CONVERGENCE THEORIES OF MULTIGRID METHODS BASED ON THE X-Z IDENTITY

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CONTENTS

Multigrid methods for solving the linear algebraic equation $Au = f$ posed on a finite dimensional Hilbert space $\mathbb{V}$ can be understood as successive subspace correction (SSC)

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method applied to nested multilevel spaces decomposition $\mathbb{V} = \sum_{i=0}^{J} \mathbb{V}_i$ with nested subspaces

$$\mathbb{V}_0 \subset \mathbb{V}_1 \ldots \subset \mathbb{V}_J = \mathbb{V}.$$

In this Chapter, we provide convergence proofs of V-cycle multigrid methods using X-Z identity [9] for SSC method. We would expect a rate independent or weakly dependent of the number of unknowns. All constants included in assumptions are presumably such constants.

Our proofs are based on various assumptions on the smoothers and space decompositions. These assumptions can be also used as a guideline to design robust multigrid methods. Generally speaking smoothers satisfies certain smoothing property and the space admits an A-stable decomposition. The smoothing property can be further build into an A-stable micro-decomposition of spaces in each level. Therefore a *stable space decomposition* is the key of a robust multigrid method interpreting as a subspace correction method.

1. X-Z IDENTITY

The problem to solve is $Au = f$ on a finite dimensional Hilbert space $\mathbb{V}$ with A being a symmetric positive definite (SPD) operator with respect to the inner product $(\cdot, \cdot)$. The SPD operator A defines a bilinear form $A(u, v) := (Au, v)$ which introduces also a new inner product $(u, v)_A = (Au, v)$.

1.1. **Notation.** Let $\mathbb{V}_i \subset \mathbb{V}$, $i = 0, \ldots, J$, be subspaces of $\mathbb{V}$. If $\mathbb{V} = \sum_{i=0}^{J} \mathbb{V}_i$, then $\{\mathbb{V}_i\}$ is called a *space decomposition* of $\mathbb{V}$. The spaces $\{\mathbb{V}_i\}$ are not necessarily nested.

We introduce the following operators for $i = 0, 1, \ldots, J$:

- $I_i : \mathbb{V}_i \hookrightarrow \mathbb{V}$ the natural inclusion;
- $Q_i : \mathbb{V} \mapsto \mathbb{V}_i$ the projection in the inner product $(\cdot, \cdot)$;
- $P_i : \mathbb{V} \mapsto \mathbb{V}_i$ the projection in the inner product $(\cdot, \cdot)_A$;
- $A_i : \mathbb{V}_i \mapsto \mathbb{V}_i$ $(A_i u_i, v_i) = (A u_i, v_i)$ corresponds to the restriction of bilinear form A on the subspace $\mathbb{V}_i \times \mathbb{V}_i$;
- $R_i : \mathbb{V}_i \mapsto \mathbb{V}_i$ an approximation of A_i^{-1} which is often called smoothers or local subspace solvers.
- $T_i : \mathbb{V} \to \mathbb{V}_i$ $T_i = R_i Q_i A = R_i A_i P_i$.

We recall some relations between these operators. By definition Q_i^t coincides with the natural inclusion I_i which is sometimes are omitted. The inclusion I_i is often called prolongation operator and $I_i^t = Q_i$ is the restriction operator. It follows from the definition that $A_i P_i = Q_i A$ and $A_i = I_i^t A I_i$.

All smoothers R_i are assumed to be non-singular but could be non-symmetric. For each $R_i, i = 0, \cdots, J$, its symmetrization $\overline{R}_i$ is an operator satisfying $I_i - \overline{R}_i A_i = (I_i - R_i^t A_i)(I_i - R_i A_i)$. That is $\overline{R}_i$ is a symmetric smoother by applying smoother R_i and R_i^t consecutively. By definition,

$$\overline{R}_i = R_i^t (R_i^{-t} + R_i^{-1} - A_i) R_i.$$

When $R_i = A_i^{-1}$, from the definition, $T_i = P_i = A_i^{-1} Q_i A$. Restricted to the subspace $\mathbb{V}_i$, the projection P_i is identity and thus $T_i|_{\mathbb{V}_i} = R_i A_i : \mathbb{V}_i \to \mathbb{V}_i$ is nonsingular. With a slight abuse of notation, we still use T_i to denote $T_i|_{\mathbb{V}_i}$ and $T_i^{-1} = (T_i|_{\mathbb{V}_i})^{-1}$. Define the symmetrization $\overline{T}_i = T_i + T_i^* - T_i T_i^* = \overline{R}_i A_i P_i$ where the adjoint * is taken with respect to the $(\cdot, \cdot)_A$ inner product. Then

$$I - \overline{T}_i = (I - T_i^*)(I - T_i). \tag{1}$$

The action of T_i and T_i^{-1} is

$$(T_i u, v)_A = (R_i A_i u, A_i v), \quad (T_i^{-1} u, v)_A = (R_i^{-1} u, v) \quad \text{for } u, v \in \mathbb{V}_i.$$

Similar relation between $\overline{T}_i$ and $\overline{R}_i$ holds. It is much easier to manipulate one single letter T_i than $R_i Q_i A = R_i A_i P_i$.

For $k = J, \cdots, 1$, in each space $\mathbb{V}_k$, we apply an effective smoother R_k, which is called pre-smoothing, to damp the high frequency relative to that level. In the coarsest space $\mathbb{V}_0$, we will use the exact solver. One post-smoothing using R_k^t from $k = 1, \cdots, J$ is supplemented to form a $V(1,1)$-cycle. In general m_1-step pre-smoothing and m_2-post-smoothing results a $V(m_1, m_2)$-cycle.

In operator form, one V-cycle iteration can be written as

$$u^{k+1} = u^k + \overline{B}(f - Au^k).$$

Let $E = (I - T_0)(I - T_1)...(I - T_J)$. The error operator of $V(1,1)$-cycle is $\overline{E} := E^* E = I - \overline{B}A$, i.e.,

$$u - u^{k+1} = \overline{E}(u - u^k) = (I - \overline{B}A)(u - u^k).$$

We want to prove the contraction

$$\|\overline{E}\|_A \leq \delta, \text{ for some } \delta \in [0, 1).$$

Ideally δ is independent of N the dimension of the space $\mathbb{V}$ and a weak dependence of $\log N$ is acceptable.

The main tool is the X-Z identity [9] for the multiplicative methods. For an elementary proof, we refer to Chen [3] or *Chapter: Subspace Correction Methods and Auxiliary Space Methods*. We will collect several versions of the X-Z identity below.

1.2. **XZ identities.** We assume each local solver is convergent restricted to the subspace.

(C) Each local solver is a contraction: $\|I_i - R_i A_i\|_{A_i} < 1$ for $i = 0, \cdots, J$

Exercise 1.1. Prove that if (C) holds, then $\|I - T_i\|_A \leq 1$ and consequently $\|\overline{E}\|_A \leq 1$ is non-expansive.

We first present identities of the V-cycle operator.

Theorem 1.2. *Suppose (C) holds. Then $\overline{B}$ is SPD, and*

$$(\overline{B}^{-1} v, v) = \inf_{\sum_{i=0}^J v_i = v} \sum_{i=0}^J \|v_i + R_i^t A_i P_i \sum_{j>i}^J v_j\|_{\overline{R}_i^{-1}}^2. \tag{2}$$

$$(\overline{B}^{-1} v, v) = \|v\|_A^2 + \inf_{\sum_{i=0}^J v_i = v} \sum_{i=0}^J \|R_i^t (A_i P_i \sum_{j=i}^J v_j - R_i^{-1} v_i)\|_{\overline{R}_i^{-1}}^2. \tag{3}$$

In particular, for $R_i = A_i^{-1}$, we have

$$(\overline{B}^{-1} v, v) = \|v\|_A^2 + \inf_{\sum_{i=0}^J v_i = v} \sum_{i=0}^J \|P_i \sum_{j=i+1}^J v_j\|_A^2. \tag{4}$$

Based on these identities, we have different versions of the X-Z identity of the error operator.

