the corresponding matrix equations for (??) and (??). Let  $\mathcal{N}(\mathcal{T})$  be the set of interior nodes of  $\mathcal{T}$  and  $N = \#\mathcal{N}(\mathcal{T})$ . Then  $\dim \mathbb{V}_{\mathcal{B}} = \dim \mathbb{V}_{\mathcal{T}} = N$ . A basis of  $\mathbb{V}_{\mathcal{B}}$  can be chosen as the characteristic function of each  $b_i, i = 1, \dots, N$ :

$$\psi_i = \chi_{b_i}(x) = \begin{cases} 1 & x \in b_i, \\ 0 & \text{otherwise.} \end{cases}$$

The nodal basis of linear finite element space  $\mathbb{V}_{\mathcal{T}}$  is the standard hat function:

$$\phi_i \in \mathbb{V}_{\mathcal{T}}, \phi_i(x_j) = \delta_{ij}, \forall x_j \in \mathcal{N}(\mathcal{T}), i = 1, \dots, N.$$

Let  $\bar{u} = \sum_{j=1}^N \bar{U}_j \phi_j$ . Choosing  $v = \psi_i, i = 1, \dots, N$  in (??), we obtain a linear algebraic equation

$$(14) \quad \bar{A}\bar{U} = \bar{F},$$

with

$$\bar{A}_{ij} = - \int_{\partial b_i} (\mathbf{K}\nabla \phi_j) \cdot \mathbf{n}, \bar{F}_i = \int_{b_i} f \, d\mathbf{x}.$$

Let  $u_L = \sum_{j=1}^N U_j \phi_j$ . Choosing  $v = \phi_i, i = 1, \dots, N$  in (??), we obtain another linear algebraic equation

$$(15) \quad AU = F,$$

with

$$A_{ij} = \int_{\Omega} (\mathbf{K}\nabla \phi_j) \cdot \nabla \phi_i, F_i = \int_{\Omega} f \phi_i \, d\mathbf{x}.$$

We shall prove that when  $\mathbf{K}(\mathbf{x})$  is piecewise constant on each triangle, then  $A = \bar{A}$ ; See [?, ?, ?]. The solution vectors are point values for  $u_L$  and  $\bar{u}$  at vertices. The only difference is the different way to compute the right hand side. For FEM,  $F_i = \int_{\omega_i} f \phi_i \, d\mathbf{x}$ , is a weighted average over the star of a vertex. For FVM,  $\bar{F}_i = \int_{b_i} f \, d\mathbf{x}$  is the average over the control volume  $b_i$ . When we choose type A control volume, i.e. choosing  $c_{\tau}$  to be the barycenter of  $\tau$ ,  $\bar{F}_i$  can be thought as an approximation of  $F_i$  using mass lumping. This modification enables the solution of linear FVM to satisfy the conservation property. It is interesting to note that on uniform grids, three methods (FDM, FEM and FVM) result the

same matrix (up to a scaling). The right hand side are chosen from very different perspective. And amazingly for all three choices of right-hand side, the resulting approximation converges to the same solution with the same order.

**Lemma 5.1.** *Given a polyhedron  $\Omega$  with  $L$ -sides in  $\mathbb{R}^n$ , let  $|F_i|$  denote the  $(n-1)$ -measure of the face  $F_i$  and  $\mathbf{n}_i$  the unit outward normal of the  $i$ -th side for  $i = 1, 2, \dots, L$ . Then*

$$(16) \quad \sum_{i=1}^L |F_i| \mathbf{n}_i = 0.$$

*Proof.* Let  $\{\mathbf{e}_k\}$  be the canonical basis of  $\mathbb{R}^n$ . Then by divergence theorem

$$\sum_{i=1}^L |F_i| \mathbf{e}_k \cdot \mathbf{n}_i = \int_{\partial\Omega} \mathbf{e}_k \cdot \mathbf{n} \, ds = \int_{\Omega} \operatorname{div} \mathbf{e}_k \, d\mathbf{x} = 0.$$

□

The following theorem is critical which says FVM and FEM have the stiffness matrix. Let us introduce an isomorphism between linear spaces  $G : \mathbb{V}_{\mathcal{B}} \rightarrow \mathbb{V}_{\mathcal{T}}$  by mapping  $\psi_i \rightarrow \phi_i, 1 \leq i \leq N$ . Then for any  $u = \sum_{i=1}^N u_i \psi_i \in \mathbb{V}_{\mathcal{B}}$ ,  $Gu = \sum_{i=1}^N u_i \phi_i \in \mathbb{V}_{\mathcal{T}}$ . Note that  $u$  and  $Gu$  share the same vector representation  $U = (u_1, \dots, u_N)^T$ . We also use a simple notation  $\bar{u}$  to denote  $Gu$ .

