Maxwells Equations

$$\begin{cases} -\frac{\partial B}{\partial t} = \nabla \times E \\ \frac{\partial D}{\partial t} + J = \nabla \times H \\ \nabla \cdot B = 0 \\ \nabla \cdot H = \rho \end{cases} \Rightarrow \quad -\frac{\partial(\mu H)}{\partial t} = \nabla \times E \qquad \frac{\partial(\varepsilon E)}{\partial t} + J = \nabla \times H$$

$$X(x,t) = \mathrm{Re}(e^{-i\omega t} \hat{X}(x))$$

$B = \mu H$ $\quad\mu$-permeability $\qquad \partial_t X = -i\omega X$

$D = \varepsilon E$ $\quad\varepsilon$-permitivity

Time harmonic

$$\begin{cases} \nabla \times E = i\omega\mu H \\ \nabla \times H = -i\omega\varepsilon E + J \end{cases} \qquad \boxed{J = \sigma E + J_a}$$

Ohms law

$\sigma > 0$ conductor

$\sigma = 0,\ \varepsilon \neq \varepsilon_0$ dielectric

$\sigma = 0,\ \varepsilon = \varepsilon_0,\ \mu = \mu_0$ vacuum

$\sigma = \infty$ perfect conductor

$$\begin{cases} \nabla \times E = i\omega\mu H \\ \nabla \times H = -i\omega(\varepsilon + i\sigma/\omega)E + J_a \end{cases}$$

$$E \Leftarrow \varepsilon_0^{1/2} E \qquad H \Leftarrow \mu_0^{1/2} H$$

$$\varepsilon_r = \frac{1}{\varepsilon_0}\left(\varepsilon + \frac{i\sigma}{\omega}\right),\ \mu_r = \mu/\mu_0,\ K = \omega\sqrt{\varepsilon_0\mu_0}$$

wavenumber

vacuum: $\varepsilon_r = 1,\ \mu_r = 1$

$$\nabla \times E = iK\mu_r H$$

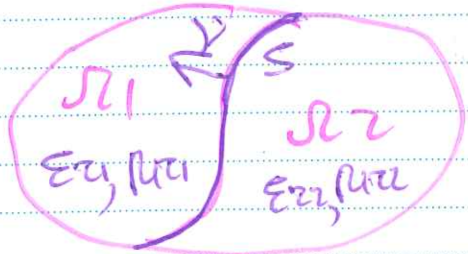
$$\nabla \times H = -iK\varepsilon_r E + \frac{1}{iK} F, \quad F = iK\underbrace{\mu_0^{1/2} J_a}_{J_a}$$

$$\boxed{\nabla \times \left(\frac{1}{\mu_r} \nabla \times E\right) - k^2 \varepsilon_r E = F}$$

$$\boxed{\nabla \times \left(\frac{1}{\varepsilon_r} \nabla \times H\right) - k^2 \mu_r H = \frac{1}{\varepsilon_r} \nabla \times J_a}$$

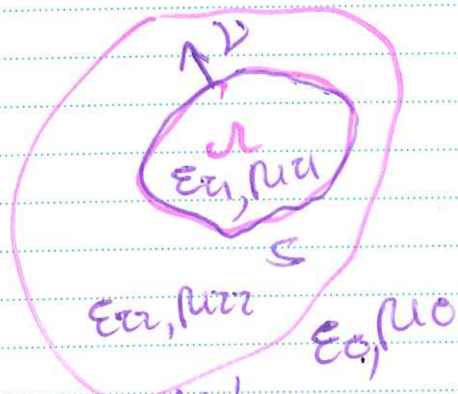
Interface and boundary conditions.

$E, H \in H(\mathrm{curl}; \Omega) = \{ \vec{v} \in (L^2(\Omega))^3, \mathrm{curl}\, \vec{v} \in (L^2(\Omega))^3 \}$



$\nu \times E_1 = \nu \times E_2$ S

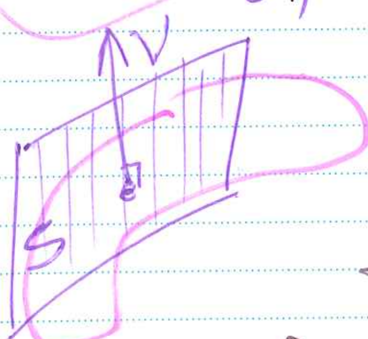
Tangential component of E should be continuous.



