VARIATIONAL FORMULATION OF MAXWELL EQUATIONS

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In this note, we consider the variational formulation of Maxwell's equations. We first introduces the Sobolev spaces $H(\operatorname{curl};\Omega)$ and $H(\operatorname{div};\Omega)$, pertinent to the fields of electromagnetic theory. We addresses interface and boundary conditions, traces within Sobolev spaces, and the well-posedness of weak formulations.

1. INTRODUCTION

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. We introduce the Sobolev spaces

$$H(\operatorname{curl};\Omega) = \boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \operatorname{curl} \boldsymbol{v} \in \boldsymbol{L}^2(\Omega),$$
$$H(\operatorname{div};\Omega) = \boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \operatorname{div} \boldsymbol{v} \in \boldsymbol{L}^2(\Omega).$$

The intensity fields $(\boldsymbol{E}, \boldsymbol{H})$ belong to $H(\operatorname{curl};\Omega)$, while the flux fields $(\boldsymbol{D}, \boldsymbol{B})$ belong to $H(\operatorname{div};\Omega)$. We use the unified notation $H(\mathrm{d};\Omega)$ with $\mathrm{d} = \operatorname{grad}, \operatorname{curl}$, or div. Note that $H(\operatorname{grad};\Omega)$ is the familiar $H^1(\Omega)$ space. One can verify that $H(\mathrm{d};\Omega)$ is a Hilbert space with respect to the inner product

$$(u, v) + (\mathrm{d}u, \, \mathrm{d}v).$$

The norm for $H(\mathrm{d};\Omega)$ is the graph norm

$$\|u\|_{\mathrm{d},\Omega} := \left(\|u\|^2 + \|\mathrm{d}u\|^2\right)^{1/2}.$$

We recall the integration by parts for vector functions below. Formally, the boundary term is obtained by replacing the Hamilton operator ∇ with the unit outward normal vector $\boldsymbol{n}$. For example,

$$\int_\Omega \nabla u \cdot \boldsymbol{\phi} \, \mathrm{d}x = -\int_\Omega u \nabla \cdot \boldsymbol{\phi} \, \mathrm{d}x + \int_{\partial\Omega} \boldsymbol{n} u \cdot \boldsymbol{\phi} \, \mathrm{d}S,$$
$$\int_\Omega \nabla \times \boldsymbol{u} \cdot \boldsymbol{\phi} \, \mathrm{d}x = \int_\Omega \boldsymbol{u} \cdot \nabla \times \boldsymbol{\phi} \, \mathrm{d}x + \int_{\partial\Omega} \boldsymbol{n} \times \boldsymbol{u} \cdot \boldsymbol{\phi} \, \mathrm{d}S,$$
$$\int_\Omega \nabla \cdot \boldsymbol{u}, \phi \, \mathrm{d}x = -\int_\Omega \boldsymbol{u} \cdot \nabla \phi \, \mathrm{d}x + \int_{\partial\Omega} \boldsymbol{n} \cdot \boldsymbol{u}, \phi \, \mathrm{d}S.$$

The time-harmonic Maxwell equation for the electric field $\boldsymbol{E}$ is

$$\nabla \times (\mu^{-1} \nabla \times \boldsymbol{E}) - \omega^2 \tilde{\epsilon} \boldsymbol{E} = \tilde{\boldsymbol{J}}$$
$$\nabla \cdot (\epsilon \boldsymbol{E}) = \rho.$$

The time-harmonic Maxwell equation for the magnetic field $\boldsymbol{H}$ is

$$\nabla \times (\tilde{\epsilon}^{-1} \nabla \times \boldsymbol{H}) - \omega^2 \mu \boldsymbol{H} = \nabla \times \tilde{\boldsymbol{J}}$$
$$\nabla \cdot (\mu \boldsymbol{H}) = 0.$$

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These are obtained by the Fourier transform in time for the original Maxwell equations. Here, ω is a positive constant called the frequency. For the derivation and physical meaning of Maxwell's equations, we refer to Brief Introduction to Maxwell's Equations.