Theorem 1.3 (X-Z identity)**.** *Suppose the local iterative method satisfies* (C). *Then*

$$\|E^*E\|_A = 1 - \frac{1}{K}, \tag{5}$$

where

$$K = \sup_{\|v\|_A=1} \inf_{\sum_{i=0}^J v_i = v} \sum_{i=0}^J \|v_i + R_i^t A_i P_i \sum_{j>i}^J v_j\|_{\overline{R}_i^{-1}}^2.$$

Or

$$\|E^*E\|_A = 1 - \frac{1}{1+c_0}, \tag{6}$$

where

$$c_0 = \sup_{\|v\|_A=1} \inf_{\sum_{i=0}^J v_i = v} \sum_{i=0}^J \|R_i^t (A_i P_i \sum_{j=i}^J v_j - R_i^{-1} v_i)\|_{\overline{R}_i^{-1}}^2.$$

In particular, for $R_i = A_i^{-1}$,

$$\|E^*E\|_A = 1 - \frac{1}{1+c_0}, \tag{7}$$

where

$$c_0 = \sup_{\|v\|_A=1} \inf_{\sum_{i=0}^J v_i = v} \sum_{i=0}^J \|P_i \sum_{j=i+1}^J v_j\|_A^2.$$

Note that these identities hold for a general space decomposition, i.e., subspaces $\mathbb{V}_i, i = 0, \ldots, J$ are not necessarily nested. Estimate of constants K or c_0 will be obtained by various decomposition of an element $v \in \mathbb{V}$, and when $\{\mathbb{V}_i\}$ are nested, the decomposition can be constructed using slice operators.

1.3. **Some estimates.** We collect some useful estimates involving operators T and $\overline{T}$ in this subsection. Since the results are applied to a fixed level, the subscript of levels is skipped. We shall use the notation of comparing symmetric operators: for two SPD operators A and B, we write $A \le B$ if $(Av, v) \le (Bv, v)$ for all $v \in \mathbb{V}$.

First of all from the identity $I - \overline{T} = (I - T)^*(I - T)$, we conclude the maximal eigenvalue of $\overline{T}$ is bounded by one.

Lemma 1.4.

$$\lambda_{\max}(\overline{T}) \le 1.$$

Proof. We have $I - \overline{T} = (I - T)^*(I - T) \ge 0$, i.e., $\overline{T} \le I$ which implies $\lambda_{\max}(\overline{T}) \le 1$. □

We then give an identity to connect $\overline{T}$ and T.

Lemma 1.5. *For any* $u \in \mathbb{V}$,

$$(\overline{T}u, u)_A = \|u\|_A^2 - \|(I - T)u\|_A^2 = 2(Tu, u)_A - \|Tu\|_A^2. \tag{8}$$

Proof. It is an easy consequence of the identity $I - \overline{T} = (I - T)^*(I - T)$. □

We estimate the norm of the iterative matrix $I - T$ based on the identity (8).

Theorem 1.6.

$$\|I-T\|_A^2 = \|I-\overline{T}\|_A = 1-\lambda_{\min}(\overline{T}). \tag{9}$$

Consequently if

$$(\overline{T}^{-1}u,u)_A \le K(u,u)_A, \quad \textit{for all } u\in\mathbb{V}. \tag{10}$$

then

$$\|I-T\|_A^2 \le 1-\frac{1}{K}.$$

Proof. Rearrange the identity (8) as $\|(I-T)u\|_A^2 = \|u\|_A^2 - (\overline{T}u,u)_A$ and the desired result (9) follows from the definition of the norm and eigenvalue.

The inequality (10) implies $\lambda_{\max}(\overline{T}^{-1}) \le K$ which is equivalent to $\lambda_{\min}(\overline{T}) \ge 1/K$ and the estimate (10) then follows from (9). □

We now formulate different criterion for the convergence of the operator $I-T$.

(Tw) There exists a constant $\omega\in(0,2)$ such that

$$\|Tu\|_A^2 \le \omega(Tu,u)_A, \quad \text{for all } u\in\mathbb{V}.$$

$(\bar{T}w)$ There exists a constant $\omega\in(0,2)$ such that

$$\|Tu\|_A^2 \le \frac{\omega}{2-\omega}(\overline{T}u,u)_A, \quad \text{for all } u\in\mathbb{V}.$$

(σ) There exists a constant $\sigma>0$ such that

$$\lambda_{\max}(T\overline{T}^{-1}T^*) = \lambda_{\max}((R^{-t}+R^{-1}-A)^{-1}A) \le \sigma.$$

Exercise 1.7. Prove the assumptions (Tw) and $(\bar{T}w)$ are equivalent using the identity (8).

We now prove the assumptions $(\bar{T}w)$ and (σ) are equivalent. Using $\rho(AB)=\rho(BA)$, (σ) is equivalent to $\overline{T}^{-1}T^*T \le \sigma$ which is $T^*T \le \sigma\overline{T}$, i.e., $(\bar{T}w)$ with $\sigma = \omega/(2-\omega)$.

Theorem 1.8. *The contraction assumption* (C)*, i.e.,* $\|I-T\|_A < 1$ *is equivalent to the condition* (Tw)*.*

Proof. First by the identity (9), $\|I-T\|_A<1$ is equivalent to $\lambda_{\min}(\overline{T})>0$. Then by the identity (8), $(\overline{T}u,u)_A = 2(Tu,u)_A - \|Tu\|_A^2$, we get the equivalence of $\lambda_{\min}(\overline{T})>0$ and $2(Tu,u)_A > \|Tu\|_A^2$ which is equivalent to (Tw). □

So in later sections, we will use either (C), (Tw), $(\bar{T}w)$, or (σ).

Remark 1.9. By a simple change of variable $v=Tu$, the condition (Tw) is equivalent to: there exists a number $\omega\in(0,2)$ such that

$$(Au,u) \le \omega(R^{-1}u,u), \quad \text{for all } u\in\mathbb{V}. \tag{11}$$

When the smoother R is symmetric, T is symmetric in the A-inner product. The relation $I - \overline{T} = (I-T)^*(I-T)$ is reduced to

$$\overline{T} = 2T - T^2. \tag{12}$$

We could have more estimate on the eigenvalue of T and $\overline{T}$.

Theorem 1.10. *If R is symmetric and (Tw) holds. Then*

(1) $0 < \lambda_{\min}(T) \leq \lambda_{\max}(T) \leq \omega < 2$.
(2) $\lambda_{\min}(\overline{T}) = \min\{\lambda_{\min}(T)(2-\lambda_{\min}(T)), \omega(2-\omega)\}$.
(3) $(\overline{T}u, u)_A \geq (2-\omega)(Tu,u)_A \quad$ *for all* $u \in \mathbb{V}$.
(4) $(\overline{T}^{-1}u, u)_A \leq (2-\omega)^{-1}(T^{-1}u,u)_A \quad$ *for all* $u \in \mathbb{V}$

Proof. When T is symmetric, we use the equivalent condition (11) to derive the bound of $T = RA$ in (1). The identity (2) is from the relation (12). The inequality (3) follows by using (Tw) to replace $\|Tu\|_A^2$ by $(Tu,u)_A$ in the identity (8). The inequality (4) can be proved by showing

$$\overline{T}^{-1} - (2-\omega)^{-1}T^{-1} \leq 0$$

using the bound of the spectrum of T. □

When T is symmetric in the A-inner product, using the notation of comparing symmetric operators (now in $(\cdot,\cdot)_A$), we can write (3) as $\overline{T} \geq (2-\omega)T$ and (4) as $\overline{T}^{-1} \leq (2-\omega)^{-1}T^{-1}$ which is formally obtained by taking inverse of (3).

The constant ω will enter the estimate of the contraction rate through (4). Thus for a symmetric smoother, we will use the equivalent condition (Tw) instead of (C) and assume $\omega = \lambda_{\max}(T)$ is well below 2.

2. Orthogonal Telescope Decomposition

We assume the subspaces $\mathbb{V}_i$ are nested, i.e., $\mathbb{V}_0 \subset \mathbb{V}_1 \ldots \subset \mathbb{V}_J = \mathbb{V}$. Recall $P_k : \mathbb{V} \to \mathbb{V}_k$ is the orthogonal projection in the $(\cdot,\cdot)_A$ inner product. Define $P_{-1} = 0$. In this section, we will choose the orthogonal telescope decomposition $u_k = (P_k - P_{k-1})u, k = 0, ..., J$, in the X-Z identity. The analysis is simplified tremendously due to the orthogonality of operators in the A-inner product.