**Theorem 5.2.** *Assume  $\mathbf{K}(\mathbf{x})$  is piecewise constant on each  $\tau \in \mathcal{T}$  and  $\partial b_i \cap \partial\tau$  consists of middle points of edges, then*

$$a(u, v) = \bar{a}(u, \bar{v}), \quad \forall u, v \in \mathbb{V}_{\mathcal{T}}.$$

*Proof.* Since we assume  $\mathbf{K}(\mathbf{x})$  is piecewise constant, we need only show the local stiffness for Poisson equation on one triangle coincides. That reduces to prove

$$\bar{a}(\lambda_i, G\lambda_j) = a(\lambda_i, \lambda_j).$$

To be specific, let us take  $\lambda_1, \lambda_2$  as an example.

**include a figure here.**

Since  $\nabla\lambda_1$  is a constant over the triangle  $\tau$ , we can pull it out the integral and apply Lemma ?? twice to get

$$\begin{aligned} - \int_{\epsilon_1 \cup \epsilon_2} \nabla\lambda_1 \cdot \mathbf{n} \, ds &= -\nabla\lambda_1 \cdot (|e_1| \mathbf{n}_{e_1} + |e_2| \mathbf{n}_{e_2}) = \nabla\lambda_1 \cdot \frac{1}{2} (|l_1| \mathbf{n}_1 + |l_3| \mathbf{n}_3) \\ &= -\nabla\lambda_1 \cdot \frac{1}{2} |l_2| \mathbf{n}_2 = \int_{\tau} \nabla\lambda_1 \cdot \nabla\lambda_2. \end{aligned}$$

In the last step, we have used the formulae  $\nabla\lambda_i$ ; see Section 3.1.3 of *Chapter: Programming of Finite Element Methods in MATLAB*. □

## 6. ERROR ANALYSIS

The error analysis of vertex centered linear FVM (??) relies on the close relation between linear finite element method and the linear finite volume method. From previous section, linear FVM approximation  $\bar{u}$  can be thought as a perturbation of the FEM approximation  $u_L$ . First order optimal convergence rate in the energy norm can be obtained using this relation.

Note that the right hand sides may be quite different for type B dual mesh. For example, let  $f = 1$  and consider the control volume in Figure ?. Then  $F_i = |\omega_i|/3$  while  $\bar{F}_i =$

$|\omega_i|/4$ . Nevertheless optimal first order convergence in  $H^1$  norm can be still derived by comparing them in  $H^{-1}$  norm [?].

Furthermore if we choose type A dual mesh, then optimal second order convergence in  $L^2$ -norm can be also derived [?, ?, ?, ?]. Note that for general choice of control volumes, finite volume approximation may not lead to optimal  $L^2$ -norm convergent rate. See [?] for such an example on type B dual mesh. Optimal  $L^\infty$  norm estimate can be also obtained by treating it as a perturbation of finite element methods; See [?, ?]. We are not going to discuss  $L^2$  or  $L^\infty$  error estimate.

**Theorem 6.1.** *Assume  $\mathbf{K}(x)$  is piecewise constant on each  $\tau \in \mathcal{T}$ , the solution  $u$  of the diffusion equation is in the space  $H_0^1(\Omega) \cap H^2(\Omega)$  and the mesh is quasi-uniform with mesh size  $h$ , then the finite volume approximation  $u_h$  has optimal approximation order*

$$(17) \quad \|u - u_h\|_{1,\Omega} \lesssim h(\|u\|_{2,\Omega} + \|f\|).$$

*Proof.* For any  $f \in L^2(\Omega)$ , we define  $\Pi_h f \in \mathbb{V}'_{\mathcal{T}}$  as

$$\langle \Pi_h f, v_h \rangle = (f, Gv_h), \quad \text{for all } v_h \in \mathbb{V}_{\mathcal{T}}.$$

and  $Q_h f \in \mathbb{V}'_{\mathcal{T}}$  as

$$\langle Q_h f, v_h \rangle = (f, v_h), \quad \text{for all } v_h \in \mathbb{V}_{\mathcal{T}}.$$