To simplify the discussion, we consider the following model problems:

Symmetric and positive definite problem:

$$\nabla \times (\alpha \nabla \times \boldsymbol{u}) + \beta \boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega, \qquad \boldsymbol{u} \times \boldsymbol{n} = 0 \quad \text{on } \partial\Omega \tag{1}$$

Saddle point system:

$$\nabla \times (\alpha \nabla \times \boldsymbol{u}) = \boldsymbol{f} \quad \text{in } \Omega, \qquad \nabla \cdot (\beta \boldsymbol{u}) = 0 \quad \text{in } \Omega, \qquad \boldsymbol{u} \times \boldsymbol{n} = 0 \quad \text{on } \partial\Omega. \tag{2}$$

where α and β are uniformly bounded, positive, and real coefficients. The right-hand side $\boldsymbol{f}$ is divergence-free, i.e., $\operatorname{div} \boldsymbol{f} = 0$ in the distribution sense.

2. Interface and Boundary Conditions

For a vector $\boldsymbol{u} \in \mathbb{R}^3$ and a unit norm vector $\boldsymbol{n}$, we can decompose $\boldsymbol{u}$ into the normal component and the tangential component as

$$\boldsymbol{u} = (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n} + \boldsymbol{n} \times (\boldsymbol{u} \times \boldsymbol{n}) = \boldsymbol{u}_n + \boldsymbol{u}_\tau.$$

The vector $\boldsymbol{u} \times \boldsymbol{n}$ is also on the tangent plane and orthogonal to the tangential component $\boldsymbol{u}_\tau$, which is a clockwise 90° rotation of $\boldsymbol{u}_\tau$ on the tangent plane. Consequently, $\boldsymbol{u} \times \boldsymbol{n}, \boldsymbol{u}_\tau, \boldsymbol{n}$ forms an orthogonal basis of $\mathbb{R}^3$; see Fig. 1.

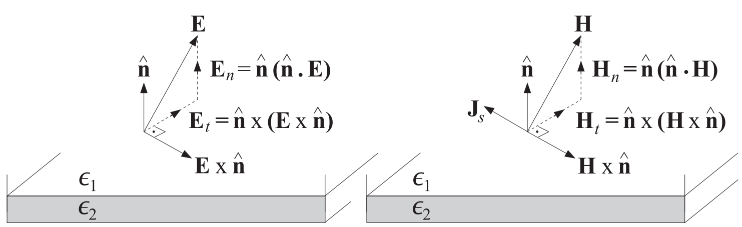

FIGURE 1. Field directions at boundary. Extract from *Electromagnetic Waves and Antennas* by Orfanidis [5].

The interface condition can be derived from the continuity requirement for piecewise smooth functions to be in $H(\mathrm{d};\Omega)$. Let $\Omega = K_1 \cup K_2 \cup S$ with interface $S = \bar{K}_1 \cap \bar{K}_2$. Let $u_i \in H(\mathrm{d}; K_i)$. Define $u \in L^2(\Omega)$ as

$$u = \begin{cases} u_1 & x \in K_1, \\ u_2 & x \in K_2. \end{cases}$$

We can always define the derivative $\mathrm{d}u$ in the distribution sense. To be a weak derivative, we need to verify that it coincides with the piecewise derivative, i.e.,

$$\mathrm{d}u = \begin{cases} \mathrm{d}u_1 & x \in K_1, \\ \mathrm{d}u_2 & x \in K_2. \end{cases}$$

To do so, let $\phi \in \mathcal{D}(\Omega)$, by the definition of the derivative of a distribution

$$\begin{aligned} \langle \mathrm{d}u, \phi \rangle &:= \langle u, \mathrm{d}^\phi \rangle = (u_1, \mathrm{d}^\phi) + (u_2, \mathrm{d}^\phi) \\ &= (\mathrm{d}u_1, \phi) + (\mathrm{d}u_2, \phi) + \langle \gamma_S(u_1 - u_2), \phi \rangle_S, \end{aligned}$$