2.1. Properties and Assumptions.

Since the spaces are nested, we have the following properties of the projections

(P1) $P_k P_l = P_k$ for $l \geq k$;
(P2) $P_k(P_l - P_{l-1}) = 0$ for $l > k$;
(P3) $(P_k - P_{k-1})(P_l - P_{l-1}) = 0$ for $l > k$ and $(P_k - P_{k-1})^2 = P_k - P_{k-1}$;
(P4) $P_k = A_k^{-1}Q_k A$ is symmetric in the A-inner product.

As a consequence of the properties (P3) and (P4), we have

$$\sum_{k=0}^{J} \|u_k\|_A^2 = \|u\|_A^2. \tag{13}$$

As a consequence of the property (P2), the first version of X-Z identity (5) can be simplified to

$$\sum_{k=0}^{J} \left\| u_k + R_k^t A_k P_k \left(\sum_{l>k} u_l \right) \right\|_{\overline{R}_k^{-1}}^2 = \sum_{k=0}^{J} \|u_k\|_{\overline{R}_k^{-1}}^2. \tag{14}$$

Comparing (13) and (14), it is natural to make the following smoothing assumption on the smoother.

($\bar{S}_P$) The smoother $\overline{R}_k$ will smooth the frequency $(P_k - P_{k-1})\mathbb{V}$, i.e., there exists $c_R \geq 0$

(15) $$(\overline{R}_k^{-1} u_k, u_k) \leq c_R(Au_k, u_k), \quad \text{for all } u_k \in (P_k - P_{k-1})\mathbb{V},\ k = 0, \cdots, J.$$

Exercise 2.1. Prove if ($\bar{S}_P$) holds, then the constant $c_R \geq 1$. *Hint: use* $\lambda_{\max}(\overline{R}A) \leq 1$.

Consider the Richardson smoother $R_k = \omega I_k$. Then the contraction rate $\|I_k - R_k A_k\|_{A_k}$ could be very close to 1, say $1 - Ch_k^2$. But if the smoother $R_k = \omega I_k$ satisfies ($\bar{S}_P$), one can show

$$\|(I_k - R_k A_k)|_{\mathbb{W}_k}\|_A^2 = \|(I_k - \overline{R}_k A_k)|_{\mathbb{W}_k}\|_A \leq 1 - \frac{1}{c_R}.$$

For this example, ($\bar{S}_P$) implies that restricted to the subspace $\mathbb{W}_k = (P_k - P_{k-1})\mathbb{V}$, the contraction rate is well below 1.

One may wonder why the rate of convergence on the whole space $\mathbb{V}_k$ is worse than on a subspace. If we explicitly apply the iteration to the subspace $\mathbb{W}_k$, then from the minimization of energy point of view, of course, the reduction of the energy on $\mathbb{V}_k$ is not worse than $1 - 1/c_R$. The point is: in the smoothing, we *do not* form the space $\mathbb{W}_k$ explicitly. If we knew a bases of $(P_k - P_{k-1})\mathbb{V}$, we would have rewritten the operator A using the orthogonal decomposition

$$\mathbb{V} = \bigoplus_{k=0}^{J} (P_k - P_{k-1})\mathbb{V}.$$

The corresponding matrix will be a block diagonal matrix and inverting this block diagonal matrix is relatively easy. The best example is the Fourier bases. Then for Laplacian operator, the corresponding matrix is diagonal. Such nice bases in general, (for example for variable coefficients, complex domains, and unstructured triangulations), is difficult, if not impossible, to construct. In multigrid methods, however, we do not form the decomposition of frequency but just relax on a larger set of basis. The redundancy really helps.

Exercise 2.2. For linear finite element discretization of 1-D Poisson equation, show that the hierarchical basis (HB) is A-orthogonal and the corresponding matrix is diagonal.

When the smoother is symmetric, we can use the smoothing property of R_k, which is easier to verify for symmetric smoothers, instead that of $\overline{R}_k$.

(S_P) The smoother R_k will smooth the frequency $(P_k - P_{k-1})\mathbb{V}$, i.e., there exists $c_R > 0$

(16) $$(R_k^{-1} u_k, u_k) \leq c_R(Au_k, u_k), \quad \text{for all } u_k \in (P_k - P_{k-1})\mathbb{V}, k = 0, \cdots, J.$$

Using the inequality (4) in Theorem 1.10, we can get ($\bar{S}_P$) from (S_P).

Lemma 2.3. *For a symmetric smoother R, if (C) holds, then*

(17) $$(\overline{R}^{-1} u, u) \leq \frac{1}{2 - \omega}(R^{-1} u, u), \quad \textit{with } \omega = \lambda_{\max}(RA).$$

As a direct consequence of Lemma 2.3, we have the following lemma which says: $(Tw) + (S_P) \Rightarrow (\bar{S}_P)$.

Lemma 2.4. *For a symmetric smoother R, if it satisfies (Tw) and (S_P) with constant c_R, then it satisfies $(\bar{S}_P)$ with constant $c_R/(2-\omega)$.*

Example 2.5. Consider Richardson smoother $R_i = \lambda_{\max}^{-1}(A_i) I_i$. Then $\omega = \lambda_{\max}(R_i A_i) = 1$ which also implies the convergence of the smoother. To simplify notation, we write $\mathbb{V}$ and $\mathbb{V}_H$ for two consecutive subspaces $\mathbb{V}_k$ and $\mathbb{V}_{k-1}$ and $I - P_H$ for $P_k - P_{k-1}$. Then (S_P) becomes

$$\|(I-P_H)u\|^2 \leq \frac{c_R}{\lambda_{\max}(A)} \|(I-P_H)u\|_A^2. \tag{18}$$

In the setting of FEM for elliptic equations, (18) can be proved by the duality argument using H^2-regularity assumption.

When (18) fails to hold, which means the point-wise smoother fails, it is still possible to design block-wise smoothers to satisfy (S_P) or $(\bar{S}_P)$ by choosing a stable decomposition of the frequency $(I-P_H)\mathbb{V}$; see e.g. [1]. We shall discuss this more in Section 5.

2.2. **Convergence.** We summarize our first convergence proof of V-cycle multigrid as follows.

Theorem 2.6. *Suppose the smoother R_k satisfies assumptions (C) and $(\bar{S}_P)$. Then the corresponding $V(1,1)$-cycle is uniform convergent with a rate $1 - 1/c_R$, i.e.*

$$\|E^* E\|_A \leq 1 - \frac{1}{c_R}.$$

Proof. The proof is straightforward using assumptions and the X-Z identity (5):

$$\sum_{k=0}^{J} \|u_k + R_k^t A_k P_k (\sum_{l>k} u_l)\|_{\overline{R}_k^{-1}}^2 = \sum_{k=0}^{J} \|u_k\|_{\overline{R}_k^{-1}}^2 \leq c_R \sum_{i=0}^{J} \|v_i\|_A^2 = c_R \|v\|_A^2.$$

□

Theorem 2.7. *Suppose the symmetric smoother R_k satisfies assumptions (Tw) and (S_P). Then the corresponding $V(1,1)$-cycle is uniform convergent with a rate $1 - (2-\omega)/c_R$*

$$\|E^* E\|_A \leq 1 - \frac{2-\omega}{c_R}.$$

The assumption (S_P) is easier to verify than $(\bar{S}_P)$ for symmetric smoothers such as Richardson or weighted Jacobi smoothers. But one needs to check one more constant ω is well below 2.

3. Approximation and Smoothing Property

We shall break the assumption $(\bar{S}_P)$ into two assumptions which are easier to verify. Recall that the two ingredients of multigrid methods are

(1) use a smoother to damp the high frequency;
(2) use the coarse grid correction to approximate the low frequency.

The smoother has the smoothing property which can damp the high frequency effectively. The approximation property ensures the subspaces $(P_k - P_{k-1})\mathbb{V}$ consists of high frequency functions. The analysis based on the approximation and smoothing property is firstly developed by Hackbusch [7]. The presentation here is simplified using X-Z identity.