Following the notation of Hackbusch [?], we denoted by  $u_h^G$  as the standard Galerkin (finite element) approximation and  $u_h^B$  is the box (finite volume) approximation. The equivalence of the stiffness matrices means

$$L_h u_h^G = Q_h f, \quad L_h u_h^B = \Pi_h f.$$

Therefore by the stability of  $L_h^{-1}$ , we have

$$|u_h^G - u_h^B|_1 = \sup_{v_h \in \mathbb{V}_{\mathcal{T}}} \frac{\langle Q_h f - \Pi_h f, v_h \rangle}{|v_h|_1}.$$

By the definition

$$\langle Q_h f - \Pi_h f, v_h \rangle = (f, v_h - Gv_h).$$

Denote the support the hat basis function at  $x_i$  as  $\omega_i$ . Note that  $b_i \subset \omega_i$  and the operator  $I - G$  preserve constant function in the patch  $\omega_i$  and thus

$$(f, v_h - Gv_h)_{b_i} \leq \|f\|_{b_i} \|v_h - Gv_h\|_{\omega_i} \leq Ch \|f\|_{b_i} |v_h|_{1,\omega_i}.$$

Summing up and using Cauchy Schwarz inequality, we get the first order convergence

$$|u_h^G - u_h^B|_1 \leq Ch \|f\|.$$

The estimate (??) the follows from the triangle inequality and the estimate of the finite element method.  $\square$

## REFERENCES

- [1] R. E. Bank and D. J. Rose. Some error estimates for the box method. *SIAM J. Numer. Anal.*, 24:777–787, 1987.
- [2] Z. Cai. On the finite volume element method. *Numer. Math.*, 58(1):713–735, 1990.
- [3] Z. Cai, J. Mandel, and S. F. McCormick. The finite volume element method for diffusion equations on general triangulations. *SIAM J. Numer. Anal.*, 28:392–402, 1991.
- [4] P. Chatzipantelidis. Finite volume methods for elliptic PDE's: a new approach. *M2AN Math. Model. Numer. Anal.*, 36(2):307–324, 2002.
- [5] L. Chen. A new class of high order finite volume methods for second order elliptic equations. *SIAM J. Numer. Anal.*, 47(6):4021–4043, 2010.
- [6] S.-H. Chou and Q. Li. Error estimates in  $L^2$ ,  $H^1$  and  $L^\infty$  in covolume methods for elliptic and parabolic problems: a unified approach. *Math. Comp.*, 69(229):103–120, 2000.

- [7] R. E. Ewing, T. Lin, and Y. Lin. On the accuracy of the finite volume element method based on piecewise linear polynomials. *SIAM J. Numer. Anal.*, 39(6):1865–1888, 2002.
- [8] R. Eymard, T. Gallouët, and R. Herbin. Finite volume methods. In *Handbook of numerical analysis, Vol. VII*, Handb. Numer. Anal., VII, pages 713–1020. North-Holland, Amsterdam, 2000.
- [9] W. Hackbusch. On first and second order box schemes. *Computing*, 41(4):277–296, 1989.
- [10] J. Huang and S. Xi. On the finite volume element method for general self-adjoint elliptic problems. *SIAM J. Numer. Anal.*, 35(5):1762–1774, 1998.
- [11] R. Li, Z. Chen, and W. Wu. *Generalized difference methods for differential equations*, volume 226 of *Mono-graphs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 2000. Numerical analysis of finite volume methods.
- [12] R. H. Li. On the generalized difference method for elliptic and parabolic differential equations. In *Proc. of the Symposium on the Finite Element Method between China and France, Beijing, China*, 1982.
- [13] T. Russell and M. Wheeler. Finite element and finite difference method for continuous flows in porous media. *Frontiers in Applied Mathematics*, 1:35, 1983.
- [14] E. Süli. Convergence of finite volume schemes for Poisson’s equation on nonuniform meshes. *SIAM J. Numer. Anal.*, 28(5):1419–1430, 1991.
- [15] J. Xu and Q. Zou. Analysis of linear and quadratic simplicial finite volume methods for elliptic equations. *Numer. Math.*, 111(3):469–492, 2009.