where d^* is the adjoint of d in the L^2-inner product, and γ_S is an appropriate restriction of functions on the interface depending on the differential operators. The negative sign in front of u_2 is from the fact that the outward normal direction of K_2 is opposite to that of K_1. Then $u \in H(\mathrm{d};\Omega)$ if and only if

$$\begin{cases} u_1|_S = u_2|_S & \text{for } \mathrm{d} = \mathrm{grad}\,, \\ \boldsymbol{n} \times \boldsymbol{u}_1|_S = \boldsymbol{n} \times \boldsymbol{u}_2|_S & \text{for } \mathrm{d} = \mathrm{curl}\,, \\ \boldsymbol{n} \cdot \boldsymbol{u}_1|_S = \boldsymbol{n} \cdot \boldsymbol{u}_2|_S & \text{for } \mathrm{d} = \mathrm{div}\,. \end{cases}$$

Here, strictly speaking, the restriction operator $(\cdot)|_S$ should be replaced by appropriate trace operators, which will be discussed in the next section. So, for a function in $H(\mathrm{curl}\,;\Omega)$, its tangential component should be continuous across the interface, and for a function in $H(\mathrm{div};\Omega)$, its normal component should be continuous. This will be the key to constructing finite element spaces for these Sobolev spaces.

When the interface S contains surface charge ρ_S and surface current J_S, the interface condition for $\boldsymbol{H}$ and $\boldsymbol{D}$ is changed to

$$(\boldsymbol{H}_1 - \boldsymbol{H}_2) \times \boldsymbol{n} = \boldsymbol{J}_S, \qquad (\boldsymbol{D}_1 - \boldsymbol{D}_2) \cdot \boldsymbol{n} = \rho_S.$$

The interface condition for $\boldsymbol{H}$ can be built into the right-hand side of the weak formulation using a surface integral on S. The boundary condition can be thought of as an interface condition when one side of the interface is free space. The following are popular boundary conditions for Maxwell-type equations.

- If one side is a perfect conductor, then $\sigma = \infty$. By Ohm's law, to have a finite current, the electric field $\boldsymbol{E}$ should be zero. So we obtain the boundary condition $\boldsymbol{E} \times \boldsymbol{n} = 0$ for a perfect conductor.
- Impedance boundary condition •1

•1 *more on this*

$$n \times \boldsymbol{H} - \lambda \boldsymbol{E}_t = g.$$

3. Traces

The trace of functions in $H(\mathrm{d};\Omega)$ is not simply the restriction of the function values since the differential operator div or curl controls only a partial component of the vector function. The best way to look at the trace is, again, through integration by parts.

Recall that $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is the trace operator for H^1 functions. It is continuous and surjective. When u is also continuous on $\bar{\Omega}$, $\gamma u = u|_{\partial\Omega}$.

3.1. $H(\mathrm{div};\Omega)$ **space.** For functions $\boldsymbol{v} \in \boldsymbol{C}^1(\Omega), \phi \in C^1(\Omega)$ and Ω is a domain with a smooth boundary, we have the following integration by parts:

$$\int_\Omega \mathrm{div}\, \boldsymbol{v}\phi \,\mathrm{d}x = -\int_\Omega \boldsymbol{v} \cdot \mathrm{grad}\,\phi \,\mathrm{d}x + \int_{\partial\Omega} \boldsymbol{n} \cdot \boldsymbol{v}, \phi, \mathrm{dS}. \tag{3}$$

Then we relax the smoothness of functions and the domain such that (3) still holds. First, since for Lipschitz domains, the normal vector $\boldsymbol{n}$ of $\partial\Omega$ is well-defined almost everywhere, we can relax the smoothness of the domain Ω to be a bounded Lipschitz domain only. Second, we only need $\boldsymbol{v} \in H(\mathrm{div};\Omega)$ and $\phi \in H^1(\Omega)$ so that the volume integral is finite. Then (3) can be used to define the trace of $\boldsymbol{v} \in H(\mathrm{div};\Omega)$:

$$\langle \boldsymbol{n} \cdot \boldsymbol{v}, \gamma\phi \rangle_{\partial\Omega} := \int_\Omega \mathrm{div}\, \boldsymbol{v}\phi \,\mathrm{d}x + \int_\Omega \boldsymbol{v} \cdot \mathrm{grad}\,\phi \,\mathrm{d}x, \text{ for all } \phi \in H^1(\Omega). \tag{4}$$