The advantage of this approach is the relatively simple proof and stronger results involving smoothing steps; see Section 7. The limitation of this approach is:

- the smoother is essentially point-wise.
- duality argument involving full regularity assumption is needed to estimate L^2-type norm of the A-projection.

Therefore this approach is not easy to handle elliptic equations with singularity, say, discontinuous diffusion coefficients or concave domains, and elliptic operators with nontrivial kernel for which point-wise smoothers are not adequate.

3.1. **High frequency.** We begin with the definition of high frequency. Let $\rho(A) = \lambda_{\max}(A)$ be the spectral radius of A and define a scaled norm $\|u\|_\rho = \sqrt{\rho(A)(u,u)}$. For every $u \in \mathbb{V}$, by definition,

$$\|u\|_A^2 = (Au, u) \leq \rho(A)(u,u) = \|u\|_\rho^2. \tag{19}$$

In FEM setting, (19) is known as the inverse inequality $|v|_1 \lesssim h^{-1}\|v\|$ with $\rho(A) = Ch^{-2}$.

An element $v \in \mathbb{V}$ is called *high frequency* if

$$\|u\|_\rho^2 \leq C\|u\|_A^2. \tag{20}$$

Consider the decomposition of u using the eigen-vector bases of A. The constant C in (20) is introduced to include not only the highest frequency but a range of frequencies comparable to the highest one. In FEM setting, (20) reads as $h^{-1}\|v\| \lesssim |v|_1$, i.e., the function oscillates with frequency h. In other words, for high frequency functions, the inverse of the inverse inequality holds.

3.2. **Approximation property.**

(A_P) Approximation property of P_k. There exists a constant c_a such that:

$$\|(I - P_{k-1})u_k\|^2 \leq \frac{c_a}{\rho(A_k)}\|u_k\|_A^2 \quad \text{for all } u_k \in \mathbb{V}_k, k = 1, \dots, J. \tag{21}$$

The approximation property (A_P) is equivalent to the difference $(I - P_H)u$ is high frequency, i.e.

$$\rho(A)\|(I - P_H)u\|^2 \leq c_a\|(I - P_H)u\|_A^2, \tag{22}$$

The equivalence can be easily verified by noting that $(I-P_H)u = (I-P_H)^2u$. Interesting enough it is also equivalent to Richardson smoother satisfies (S_P); see Example 2.5.

Example 3.1. In the example of FEM for elliptic equations, (A_P) can be proved by the duality argument as

$$\|(I - P_H)u\| \lesssim h\|(I - P_H)u\|_A \lesssim h\|u\|_A.$$

3.3. **Smoothing property.** We formulate the smoothing property using the spectral norm.

$(\bar{S}_\rho)$ Smoothing property of high frequency. There exists a constant c_s such that

$$(\overline{R}_k^{-1}u_k, u_k) \leq c_s\rho(A_k)(u_k, u_k) \quad \text{for all } u_k \in \mathbb{V}_k, k = 0, \dots, J. \tag{23}$$

Using the notation of comparing SPDE operators, we can write $(\bar{S}_\rho)$ as $\overline{R}^{-1} \leq c_s\rho(A)I$ or $I/\rho(A) \leq c_s\overline{R}$. That is $\overline{R}^{-1}$ is dominated by a Richardson smoother. The smoother is essentially a point-wise smoother.

Recall (A_P) is equivalent to a Richardson smoother satisfies $(\bar{S}_P)$. Combining them, we have the following result.

Lemma 3.2. *The assumption* (A_P) *and* $(\bar{S}_\rho)$ *implies* $(\bar{S}_P)$ *with* $c_R = c_a c_s$.

Proof. For $u_k \in (P_k - P_{k-1})\mathbb{V}$, by the assumptions $(\bar{S}_\rho)$ and (A_P)

$$(\overline{R}_k^{-1}u_k, u_k) \leq c_s\rho(A_k)(u_k, u_k) \leq c_a c_s(Au_k, u_k).$$

In the last step, we use (P2) to write $u_k = (P_k - P_{k-1})u = (I - P_{k-1})(P_k - P_{k-1})u = (I - P_{k-1})u_k$. □

When (A_P) fails, $(\bar{S}_\rho)$ alone cannot imply $(\bar{S}_P)$ which simply means the point-wise smoother is not a good one. For example, standard G-S iteration is not a good smoother for $H(\mathrm{curl}\,)$ and $H(\mathrm{div})$ problems although G-S always satisfies $(\bar{S}_\rho)$. A good smoother satisfying $(\bar{S}_P)$ can be obtained by using block G-S smoothers.

For symmetric smoothers, as before, we can modify the smoothing property to

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(S_ρ) Smoothing property of high frequency for symmetric smoothers. There exists a constant c_s such that

$$(R_k^{-1}u_k, u_k) \leq c_s\rho(A_k)(u_k, u_k) \quad \text{for all } u_k \in \mathbb{V}_k, k = 0, \ldots, J. \tag{24}$$

Similarly (S_ρ) will imply $(\bar{S}_\rho)$ with constant $c_s/(2-\omega)$, $\omega = \lambda_{\max}(RA)$. The constant ω can be estimated by $R \leq \omega A^{-1}$ or $\omega R^{-1} \geq A$; see (11).

3.4. **Smoothing property of popular smoothers.** We now verify the smoothing property $(\bar{S}_\rho)$ of popular smoothers, including Gauss-Seidel, Richardson, and weighted Jacobi iterations. To simplify notation, we skip the subscript associated to levels.

Consider the standard bases decomposition $\mathbb{V} = \sum_{i=0}^N \mathbb{V}_i$. We assume this decomposition is stable under the norm introduced by the default inner product $(\cdot,\cdot)$. Namely for $u = \sum_j u_j$

$$c_M \sum_{i=0}^N \|u_j\|^2 \leq \|u\|^2 \leq C_M \sum_{i=0}^N \|u_j\|^2. \tag{25}$$

In FEM setting, (25) means the bases decomposition is stable in the L^2 norm which can be easily proved by the element-wise scaling argument.

One way to verify the smoothing property for multiplicative smoothers is using the formula of $\overline{R}^{-1}$ and X-Z identity again. For symmetric smoothers, alternative way is to estimate both $\lambda_{\min}(RA)$ and $\lambda_{\max}(RA)$.

Smoothing property of Gauss-Seidel iteration. Recall that Gauss-Seidel iteration can be understood as SSC apply to the basis decomposition $\mathbb{V} = \sum_{i=1}^N \mathbb{V}_i$ with $R_i = A_i^{-1}$. We then use the identity (4) to estimate

$$(\overline{B}_{\mathrm{GS}}^{-1}u, u) = \sum_{i=0}^N \|P_i \sum_{j>i} u_j\|_A^2 \leq \sum_{i=0}^N \sum_{j\in n(i)} \|u_j\|_A^2 \leq C_d\rho(A)\sum_{i=0}^N \|u_j\|^2 \leq c_s\rho(A)\|u\|^2.$$

Here we use the sparsity of A such that the repetition in the summation is bounded above by C_d, the number of neighbors of a basis which is the degree of a vertex in the associated graph of A. The final constant $c_s = C_d c_M^{-1}$.

Therefore G-S always satisfies the smoothing property $(\bar{S}_\rho)$.

Smoothing property of Richardson iteration. We choose $R = \omega\rho^{-1}(A)I$. To be a contraction, the constant $\omega \in (0,2)$. Then

$$\overline{R}^{-1} = R^{-1}(R^{-1} + R^{-1} - A)^{-1}R^{-1} = \rho(A)^2(2\omega\rho(A)I - \omega^2 A)^{-1}.$$

Therefore $(\bar{S}_\rho)$ holds, i.e.,

$$\rho(\overline{R}^{-1}) \leq \frac{\rho(A)}{\omega(2-\omega)}. \tag{26}$$

The optimal parameter to minimize the constant $1/(\omega(2-\omega))$ is $\omega = 1$. That is Richardson iteration $R = \rho^{-1}(A)$ is a good smother.

Richardson iteration needs an estimate of $\rho(A)$ while G-S is parameter free. One tool for estimating eigenvalues is the Gershgorin circle theorem.