$\vec{u} = \vec{u}_{\parallel} \oplus \vec{u}_{\perp}$

$\vec{u}_{\perp} = (\vec{u} \cdot \vec{\nu}) \vec{\nu}$

$\vec{u}_{\parallel} = \vec{u} - (\vec{u} \cdot \vec{\nu}) \vec{\nu}$

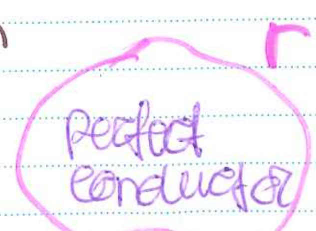


$(\nu \times \vec{u}) \times \nu$

$\nu \times (H_1 - H_2) = J_S$

$J_S = 0$ in most cases

Boundary condition
Perfect conductor.



$\nu \times E = 0 \quad \Gamma$

Impedance B.C. Robin B.C.

$\nu \times H - \eta E_T = 0 \quad \left(\alpha \frac{\partial u}{\partial n} + \beta u = g \right)$

Boundary Value Problems

1) Time harmonic problem in a cavity

$$\begin{cases} \nabla\times(\mu_r^{-1}\nabla\times E) - k^2\varepsilon_r E = F & \Omega \\ \nabla\times E = 0 & \Gamma \\ \mu_r^{-1}(\nabla\times E)\times\nu - ik\lambda E_T = g & \Sigma \end{cases}$$

2) Cavity resonator. Maxwell eigenvalue problem

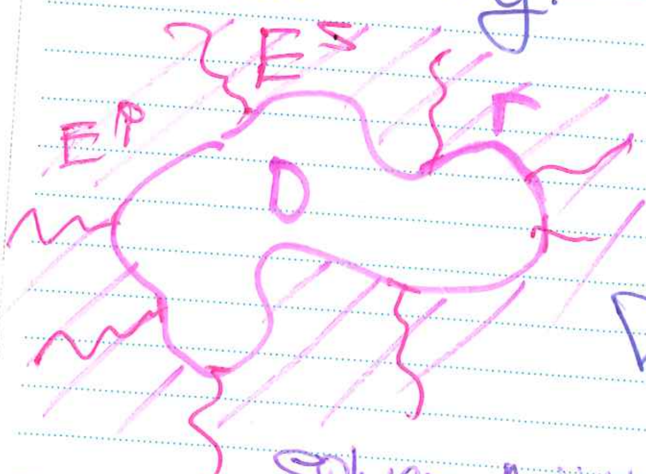
$$\nabla\times(\mu_r^{-1}\nabla\times E) - k^2\varepsilon_r E = 0 \quad \Omega$$

$$\nabla\times E = 0 \quad \Gamma$$

$$\nabla\cdot(\varepsilon_r E) = 0$$

$k = \omega\sqrt{\varepsilon_0\mu_0}$, (ω, E_ω)

3) Scattering.



$E = E^i + E^s$

incident field, scattered field

$$\nabla\times(\mu_r^{-1}\nabla\times E) - k^2\varepsilon_r E = F, \quad \mathbb{R}^3\setminus D$$

$$E\times\nu = 0, \quad \Gamma$$

Silver-Müller radiation condition

$$\lim_{\rho\to\infty} \rho\left((\nabla\times E^s)\times\frac{\vec{x}}{|\vec{x}|} - ikE^s\right) = 0$$

$\rho = |\vec{x}|$

Variational Formulation

$$\begin{cases} \nabla \times (\nabla \times u) + u = f & \text{in } \Omega \\ u \times \nu = 0 & \text{on } \Gamma \end{cases}$$

$$\int_\Omega (\nabla \times \nabla \times \vec{u}) \cdot \vec{v} + \int_\Omega \vec{u} \cdot \vec{v} = \int_\Omega \vec{f} \cdot \vec{v}$$

integration by parts in $\mathbb{R}^3$

$$\int_\Omega (\nabla \times \vec{v}) \cdot \vec{\varphi}\, dx = \int_\Omega \vec{v} \cdot (\nabla \times \vec{\varphi}) + $$

$$+ \int_{\partial\Omega} (\nu \times \vec{v}) \cdot \vec{\varphi}\, ds$$

exercise 2D

$$(\nabla \times \vec{u}, \nabla \times \vec{v}) + (\vec{u}, \vec{v}) = (\vec{f}, \vec{v}) + \int_{\partial\Omega} \nu \times \nabla \times \vec{u} \cdot \vec{v}$$

$$= \int_{\partial\Omega} (\nu \times \nabla \times \vec{u}) \cdot ((\nu \times \vec{v}) \times \nu)$$

$$\nu \times \vec{v} = 0 \qquad \Gamma = \partial\Omega$$

find $\vec{u} \in H_0(\text{curl}; \Omega)$ s.t.