In the left-hand side of (4), we change from a boundary integral to an abstract duality action, and $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is the trace operator for H^1 functions. Since γ is an onto map, $\gamma\phi$ will run over all $H^{1/2}(\partial\Omega)$ when ϕ runs over $H^1(\Omega)$. That is, $\boldsymbol{n}\cdot\boldsymbol{v}$ is a dual of $H^{1/2}(\partial\Omega)$. Note that $\partial(\partial\Omega)=0$. So the right space for $\boldsymbol{n}\cdot\boldsymbol{v}$ is $H^{-1/2}(\partial\Omega)$. We summarize as the following theorem.

Theorem 3.1 (Trace of $H(\mathrm{div};\Omega)$)**.** *Let $\Omega\subset\mathbb{R}^3$ be a bounded Lipschitz domain in $\mathbb{R}^3$ with unit outward normal $\boldsymbol{n}$. Then the mapping $\gamma_n : C^\infty(\bar\Omega)\to C^\infty(\partial\Omega)$ with $\gamma_n\boldsymbol{v}=\boldsymbol{n}\cdot\boldsymbol{v}|_{\partial\Omega}$ can be extended to a continuous linear map γ_n from $H(\mathrm{div};\Omega)$ onto $H^{-1/2}(\partial\Omega)$, namely*

$$\|\gamma_n\boldsymbol{v}\|_{-1/2,\partial\Omega}\lesssim\|\boldsymbol{v}\|_{\mathrm{div},\Omega}. \tag{5}$$

and the following Green's identity holds for functions $\boldsymbol{v}\in H(\mathrm{div};\Omega)$ and $\phi\in H^1(\Omega)$

$$\langle\gamma_n\boldsymbol{v},\gamma\phi\rangle_{\partial\Omega}=\int_\Omega\mathrm{div}\,\boldsymbol{v}\phi\,\mathrm{d}x+\int_\Omega\boldsymbol{v}\cdot\mathrm{grad}\,\phi\,\mathrm{d}x. \tag{6}$$

The space $H_0(\mathrm{div};\Omega)$ can be defined as

$$H_0(\mathrm{div};\Omega)=\{\boldsymbol{v}\in H(\mathrm{div};\Omega):\gamma_n\boldsymbol{v}=0\}.$$

Proposition 3.2. *The trace operator γ_n from $H(\mathrm{div};\Omega)$ onto $H^{-1/2}(\partial\Omega)$ is surjective and there exists a continuous right inverse. Namely for any $g\in H^{-1/2}(\partial\Omega)$, there exists a function $\boldsymbol{v}\in H(\mathrm{div};\Omega)$ such that $\gamma_n\boldsymbol{v}=g$ in $H^{-1/2}(\partial\Omega)$ and $\|\boldsymbol{v}\|_{\mathrm{div},\Omega}\lesssim\|g\|_{-1/2,\partial\Omega}$.*

Proof. For a given $g\in H^{-1/2}(\partial\Omega)$, let $f=-|\Omega|^{-1}\langle g,1\rangle$. We solve the Poisson equation $-\Delta p=f$ with Neumann boundary condition $\partial_n p=g$:

$$(\nabla p,\nabla\phi)=(f,\phi)+\langle g,\gamma\phi\rangle_{\partial\Omega}\quad\text{for all }\phi\in H^1(\Omega).$$