Smoothing property for weighted Jacobi iteration. Jacobi iteration itself may not have the smoothing property. For example, for 1-D discretization Poisson on uniform grids, the Jacobi method is the Richardson method with $\omega = 2$ and thus no smoothing property in view of (26).

Consider the weighted Jacobi smoother $R = \tau D^{-1}$. Then the smoothing property is easy to show

$$(R^{-1}u, u) = \tau^{-1}\sum_{i=1}^{N}\|u_i\|_A^2 \leq \tau^{-1}\rho(A)\sum_{i=1}^{N}\|u_i\|^2 \leq \tau^{-1}c_M^{-1}\rho(A)\|u\|^2.$$

However, the final smoothing effect $(\bar{S}_\rho)$ will be weighted by $1/(2-\omega)$ with

$$\omega = \lambda_{\max}(RA) = \tau\lambda_{\max}(D^{-1}A) = \tau\lambda_{\max}(A_D),$$

where $A_D = D^{-1/2}AD^{-1/2}$. Therefore

$$(\overline{R}^{-1}u, u) \leq (2 - \tau\lambda_{\max}(A_D))^{-1}\tau^{-1}c_M^{-1}\rho(A)\|u\|^2.$$

To maximize the constant $(2 - \tau\lambda_{\max}(A_D))\tau$, we can chose $\tau = 1/\lambda_{\max}(A_D)$.

For the scaled SPD matrix A_D, the diagonal is always 1. When the matrix A is diagonal dominate, by the Gershgorin circle theorem, $\lambda_{\max}(A_D) \leq 2$. Therefore in practice, $\tau = 0.5$ is a recommend for weighted Jacobi iteration when used as a smoother.

Exercise 3.3. Prove the smoothing property $(\bar{S}_\rho)$ for SOR iteration $R = \omega(D + \omega L)^{-1}$ with a suitable parameter ω.

3.5. **Convergence.** We state the convergence of V-cycle using the approximation and smoothing property.

Theorem 3.4. *Suppose the nested space decomposition satisfies (A_P) and the smoother R_k satisfies assumptions (C) and $(\bar{S}_\rho)$. Then the corresponding $V(1,1)$-cycle converges with rate $1 - 1/(c_a c_s)$.*

Theorem 3.5. *Suppose the nested space decomposition satisfies (A_P) and the symmetric smoother R_k satisfies assumptions (Tw) and (S_ρ). Then the corresponding $V(1,1)$-cycle converges with rate $1 - (2-\omega)/(c_a c_s)$.*

4. Stable Decomposition and Quasi-Orthogonality

Verification of $(\bar{S}_P)$ or (A_P) requires the duality argument which in turn needs the regularity result of elliptic problems. This is a serve restriction in application. In this section we switch from the subspace $(P_k - P_{k-1})\mathbb{V}$ to a more general subspace $\mathbb{W}_k \subset \mathbb{V}_k$ and assume $\mathbb{V} = \sum_{k=0}^J \mathbb{W}_k$. This decomposition should be stable and quasi-orthogonal. We emphasize again that $\mathbb{W}_k$ is introduced for the analysis and do not need to be explicitly formed in the algorithm.

This framework is introduced by Xu [8] and the proof is simplified using X-Z identity.

4.1. Assumptions.

We need the following assumptions on the space decomposition.

(D_W): The space decomposition $\mathbb{V} = \sum_{k=0}^J \mathbb{W}_k$ is A-stable:

$$\inf_{\sum_{i=0}^J v_i = v, v_i \in \mathbb{W}_i} \sum_{i=0}^J \|v_i\|_A^2 \le K_1 \|v\|_A^2. \tag{27}$$

(O_W): Quasi-Orthogonality. The following Strengthened Cauchy Schwarz (SCS) inequality holds for any $u_i \in \mathbb{V}_i, v_i \in \mathbb{W}_i, i = 0, \cdots, J$

$$\sum_{i=0}^J \sum_{j=i+1}^J (u_i, v_j)_A \le K_2 \left(\sum_{i=0}^J \|u_i\|_A^2\right)^{1/2} \left(\sum_{i=0}^J \|v_i\|_A^2\right)^{1/2}. \tag{28}$$

If we chose $\mathbb{W}_i = (P_i - P_{i-1})\mathbb{V}$, then (D_W) holds with constant $K_1 = 1$ and (O_W) holds with $K_2 = 0$. The assumption (D_W) relax the A-orthogonal decomposition to a stable one and (O_W) means the subspace $\mathbb{W}_j$, which is often of high frequency, is quasi-orthogonal to the coarser space $\mathbb{V}_i$ for $i < j$.

We revise the smoothing property associated to $\mathbb{W}$.

$(\bar{S}_W)$ The smoother $\overline{R}_k$ will smooth the frequency in $\mathbb{W}_k$, i.e., there exists $c_R \ge 0$

$$(\overline{R}_k^{-1} u_k, u_k) \le c_R (A u_k, u_k), \quad \text{for all } u_k \in \mathbb{W}_k, k = 0, \cdots, J. \tag{29}$$

4.2. Convergence.

In the estimate below, we shall use the equivalent condition (σ) instead of (C) since the constant $\sigma = \omega/(2-\omega)$ will enter the estimate.

Theorem 4.1. *Suppose the smoother R_k satisfies assumptions (σ), and $(\bar{S}_W)$ and the space decomposition satisfies (D_W) and (O_W). Then the corresponding $V(1,1)$-cycle is uniform convergent with a rate bounded by*

$$1 - \frac{1}{K_1(\sqrt{c_R} + \sqrt{\sigma} K_2)^2}.$$

Proof. We consider the first X-Z identity (5). Let $w_i = v_i + R_i^t A_i P_i (\sum_{j>i} v_j)$. We split as

$$\sum_{i=0}^J \|w_i\|_{\overline{R}_i^{-1}}^2 = \sum_{i=0}^J (w_i, v_i)_{\overline{R}_i^{-1}} + \sum_{i=0}^J (w_i, R_i^t A_i P_i \sum_{j>i} v_j)_{\overline{R}_i^{-1}} = I_1 + I_2.$$

Using Cauchy-Schwarz inequality and assumption $(\bar{S}_W)$, we can bound the first term

$$I_1 \le \left(\sum_{i=0}^J \|w_i\|_{\overline{R}_i^{-1}}^2\right)^{1/2} \left(\sum_{i=0}^J \|v_i\|_{\overline{R}_i^{-1}}^2\right)^{1/2} \le \sqrt{c_R}\left(\sum_{i=0}^J \|w_i\|_{\overline{R}_i^{-1}}^2\right)^{1/2} \left(\sum_{i=0}^J \|v_i\|_A^2\right)^{1/2}.$$

To bound the second term, we denote by $u_i = T_i\overline{T}_i^{-1}w_i$ and bound the norm $\|u_i\|_A$ using (σ) as

$$\|u_i\|_A^2 = \|T_i\overline{T}_i^{-1}w_i\|_A^2 \le \sigma(\overline{T}_i\overline{T}_i^{-1}w_i, \overline{T}_i^{-1}w_i)_A = \sigma(\overline{T}_i^{-1}w_i, w_i)_A = \sigma\|w_i\|_{\overline{R}_i^{-1}}^2.$$

We then rewrite I_2 using operator T_i and $\overline{T}_i$

$$I_2 = \sum_{i=0}^J(\overline{T}_i^{-1}w_i, T_iP_i\sum_{j>i}v_j)_A = \sum_{i=0}^J(T_i\overline{T}_i^{-1}w_i, \sum_{j>i}v_j)_A = \sum_{i=0}^J\sum_{j=i+1}^J(u_i, v_j)_A.$$

We now apply SCS, i.e., (O_W) to get

$$I_2 \le K_2\left(\sum_{i=0}^J \|u_i\|_A^2\right)^{1/2} \left(\sum_{i=0}^J \|v_i\|_A^2\right)^{1/2} \le K_2\sqrt{\sigma}\left(\sum_{i=0}^J \|w_i\|_{\overline{R}_i^{-1}}^2\right)^{1/2} \left(\sum_{i=0}^J \|v_i\|_A^2\right)^{1/2}.$$

Combing the bound of I_1 and I_2 and choosing a stable decomposition $v = \sum_i v_i, v_i \in \mathbb{W}_i$, we get the desirable inequality

$$\sum_{i=0}^J \|w_i\|_{\overline{R}_i^{-1}}^2 \le (\sqrt{c_R} + K_2\sqrt{\sigma})^2\sum_{i=0}^J \|v_i\|_A^2 \le K_1(\sqrt{c_R} + K_2\sqrt{\sigma})^2\|v\|_A^2.$$

□

Note that (O_W) hold for $K_2 = J + 1 = O(\log N)$ by a naive application of Cauchy-Schwarz inequality. The focus in a stable decomposition (D_W). The smoothing property $(\bar{S}_W)$ can be also verified using a stable decomposition of subspace $\mathbb{W}$; see Section 6.