$$(\nabla \times \vec{u}, \nabla \times \vec{v}) + (\vec{u}, \vec{v}) = (\vec{f}, \vec{v})$$

for any $\vec{v} \in H_0(\text{curl}, \Omega)$

$$H_0(\text{curl}; \Omega) = \{ \vec{v} \in H(\text{curl}, \Omega),\ \nu \times \vec{v}|_\Gamma = 0 \}$$

$$(\nabla \times \cdot, \nabla \times \cdot) + (\cdot, \cdot)$$

defines an inner product in $H(\text{curl}; \Omega)$

By Riesz representation theorem,

$\exists!$ solution.

* Sobolev Spaces $\Omega \subset \mathbb{R}^3$

D = grad : $H^1(\Omega)$

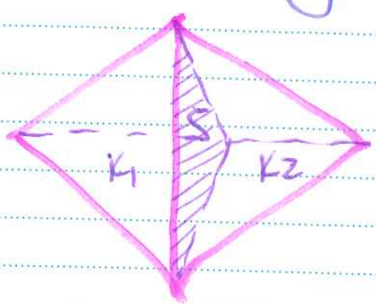
D = curl : $H(\mathrm{curl};\Omega)$

D = div : $H(\mathrm{div};\Omega)$

$$H(D;\Omega) = \{v \in L^2(\Omega),\ Dv \in L^2(\Omega)\}$$

$$\langle Du, \varphi\rangle \underset{def}{=} \langle u, D^*\varphi\rangle$$ continuity condition.

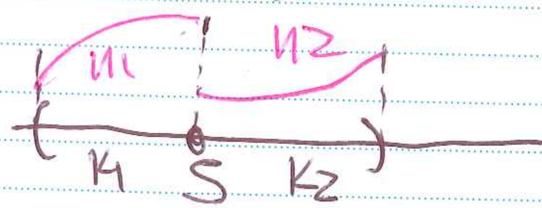
Continuity condition



$K_1, K_2 \quad S = \overline{K_1} \cap \overline{K_2}$

$u \in H^1(K_1) \cap H^1(K_2)$

$u \in H^1(K_1 \cup K_2 \cup S)$



$u_1 \in H(D;K_1)$, $u_2 \in H(D;K_2)$

define $u \in L^2(K_1 \cup K_2 \cup S)$

$$u = \begin{cases} u_1 & x \in K_1 \\ u_2 & x \in K_2 \end{cases}$$

$u \in H(D; K_1 \cup K_2 \cup S)$

$$\begin{cases} D = \mathrm{grad} & u_1 = u_2 \text{ on } S \\ D = \mathrm{curl} & \nu \times u_1 = \nu \times u_2 \text{ on } S \\ D = \mathrm{div} & \nu \cdot u_1 = \nu \cdot u_2 \text{ on } S \end{cases}$$

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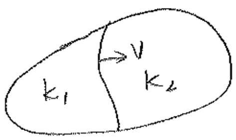
①

$H(\mathrm{curl}, \Omega) = \{ \vec{v} \in (L^2(\Omega))^3,\ \mathrm{curl}\,\vec{v} \in (L^2(\Omega))^3 \}$

$H(\mathrm{div}, \Omega) = \{ \vec{v} \in (L^2(\Omega))^3,\ \mathrm{div}\,\vec{v} \in (L^2(\Omega))^3 \}$



continunity u_1 u_2

$H(\mathrm{grad};\ K_1 \cup K_2 \cup S)$ $\quad u_1 = u_2$ on S

$H(\mathrm{curl};\ K_1 \cup K_2 \cup S)$ $\quad u_1 \times v = u_2 \times v$ on S

$H(\mathrm{div},\ K_1 \cup K_2 \cup S)$ $\quad u_1 \cdot \vec{n} = u_2 \cdot \vec{n}$ on S

for $u_i \in H(\mathrm{curl}, K_i)$ $\quad i = 1, 2$

$$\int_{K_1} \nabla \times u_1 \phi + \int_{K_2} \nabla \times u_2 \phi$$

$$= \int_{K_1} u_1 \nabla \times \phi + \int_{K_2} u_2 \nabla \times \phi + \int_S (v \times u_1 - v \times u_2)\phi$$

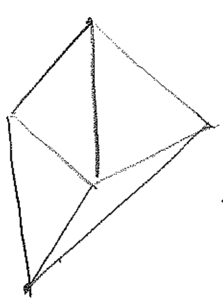
$$= \int_{K_1 \cup K_2 \cup S} u \nabla \times \phi + \int_S (v \times u_1 - v u_2)\phi \quad \text{for } \forall \phi \in C_0^\infty(K_1 \cup K_2 \cup S)$$

if $v \times u_1 = v \times u_2$ on S

then $\int \nabla \times u\, \phi = \int u \nabla \times \phi$

$$= \int_{K_1} \nabla \times u_1 \phi + \int_{K_2} \nabla \times u_2 \phi \qquad \nabla \times u \in L^2$$

Finite Element Space



← tetrahedral grid triqulation.