The existence and uniqueness of the solution $p\in H^1(\Omega)\cap L_0^2(\Omega)$ is ensured by the choice of f which satisfies the compatible condition with the boundary data g. By choosing $v\in H_0^1(\Omega)$, we conclude $-\Delta p=f$ in $L^2(\Omega)$, i.e., $\boldsymbol{v}=\nabla p$ is in $H(\mathrm{div};\Omega)$. Note that $\langle\gamma_n\boldsymbol{v},\gamma\phi\rangle=(\mathrm{div}\,\boldsymbol{v},\phi)+(\boldsymbol{v},\nabla\phi)=-(f,\phi)+(\nabla p,\nabla\phi)=\langle g,\gamma\phi\rangle$. Since $\gamma:H^1(\Omega)\to H^{1/2}(\partial\Omega)$ is surjective, we conclude $\gamma_n\boldsymbol{v}=g$ in $H^{-1/2}(\partial\Omega)$. That is, we found a function $\boldsymbol{v}\in H(\mathrm{div};\Omega)$ such that $\gamma_n\boldsymbol{v}=g$.

From the stability of $-\Delta$ operator, we have

$$\|\boldsymbol{v}\|=\|\nabla p\|\lesssim\|f\|+\|g\|_{-1/2}\lesssim\|g\|_{-1/2,\partial\Omega}.$$

Together with the identity $\|\mathrm{div}\,\boldsymbol{v}\|=\|f\|$, we obtain $\|\boldsymbol{v}\|_{\mathrm{div},\Omega}\lesssim\|g\|_{-1/2,\partial\Omega}$. □

3.2. $H(\mathrm{curl}\,;\Omega)$ **space.** Similarly, we can use the integration by parts

$$\int_\Omega\mathrm{curl}\,\boldsymbol{v}\cdot\phi\,\mathrm{d}x=\int_\Omega\boldsymbol{v}\cdot\mathrm{curl}\,\boldsymbol{\phi}\,\mathrm{d}x-\int_{\partial\Omega}(\boldsymbol{v}\times\boldsymbol{n})\cdot\boldsymbol{\phi}\,\mathrm{d}S$$

to define the trace of $H(\mathrm{curl}\,;\Omega)$. The trace only controls the tangential part of $\boldsymbol{v}|_{\partial\Omega}$.

Theorem 3.3 (Trace of $H(\mathrm{curl}\,;\Omega)$)**.** *Let $\Omega\subset\mathbb{R}^3$ be a bounded Lipschitz domain in $\mathbb{R}^3$ with unit outward normal $\boldsymbol{n}$. Then the mapping $\gamma_\tau : C^\infty(\bar\Omega)\to C^\infty(\partial\Omega)$ with $\gamma_\tau\boldsymbol{v}=\boldsymbol{v}|_{\partial\Omega}\times\boldsymbol{n}$ can be extended by continunity to a continuous linear map γ_τ from $H(\mathrm{curl}\,;\Omega)$ to $H^{-1/2}(\partial\Omega)$, namely*

$$\|\gamma_\tau\boldsymbol{v}\|_{-1/2,\partial\Omega}\lesssim\|\boldsymbol{v}\|_{\mathrm{curl}\,,\Omega}. \tag{7}$$

and the following Green's identity holds for functions $\boldsymbol{v}\in H(\mathrm{curl}\,;\Omega)$ and $\boldsymbol{\phi}\in\boldsymbol{H}^1(\Omega)$

$$\langle\gamma_\tau\boldsymbol{v},\gamma\phi\rangle_{\partial\Omega}=\int_\Omega\boldsymbol{v}\cdot\mathrm{curl}\,\boldsymbol{\phi}\,\mathrm{d}x-\int_\Omega\mathrm{curl}\,\boldsymbol{v}\cdot\boldsymbol{\phi}\,\mathrm{d}x. \tag{8}$$

The trace γ_τ from $H(\mathrm{curl}\,;\Omega)$ to $H^{-1/2}(\partial\Omega)$, however, is not surjective since in (8) the test function ϕ can be further extended from $H^1(\Omega)$ to $H(\mathrm{curl}\,;\Omega)$. To characterize the trace space exactly, we look closely at the boundary interaction. Let us denote by $\Gamma = \partial\Omega$ and introduce the tangential component trace π_τ as $\pi_\tau \boldsymbol{v} = \boldsymbol{v}_\tau = \boldsymbol{n} \times (\boldsymbol{v} \times \boldsymbol{n})$. The boundary pair can be written as

$$\langle \boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{\phi} \rangle_\Gamma = \langle \boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{\phi}_\tau \rangle_\Gamma = \langle \gamma_\tau \boldsymbol{v}, \pi_\tau \boldsymbol{\phi} \rangle_\Gamma. \tag{9}$$