4.3. **Point-wise smoothers.** We can use the spectral norm to form slightly different assumptions. To unify the notation, we understand $\|u_0\|_{\rho_0} = \|u_0\|_A$.

(D_{W_ρ}): The space decomposition $\mathbb{V} = \sum_{k=0}^J \mathbb{W}_k$ is ρ-stable:

$$\inf_{\sum_{i=0}^J v_i = v, v_i\in\mathbb{W}_i}\sum_{i=0}^J \|v_i\|_{\rho(A_i)}^2 \le K_1\|v\|_A^2. \tag{30}$$

(O_{W_ρ}): Quasi-Orthogonality. The following Strengthened Cauchy Schwarz (SCS) inequality holds for any $u_i \in \mathbb{V}_i, v_i \in \mathbb{W}_i, i = 0, \cdots, J$

$$\sum_{i=0}^J\sum_{j=i+1}^J(u_i, v_j)_A \le K_2\left(\sum_{i=0}^J \|u_i\|_A^2\right)^{1/2}\left(\sum_{i=0}^J \|v_i\|_{\rho(A_i)}^2\right)^{1/2}. \tag{31}$$

Using the inequality $\|v_i\|_A \le \|v_i\|_{\rho(A_i)}$, one can easily verify that (D_{W_ρ}) implies (D_W) but (O_W) implies (O_{W_ρ}). Therefore the difficulty of verifying assumptions $(D_W)-(O_W)$ or $(D_{W_\rho})-(O_{W_\rho})$ are balanced. These two set of assumptions are equivalent if $\mathbb{W}_i$ consists of high frequency functions since then $\|v_i\|_A^2 \eqsim \|v_i\|^2_{\rho(A_i)}$. The benefit of using $(D_{W_\rho})-(O_{W_\rho})$ is that we only need to check the weaker smoothing property $(\bar{S}_\rho)$ which holds for most popular point-wise smoothers.

With a minor modification of the proof, we can obtain the following results using assumptions on the spectral norm.

Theorem 4.2. *Suppose the smoother R_k satisfies assumptions (σ), and $(\bar{S}_\rho)$ and the space decomposition satisfies (D_{W_ρ}) and (O_{W_ρ}). Then the V(1,1)-cycle is uniform convergent with a rate bounded by*

$$1-\frac{1}{K_1(\sqrt{c_R}+\sqrt{\sigma}K_2)^2}.$$

Example 4.3. The above results requires an estimate of σ. For symmetric smoother, it is equivalent to assume $\omega = \lambda_{\max}(R_iA_i)$ is uniformly bounded below 2 which can be ensured by imposing a suitable weight. In this example we verify the equivalent assumption (σ) for the most popular non-symmetric smoother: G-S iteration. For $R=(D+L)^{-1}$ or $R=(D+U)^{-1}$, $(R^{-t}+R^{-1}-A)^{-1}=D^{-1}$. Therefore $\rho(T\overline{T}^{-1}T^*)=\rho(A_D)\le\sigma$, where $\sigma=2$ for diagonal dominate matrices and finite for general sparse SPD matrices.

A way to verify the stable decomposition using a stable quasi-interpolation will be discussed below.

4.4. **Decomposition using a stable quasi-interpolation.** The assumptions on the space decomposition (D_W) and (O_W) are not easy to verify, especially to get a uniform bounded constants K_1 and K_2. In this subsection we present a simple but nearly optimal convergence proof which only requires the construction of a quasi-interpolation operator with the following stability and approximation property.

(B_Π): The operator Π_k is stable (bounded) in A-norm, i.e., for $k=1,\dots,J$:

$$\|\Pi_k u\|_A^2 \le c_b\|u\|_A^2, \text{ for all } u\in\mathbb{V}.$$

(A_Π) Approximation property of Π_k. For $k=1,\dots,J$:

$$\|(I-\Pi_k)u\|^2 \le \frac{c_a}{\rho(A_k)}\|u\|_A^2, \text{ for all } u\in\mathbb{V}. \tag{32}$$

Theorem 4.4. *Suppose the smoother R_k satisfies assumptions (σ), and $(\bar{S}_\rho)$. Suppose there exist linear operator Π_k satisfying the approximation property (A_Π). Suppose $\rho(A_k)\le c_r\rho(A_{k-1})$. Then the V(1,1)-cycle is nearly uniform convergent with a rate* •1

•1 *check the constant. not quite right.*

$$1-\frac{1}{4c_a(1+c_r)J(c_R+\sigma J)}.$$

Proof. We consider the decomposition $\mathbb{V}=\sum_k\mathbb{W}_k$, with $\mathbb{W}_k=(\Pi_k-\Pi_{k-1})\mathbb{V}$ for $k=0,\dots,J$. Here for the convenience of notation, $\Pi_J := I, \Pi_{-1}:=0$. We verify the assumptions (D_Π) and (O_Π). First the assumption (O_Π) holds for $K_2=J$.

Using the triangle inequality and (A_Π), we have

$$\sum_{k=1}^{J}\rho(A_k)\|(\Pi_k-\Pi_{k-1})u\|^2 \le 2\sum_{k=1}^{J}\rho(A_k)(\|(I-\Pi_k)u\|^2+\|(I-\Pi_{k-1})u\|^2)$$
$$\le 2c_a(1+c_r)J\|u\|_A^2.$$

For $k=0$, we have to use the stability of Π_0 to bound $\|\Pi_0 u\|_A \le c_b\|u\|_A$.

Apply Theorem 4.2 to $\mathbb{W}_k=(\Pi_k-\Pi_{k-1})\mathbb{V}$ to get the desirable result. □

Example 4.5. As an example, we consider linear finite element methods for elliptic problems based on a sequence of nested meshes by regular refinement. Then $\rho(A_k)\eqsim h_k^{-2}$ and $c_r=4$.

We have verified the smoothing property and the assumption (σ) for three popular smoothers: Richardson, Jacobi, and Gauss-Seidel. We chose $\Pi_k=Q_k$, the L^2-projection. The approximation property (A_Q) holds by the standard L^2 error estimate. The L^2 projection Q_k is stable in H^1-norm on quasi-uniform grids. Note that we cannot chose the nodal interpolation I_k since it is not stable in H^1-norm.

We thus have proved V-cycle multigrid will converge in a nearly optimal rate $1-1/\log^2 N$.

Removing the $\log N$ factor is technical and will be discussed in somewhere else.

5. Block Smoothers

If using the orthogonal telescope decomposition, the crucial assumption is the smoothing property $(\bar{S}_P)$. In general, suppose Π_k is a stable operator, which is not difficulty to construct, define $\mathbb{W}_k=(\Pi_k-\Pi_{k-1})\mathbb{V}$, assumptions (D_W) and (O_W) will hold with constant $J+1$. Again the crucial assumption is the smoothing property $(\bar{S}_W)$.

If the approximation property holds, then point-wise smoother is adequate. When it fails, block smoothers can be designed to still satisfy the required smoothing property.

A smoother in each level can be further treat as a subspace correction method based on a decomposition of the space. In this section, we discuss smoothers based on a stable micro-decomposition.

5.1. Smoothers based on micro-decomposition.

We shall construct effective smoother based a micro-decomposition of each space $\mathbb{W}_k$. The stable decomposition $\mathbb{W}_k=\sum_{i=1}^{N_k}\mathbb{W}_{k,i}$ and the quasi-orthogonality will implies a multiplicative subspace correction method based on this decomposition is an effective one in the sense that the smoothing property $(\bar{S}_W)$ will holds.