↖ Conforming $\tau_1 \cap \tau_2$ is a common face / edge / vertex

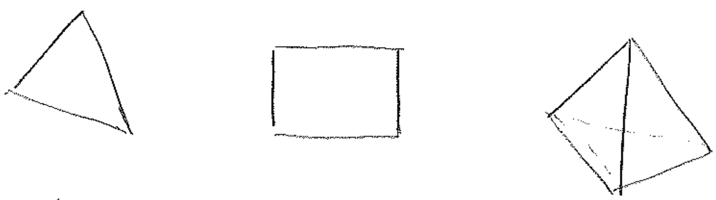
FE triple (K, P_K, Σ_K)

K: geometric domain

P_K: Space of functions

Σ_K degree of freedom $(\Sigma_K \subset P_K')$

K



P_1: linear poly

Σ_1:

$l_i(p) = p(x_i)$

nodal value at three vertices

(P_K, Σ_K) uni-solvent

$P_K = \operatorname{span}\{\phi_1, \cdots \phi_n\}$

$\Sigma_K = \operatorname{span}\{l_1 \cdots l_n\}$

s.t. if $l_i(p) = 0$ for $\forall i = 1 \cdots n \Rightarrow p \equiv 0$

$p = \sum_{i=1}^{n} c_i \phi_i \quad l_i(p) = \sum_{i=1}^{n} c_i \, l_j(\phi_i)$

$$\underset{\text{non-singular}}{\underset{\uparrow}{\begin{pmatrix} l_i(\phi_j) \end{pmatrix}}} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} l_1(p) \\ \vdots \\ l_n(p) \end{pmatrix}$$

Example 1

$P_1 = \operatorname{span}\{\lambda_1\ \lambda_2\ \lambda_3\}$

$\Sigma = \operatorname{span}\{l_1\ l_2\ l_3\}$

$\Rightarrow \quad (l_i(\phi_j)) = I_{3\times 3}$

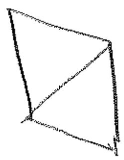
Example 2

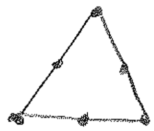
$P_2 = \operatorname{span}\{\lambda_1, \lambda_2, \lambda_3, \lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_1\lambda_3\}$

$\Sigma_2 = \operatorname{span}\{l_1\ l_2\ l_3\ l_1l_2\ l_1l_3\ l_2l_3\}$

In practice, it's better to construct a basis s.t. $l_i(\phi_j) = \delta_{ij}$ ③

construct a basis which is dual to Σ_2



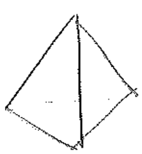


H^1 conforming

not in H^1
non-conforming

Edge element

K



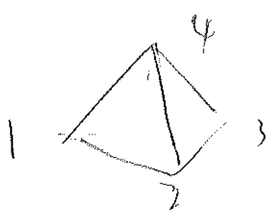
$P_K = \text{span}\{\lambda_i \nabla\lambda_j - \lambda_j \nabla\lambda_i \quad i<j\}$ ... 4 ?

$\Sigma_K = \text{span}\{ l_{ij} \quad i<j \}$ $\qquad l_{ij}(p) = \int_{E_{ij}} p\cdot t \, ds$

Direction E_{ij} is from smaller index to bigger index

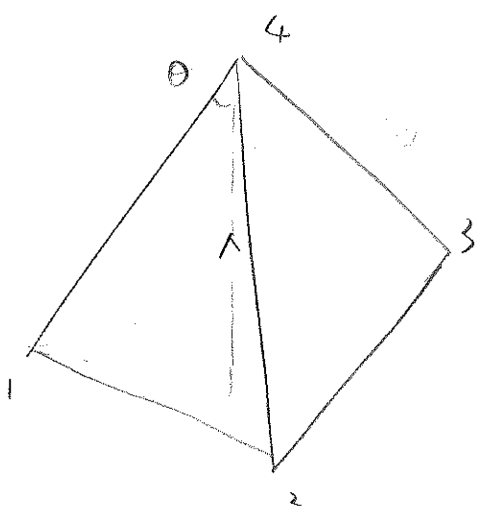
$\phi_1 \cdot t|_E = \text{constant}$

$\phi_1 = P_1^2(K)$



$l(i,j) = \phi_l = \lambda_i \nabla\lambda_j - \lambda_j \nabla\lambda_i$

$\nabla\lambda_1 \, t_{21} = 0 \qquad \lambda_1|_{t_{24}} = 0 \qquad \phi_1 \cdot t_1$