Let $\mathrm{curl}_\Gamma, \mathrm{div}_\Gamma$ be the $\mathrm{curl}\,, \mathrm{div}$ operators on the boundary surface Γ, which can be defined intrinsically using metrics on the tangent planes. It is, however, advantageous to define through the operator ∇ in space and operations with the normal vector

$$\boldsymbol{n} \cdot (\nabla \times \boldsymbol{v}) = \mathrm{curl}_\Gamma(\pi_\tau \boldsymbol{v}) = \mathrm{div}_\Gamma(\gamma_\tau \boldsymbol{v}).$$

For a function $\boldsymbol{v} \in H(\mathrm{curl};\Omega)$, $\mathrm{curl}\,\boldsymbol{v} \in H(\mathrm{div};\Omega)$ since $\mathrm{div}\,\mathrm{curl}\,\boldsymbol{v} = 0$. Hence, $\gamma_n(\nabla \times \boldsymbol{v}) = \boldsymbol{n} \cdot (\nabla \times \boldsymbol{v}) = \mathrm{curl}_\Gamma(\pi_\tau \boldsymbol{v}) \in H^{-1/2}(\Gamma)$, implying $\pi_\tau \boldsymbol{v} \in H^{-1/2}(\mathrm{curl}_\Gamma;\Gamma)$. As its rotation, $\gamma_\tau \boldsymbol{v} \in H^{-1/2}(\mathrm{div}_\Gamma;\Gamma)$.

The duality pair in (9) is $\langle H^{-1/2}(\mathrm{div}_\Gamma, \Gamma), H^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)\rangle$. The exact characterization of the trace operator is given by

$$\gamma_\tau : H(\mathrm{curl};\Omega) \to H^{-1/2}(\mathrm{div}_\Gamma;\Gamma)$$

and this mapping is surjective. Detailed explanations can be found in [4] (pages 58–60) and [2, 1]. To verify the surjectivity of the mapping, a lifting operator analogous to Proposition 3.2 needs to be constructed for a given trace in $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. The construction of such a lifting operator is technical and was introduced by Tartar [6], also discussed in [1].

The space $H_0(\mathrm{curl}\,;\Omega)$ can be defined as

$$H_0(\mathrm{curl}\,;\Omega) = \{\boldsymbol{v} \in H(\mathrm{curl}\,;\Omega) : \gamma_\tau \boldsymbol{v} = 0\}.$$

4. Well-posedness of Weak Formulations

Let $V = H_0(\mathrm{curl};\Omega)$. The weak formulation of (1) is: given an $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, find $\boldsymbol{u} \in V$ such that

$$(\alpha\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}) + (\beta \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in V. \tag{10}$$

In (10), the first term arises from integration by parts

$$(\alpha\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}) = (\nabla \times (\alpha \times \nabla \times \boldsymbol{u}), \boldsymbol{v}) + (\alpha\nabla \times \boldsymbol{u}, \boldsymbol{n} \times \boldsymbol{v})_{\partial\Omega}$$

and choosing the test function $\boldsymbol{v} \in V$ to eliminate the boundary term. The boundary condition for $\boldsymbol{u}$ is of Dirichlet type: $\boldsymbol{u} \times \boldsymbol{n} = 0$ on $\partial\Omega$, or more precisely $\gamma_\tau \boldsymbol{u} = 0$.

Assuming the positive coefficients α and β are uniformly bounded below and above, the well-posedness of (10) is trivial since the bilinear form is equivalent to the inner product of $H(\mathrm{curl}\,;\Omega)$. The existence and uniqueness of the solution to (10) can be obtained by the Riesz representation theorem. However, the stability constant will be proportional to $1/\beta$ and thus will blow up as $\beta \to 0$. Unlike the Poisson equation, where $(\nabla u, \nabla v)$ defines an inner product on $H_0^1(\Omega)$, for the space $H_0(\mathrm{curl}\,;\Omega)$, the zero trace cannot handle the much larger kernel space of the curl operator, which consists of the image of ∇ for simply connected domains Ω. We will revisit this issue (robustness as $\beta \to 0^+$) after discussing the saddle point formulation.