(D_{W_k}): The space decomposition $\mathbb{W}_k=\sum_{i=1}^{N_k}\mathbb{W}_{k,i}$ is A-stable: for all $v\in\mathbb{W}_k$

$$\inf_{\sum_{i=0}^{N_k}v_i=v,\,v_i\in\mathbb{W}_{k,i}}\sum_{i=0}^{N_k}\|v_i\|_A^2 \le K_1\|v\|_A^2. \tag{33}$$

(O_{W_k}): Quasi-Orthogonality. The following Strengthened Cauchy Schwarz (SCS) inequality holds for any $u_i \in \mathbb{W}_{k,i}, v_i \in \mathbb{W}_{k,i}, i = 0, \cdots, N_k$

$$\sum_{i=0}^{N_k} \sum_{j=i+1}^{N_k} (u_i, v_j)_A \leq K_2 \left(\sum_{i=0}^{N_k} \|u_i\|_A^2 \right)^{1/2} \left(\sum_{i=0}^{N_k} \|v_i\|_A^2 \right)^{1/2}. \tag{34}$$

Theorem 5.1. *If there exists a micro-decomposition for each subspace $\mathbb{W}_k$, then the SSC smoother based on this decomposition will satisfy the smoothing assumption $(\bar{S}_W)$.*

The proof is straightforward and left as an exercise.

Remark 5.2. For additive smoother, a scaling is needed.

The quasi-orthogonality is easy to verify. Usually $\mathbb{W}_{k,i}$ is spanned by few bases of $\mathbb{V}_k$ and thus by the finite overlapping property, (O_{W_k}) holds. Therefore the crucial thing is a stable decomposition.

5.2. **Point-wise smoother.** Consider the standard basis decomposition $\mathbb{V} = \sum_{i=0}^{N} \mathbb{V}_i$. We assume this decomposition is stable under the norm introduced by the default inner product $(\cdot, \cdot)$; see (25). In FEM setting, that is the bases decomposition is stable in the L^2 norm.

The decomposition is in general not stable in the A-norm. Namely for any $v \in \mathbb{V}$, write $v = \sum_{i=1}^{N} v_i$, the inequality

$$\sum_{i=1}^{N} \|v_i\|_A^2 \leq C\|v\|_A^2 \tag{35}$$

does not hold for a constant C independent of N due to the existence of low frequency. As an extreme example, one can choose $v = \sum_{i=1}^{N} \phi_i$. Then v is flat except in a band near the boundary. The derivative of v is zero in most region while $|\nabla v_i|$ is always of order $1/h$.

If we use the stability in $(\cdot, \cdot)$, we can get

$$\sum_{i=1}^{N} \|v_i\|_A^2 \leq \rho(A) \sum_{i=1}^{N} \|v_i\|^2 \leq C\rho(A)\|v\|^2.$$

From which we immediately conclude that (35) holds for high frequency.

When $\mathbb{W}_k = (\Pi_k - \Pi_{k-1})\mathbb{V}$ and Π_k satisfies the approximation property (A_Π), then the difference $(\Pi_k - \Pi_{k-1})\mathbb{V}$ will be high frequency and thus point-wise smoother is effective. When the approximation fails, block smoothers can be designed to still satisfy the required smoothing property. Another direction is to enrich the coarse space which is exploited mostly in algebraic multigrid methods.

6. DECOMPOSITION INTO LEAFS

The previous theories requires the nested-ness of the space decomposition which is not suitable for adaptive grids. In this section, we present another framework developed in [4]; see also [6].

6.1. **Assumptions.** For each subspace $\mathbb{V}_k$, we can further decompose into micro pieces $\mathbb{V}_k = \sum_{i=1}^{N_k} \mathbb{V}_{k,i}$ and the dimension of each leaf $\mathbb{V}_{k,i}$ is small such that direct solvers for problems restricted on leafs are applicable. For example, the basis decomposition leads to one dimensional leafs and exact solvers can be applied. For block Gauss-Seidel iterations, each leaf corresponds to one block of several unknowns. The micro decomposition itself can be used to define a smoother and thus the assumption of the smoother can be build into the decomposition.

Consider subspace correction methods for the decomposition $\mathbb{V} = \sum_{k=0}^{J} \sum_{i=1}^{N_k} \mathbb{V}_{k,i}$. To clean the notation, we merge the index (k, i) to one subscript l and consider the following decomposition

$$\mathbb{V} = \sum_{l=0}^{L} \mathbb{V}_l.$$

In this decomposition, subspaces are not necessarily nested.

(D_L): The space decomposition $\mathbb{V} = \sum_{l=0}^{L} \mathbb{V}_l$ is A-stable:

$$\inf_{\sum_{l=0}^{L} v_l = v, v_l \in \mathbb{V}_l} \sum_{l=0}^{L} \|v_l\|_A^2 \leq K_1 \|v\|_A^2. \tag{36}$$

(O_L): Quasi-Orthogonality. For any $u_i \in \mathbb{V}_i, v_i \in \mathbb{V}_i, i = 0, \cdots, L$

$$\sum_{i=0}^{L} \sum_{j=i+1}^{L} (u_i, v_j)_A \leq K_2 \left(\sum_{i=0}^{L} \|u_i\|_A^2 \right)^{1/2} \left(\sum_{i=0}^{L} \|v_i\|_A^2 \right)^{1/2}. \tag{37}$$

6.2. **Example: Hierarchical Basis.** Define HB. The decomposition is unique. Such decomposition satisfies (O_L) due to the strengthened Cauchy-Schwarz but only nearly stable in 2-D and not stable in 3-D.

Prove the nearly stable decomposition. Good for jump coefficients and adaptive grids.

6.3. **Smoothing on leaves.** Since each leaf is of small dimension, the smoothing property can be derived from the contraction rate of each local solver. We state the contraction assumption again with an explicit constant ρ.

(C_ρ) Each local solver is a contraction $\|I_l - R_l A_l\|_{A_l} \leq \rho < 1$ for each $l = 0, \cdots, L$.

Lemma 6.1. *Suppose each local solver satisfy (C_ρ). Then we have the estimate of spectrum of* $\overline{T}_l$

$$1 \leq \lambda_{\min}(\overline{T}_l^{-1}) \leq \lambda_{\max}(\overline{T}_l^{-1}) \leq \frac{1}{1-\rho^2}.$$

Consequently the smoothing property ($\bar{S}$) holds with $c_R = 1/(1-\rho^2)$, *i.e,*

$$(\overline{R}_l^{-1} u, u) = (\overline{T}_l^{-1} u, u)_A \leq \frac{1}{1-\rho^2} (u, u)_A, \quad \textit{for all } u \in \mathbb{V}_l.$$

Proof. First $\lambda_{\max}(\overline{T}_i) \leq 1$ since it is a symmetrization. Then

$$1 - \lambda_{\min}(\overline{T}_l) = \|I_l - \overline{T}_l\|_A = \|I_l - T_l\|_{A_l}^2 \leq \rho^2. \tag{38}$$

which is equivalent to $\lambda_{\min}(\overline{T}_l) \geq 1 - \rho^2$. □

One can apply Lemma 6.1 to the macro decomposition $\mathbb{V} = \sum_{k=0}^J \mathbb{V}_k$. But each smoother only take care of certain frequency and the overall rate ρ could be very close to one, say $1 - Ch^2$, and thus Lemma 6.1 is useless for macro decomposition. For small dimensional space $\mathbb{V}_l$, however, one can easily construct local solver with rate $\rho < 1$ uniformly. One such example is the exact solver with $\rho = 0$.

6.4. **Converges.** We first present a simple proof using the simplest X-Z identity for the exact solver $R_l = A_l^{-1}$ which is practical since the problem on leaves is of small size.