$\phi_l \, t_k = 0 \quad \text{if } k \neq l$

$\phi_l t_l$

$\nabla\lambda_4 \, t_{14} = |\nabla\lambda_4|\,|t_{14}| \cos\theta = 1$

$\nabla\lambda_1 \, t_{14} = -1$

$l_k(\phi_l) = 0 \quad \text{if } k \neq l$

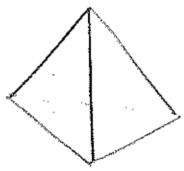
$l_l(\phi_l) = \int_{E_{ij}} ds = 1$

Edge-Element
dof is associated to edges

$\left(l_k(\phi_l) \right) = I_{6\times 6}$

Face Element

K:



$P_K = \text{span}\{ \lambda_i \nabla\lambda_j \times \nabla\lambda_k + \lambda_j \nabla\lambda_k \times \nabla\lambda_i + \lambda_k \nabla\lambda_i \times \nabla\lambda_j \}$

$\Sigma_K = \text{span}\{ l_{ijk},\ l_{ijk}(p) = \int_{F_{ijk}} p \cdot n\, ds \}$

$V(\text{curl}, \mathcal{T}) = \{ v|_K \in \text{edge element},\ \forall K \in \mathcal{T} \}$

$V(\text{curl}, \mathcal{T}) \subset H(\text{curl}, \mathcal{T})$?

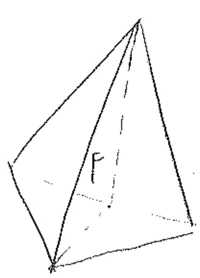
$V(\text{div}, \mathcal{T}) = \{ v|_K \in \text{face element},\ \forall K \in \mathcal{T} \}$

$V(\text{div}, \mathcal{T}) \subset H(\text{div}, \mathcal{T})$

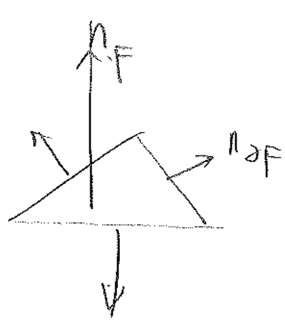
$\boxed{\begin{array}{l} p \text{ face element } \quad p\cdot n|_F = \text{constant} \quad \text{although } p \in P_1^3(K) \\ \int_F p\cdot n \text{ is continuous} \Rightarrow p\cdot n \text{ continuous} \end{array}}$

$p \cdot t_E|_E = \text{constant}$

$\int_E p \cdot t_E$ is continuous



$\Rightarrow p \times n$ is continuous ?



prove if $\int_E p \cdot t_E = 0 \quad \forall$ Edge E

then $p \times n|_F = 0$

Pf: $\int_F \nabla_F q\,(p \times n_F)$

$= -\int_F q \,\text{div}_F (p \times n_F) + \int_{\partial F} q\, n_{\partial F} (p \times n_F)$

$q \in P_0(F)$

$\int_{\partial F} q\, \underline{n_{\partial F} \cdot (p \times n_F)}$

$\stackrel{?}{=} p \cdot (n_F \times n_{\partial F}) = 0$

$\int_F \text{div}_F (p \times n_F) = 0 \Rightarrow \text{div}_F (p \times n_F) = 0$

$p \times n_F = \mathrm{curl}_F \phi$ $\quad p \cdot t|_E = (p \times n_F) \cdot n_E = \mathrm{curl}_F \phi \cdot n_E = \mathrm{grad}_F \phi \cdot t_E$

$p \cdot t|_E = 0$ implies $\mathrm{grad}_F \phi \cdot t_E = 0$

$\Rightarrow$ ϕ is constant on E $\Rightarrow$ ϕ is constant on F

$$\Rightarrow \quad p \times n_F = \mathrm{curl}_F \phi = 0$$

When local $\to$ global

be careful about the orientation of edges.

choose the unique direction for each edge and

record ~~the~~ sign global and local direction.