For the saddle point formulation of Maxwell's equation (2), the natural Sobolev space for $\boldsymbol{u}$ is again $V = H_0(\mathrm{curl}\,;\Omega)$, and the bilinear form

$$a(\boldsymbol{u}, \boldsymbol{v}) := (\alpha\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}), \quad \text{for } \boldsymbol{u}, \boldsymbol{v} \in H_0(\mathrm{curl}\,;\Omega),$$

induces an operator $A : V \to V'$ such that $\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = a(\boldsymbol{u}, \boldsymbol{v})$.

However, as a function in the $H(\operatorname{curl}; \Omega)$ space, the divergence operator cannot be directly applied. It should be understood in the weak sense, i.e.,

$$-\langle \operatorname{div}^w(\beta \boldsymbol{u}), q\rangle := (\beta \boldsymbol{u}, \operatorname{grad} q) \quad \forall q \in Q := H_0^1(\Omega).$$

We define the bilinear form

$$b(\boldsymbol{v}, q) = (\beta \boldsymbol{v}, \operatorname{grad} q) = -(\operatorname{div}^w(\beta \boldsymbol{v}), q), \qquad \text{for } \boldsymbol{v} \in H_0(\operatorname{curl}; \Omega), q \in H_0^1(\Omega),$$

which induces the operator $B : V \to Q'$ such that $\langle B\boldsymbol{u}, q\rangle = b(\boldsymbol{u}, q)$ for all $q \in H_0^1(\Omega)$, and $B' : Q \to V'$ as the dual of B.

A Lagrangian multiplier $p \in H_0^1(\Omega)$ can be introduced to impose the constraint $\operatorname{div}^w(\beta \boldsymbol{u}) = 0$. Thus, we consider the inf-sup problem

$$\inf_{\boldsymbol{u} \in V} \sup_{p \in Q} \frac{1}{2}(\alpha \nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{u}) - (\boldsymbol{f}, \boldsymbol{u}) + (\beta \boldsymbol{u}, \nabla p).$$

The Euler-Lagrange equation is the following saddle point formulation of (2): given $\boldsymbol{f} \in V'$, find $\boldsymbol{u} \in V, p \in Q$ s.t.

$$\begin{pmatrix} A & B' \\ B & O \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ 0 \end{pmatrix},$$

which is the operator form of the mixed formulation

$$(\alpha \nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}) + (\beta \boldsymbol{v}, \nabla p) = (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in V, \tag{11a}$$

$$(\beta \boldsymbol{u}, \nabla q) = 0 \qquad \forall\, q \in Q. \tag{11b}$$

The well-posedness of the saddle point system (11) is a consequence of the inf-sup condition of B and the coercivity of A in the null space $X = \ker(B) = H_0(\operatorname{curl}; \Omega) \cap \ker(\operatorname{div}^w)$; see Inf-sup conditions for operator equations.

Lemma 4.1. *For $\beta = 1$, we have the inf-sup condition*

$$\inf_{p \in Q} \sup_{\boldsymbol{v} \in V} \frac{\langle B\boldsymbol{v}, p\rangle}{\|\boldsymbol{v}\|_{\operatorname{curl}} |p|_1} = 1. \tag{12}$$