Theorem 6.2. *Suppose the space decomposition satisfy* (D_L) *and* (O_L)*. For exact local solver* $R_l = A_l^{-1}$ *for all* $l = 0, \cdots, L$*, we have*

$$\Big\| \prod_{l=0}^{L} (I - P_l) \Big\|_A^2 \leq 1 - \frac{1}{1 + K_1 K_2^2}.$$

Proof. We apply (31) with $u_i = P_i \sum_{j=i+1}^J v_j$ to obtain

$$\begin{aligned}\sum_{i=0}^J \|u_i\|_A^2 &= \sum_{i=0}^J (u_i, P_i \sum_{j=i+1}^J v_j)_A = \sum_{i=0}^J \sum_{j=i+1}^J (u_i, v_j)_A \\ &\leq K_2 \left(\sum_{i=0}^J \|u_i\|_A^2 \right)^{1/2} \left(\sum_{i=0}^J \|v_i\|_A^2 \right)^{1/2}.\end{aligned}$$

Consequently, if $v = \sum_{k=0}^J v_k$ is a stable decomposition satisfying (30), we get

$$\sum_{i=0}^J \Big\| P_i \sum_{j=i+1}^J v_j \Big\|_A^2 = \sum_{i=0}^J \|u_i\|_A^2 \leq K_2^2 \sum_{i=0}^J \|v_i\|_A^2 \leq K_1 K_2^2 \|v\|_A^2,$$

which implies $c_0 \leq K_2 K_1$. The desired result then follows from X-Z identity (7). □

Theorem 6.3. *Suppose the space decomposition satisfy* (D_L) *and* (O_L) *and the local solver satisfy* (C_ρ)*. Then* •2

•2 Check details and refine the constant in the proof

$$\Big\| \prod_{l=0}^{L} (I - T_l) \Big\|_A^2 \leq 1 - \frac{1 - \rho^2}{2K_1(1 + (1 + \rho)^2 K_2)}.$$

Proof. By Lemma 6.1, the local solver will satisfy $(\bar{S})$ with constant $c_R = 1/(1 - \rho^2)$. We can then apply Theorem 4.1 to obtain the desired result. □

For multiplicative smoothers on $\mathbb{V}_k$, it can be understood as SSC for micro scale decomposition. So the above results can be applied to multigrid with multiplicative smoothers. The assumptions (D_L) and (O_L) are verified in [4] for multigrid methods on adaptive grids where each leaf contains three nodal bases.

7. SMOOTHING STEPS

In this section we refine the analysis to include the affect of smoothing steps. Suppose we apply m steps of pre-smoothing and m steps post-smoothing, i.e., $V(m,m)$-cycle. To apply the convergence theory we have established before, we treat all m-steps smoothing as one iteration and check the corresponding smoothing property. Again in this section, since we work on smoothers on a fixed level, we shall skip the subscript of levels.

7.1. **Smoothing property of m-steps of smoothing.** Let R_m be the iterator of applying m-times of the iterative method associated with R. Let $T = RA, T_m = R_m A$ and $S = I - RA = I - T$. The relation is

$$I - R_m A = (I - RA)^m, \quad I - T_m = (I - T)^m = S^m.$$

In the sequel, we further assume the smoother R is symmetric and $\sigma(RA) \in (0,1]$ which implies the contraction assumption (C). The assumption $\lambda_{\min}(RA) > 0$ implies R is an SPD and the assumption $\lambda_{\max}(RA) \le 1$ is usually characterized by the inequality

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(R). The symmetric smoother R is non-singular and

$$(R^{-1}u, u) \ge (Au, u), \text{ for all } u \in \mathbb{V},$$

or simply $R^{-1} \ge A$.

Lemma 7.1. *Suppose the symmetric smoother R satisfy (R). Then we have*

$$(\overline{R}_m^{-1}u, u) \le (Au, u) + \frac{1}{2m}(R^{-1}u, u) \quad \textit{for all } u \in \mathbb{V}.$$

Proof. The assumption (R) implies $\sigma(T) = \sigma(RA) \in (0,1]$ and thus $\sigma(S) \in [0,1)$. For the symmetric smoother, we have $\overline{R}_m = R_{2m}$. We now estimate $(R_{2m}^{-1}u, u) = (T_{2m}^{-1}u, u)_A$. To use operator S, we first write as

$$(T_{2m}^{-1}u, u)_A = ((T_{2m}^{-1} - I)u, u)_A + (u, u)_A.$$

We manipulate the first term as

$$\begin{aligned} T_{2m}^{-1} - I &= T^{-\frac{1}{2}} T^{\frac{1}{2}} T_{2m}^{-1}(I - T_{2m}) T^{\frac{1}{2}} T^{-\frac{1}{2}} \\ &= T^{-\frac{1}{2}}(I - S)^{\frac{1}{2}}(I - S^{2m})^{-1} S^{2m} (I - S)^{\frac{1}{2}} T^{-\frac{1}{2}}. \end{aligned}$$

Therefore for $u \in \mathbb{V}$

$$((T_{2m}^{-1} - I)u, u)_A \le \max_{t \in [0,1)} \left[(1-t)(1-t^{2m})^{-1} t^{2m}\right] (T^{-\frac{1}{2}}u, T^{-\frac{1}{2}}u)_A \le \frac{1}{2m}(T^{-1}u, u)_A.$$

Consequently the inequality follows. $\square$

A formal proof can be given using the manipulation rules of symmetric operators. From the elementary inequality

$$x^{2m} \le \frac{1}{2m} \frac{1 - x^{2m}}{1 - x}, \quad \text{for all} x \in [0,1),$$

we get

$$(1 - x^{2m})^{-1} - 1 \le \frac{1}{2m}(1 - x)^{-1},$$

which implies the desirable inequality

$$(I - S^{2m})^{-1} - I \le \frac{1}{2m}(1 - S)^{-1}.$$

Corollary 7.2. *Suppose the symmetric smoother R satisfy (R) and one of the smoothing property (S_P), (S_W), or (S_ρ) with constant c_R. Then $\overline{R}_m$ satisfies the same smoothing property with constant $1 + c_R/2m$.*

7.2. **Convergence.** Combine with the convergence theory using $(\bar{S}_P)$, we thus recovery the classic result in [2].

Theorem 7.3. *Suppose the symmetric smoother R_k satisfies assumptions (R) and the smoothing approximation (S_P). Then the V(m,m)-cycle using R_k is uniform convergent:*

$$\|(I - T_J)^m (I - T_{J-1})^m ... (I - T_0)^{2m} (I - T_{J-1})^m ... (I - T_J)^m\|_A \le \frac{c_R}{c_R + 2m}.$$

What is a good choice of steps m? Note that in our notation of product, an ordering is assumed. Thus

$$\Pi_{i=0}^{J}(I - T_i)^m \neq \left[\Pi_{i=0}^{J}(I - T_i)\right]^m.$$

The left hand side is $V(m,m)$ while the right hand side is applying $V(1,1)$ m-times. Although the computation cost of operators in two sided are the same, the rate of convergence is $\mathcal{O}(\delta/m)$ v.s. δ^m. Roughly speaking if we double the smoothing steps, the rate is decreased by half. The cost is the same as that applying the original V-cycle twice. Then if $\delta^2 \le \delta/2$, i.e. $\delta < 0.5$, there is no advantage (in terms of computational cost) to increase the smoothing step. The main motivation of increasing the smoothing step is to make the V-cycle MG robust and converges with a contraction number smaller than 0.5.

Combine with the assumption on the decomposition of spaces, we can have the following convergences.

Theorem 7.4. *Suppose the symmetric smoother R_k satisfies assumptions (R) and the smoothing property (S_W). Suppose the space decomposition satisfies (D_W) and (O_W). Then the V(m,m)-cycle is uniform convergent with a rate*

$$1 - \frac{1}{2((1 + c_R/2m)K_1 + K_1 K_2)}.$$

For less smooth problems, i.e., no full regularity, then Theorem 7.3 is not applicable since the assumption $(\bar{S}_P)$ is difficult to verify. Theorem 7.4 suggests that increasing the smoothing steps will improve the rate of convergence but the rate will converge to the lower bound $1 - 1/2(K_1 + K_1 K_2)$. For H^1-elliptic problems, K_2 depends on the refinement rule of the mesh while K_1 will depends on the regularity of the solution. For $H^{1+\alpha}$-regularity, refined analysis can be given to improve the rate to $C/(C + m^\alpha)$; see *Chapter: Recursive Proof of Multigrid Methods*.

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