Proof. Here we follow the convention in the Stokes equation to write out the formulation in term of the (negative) divergence operator B. It is more natural to show the adjoint $B' = \operatorname{grad} : H_0^1(\Omega) \to H_0'(\operatorname{curl}; \Omega)$ is injective. We can interpret

$$\|\nabla p\|_{V'} = \sup_{\boldsymbol{v} \in V} \frac{\langle B\boldsymbol{v}, p\rangle}{\|\boldsymbol{v}\|_{\operatorname{curl}}} = \sup_{\boldsymbol{v} \in V} \frac{(\boldsymbol{v}, \nabla p)}{\|\boldsymbol{v}\|_{\operatorname{curl}}},$$

and it suffices to prove

$$\|\nabla p\|_{V'} = \|\nabla p\|. \tag{13}$$

First by the Cauchy-Schwarz inequality and the definition of the curl norm, we have $\|\nabla p\|_{V'} \leq \|\nabla p\|$. To prove the inequality in the opposite direction, we simply chose $\boldsymbol{v} = \nabla p$. Then $\langle B\boldsymbol{v}, p\rangle = |p|_1^2$ and $\|\boldsymbol{v}\|_{\operatorname{curl}} = \|\boldsymbol{v}\| = |p|_1$. Therefore $\|\nabla p\|_{V'} \geq \|\nabla p\|$ by the definition of sup. □

Exercise 4.2. *For coefficients $\beta_{\min} \leq \beta \leq \beta_{\max}$, prove that*

$$\beta_{\min} |p|_1 \leq \sup_{\boldsymbol{v} \in V} \frac{\langle B\boldsymbol{v}, p\rangle}{\|\boldsymbol{v}\|_{\operatorname{curl}}} \leq \beta_{\max} |p|_1.$$

The coercivity in the null space $X = \ker(B) = H_0(\operatorname{curl}; \Omega) \cap \ker(\operatorname{div}^w)$ can be derived from the following Poincaré-type inequality.

Lemma 4.3 (Poincaré Inequality. Lemma 3.4 and Theorem 3.6 in [3])**.** *When Ω is simply connected and $\partial\Omega$ consists of only one component, we have*

$$\|\boldsymbol{v}\| \lesssim \|\operatorname{curl} \boldsymbol{v}\| \quad \textit{for } \boldsymbol{v} \in X. \tag{14}$$

A heuristic argument for the above Poincaré inequality is as follows: Using the identity $-\Delta \boldsymbol{u} = \operatorname{grad} \operatorname{div} \boldsymbol{u} + \operatorname{curl} \operatorname{curl} \boldsymbol{u}$, we find $\|\boldsymbol{u}\|_1 \eqsim \|\operatorname{curl} \boldsymbol{u}\|$ for $\boldsymbol{v} \in X$. Together with the Poincaré inequality $\|\boldsymbol{u}\| \lesssim \|\boldsymbol{u}\|_1$ for H^1 functions, we obtain the desired result. The subtlety in making this argument rigorous lies in the boundary condition. For $\boldsymbol{u} \in H_0(\operatorname{curl}; \Omega)$, only the tangential component is zero, whereas to apply the Poincaré inequality for H^1 vector functions, both the tangential and normal component traces should be zero.

A sketch of a proof for (14) is as follows: First show that the operator $\operatorname{curl} : X \to H$, where $H = H_0(\operatorname{div}; \Omega) \cap \ker(\operatorname{div})$, is one-to-one and continuous. Then, by the open mapping theorem, its inverse is also continuous, which leads to (14). For each $\boldsymbol{\psi} \in H$, that is, $\operatorname{div} \boldsymbol{\psi} = 0$, given the assumption of the domain Ω, there exists a vector potential $\boldsymbol{v}$ such that $\boldsymbol{\psi} = \operatorname{curl} \boldsymbol{v}$, which is not unique. However, if we further require that $\operatorname{div} \boldsymbol{v} = 0$ and impose the boundary condition $\boldsymbol{v} \times \boldsymbol{n} = 0$, then the potential is unique. Details can be found in [3, Chapter 1, Theorem 3.6]. The condition that Ω is simply connected and $\partial\Omega$ consists of only one component is necessary to eliminate the presence of non-trivial harmonic forms. We will refer to this condition as the "trivial topology" condition.

Another approach is through the compact embedding. By modifying the proof in [3, Chapter 1, Section 3.4], that is, using H^s-regularity instead of H^2-regularity of the Poisson equation, we can prove the following result.

Lemma 4.4. *For a Lipschitz polyhedron domain Ω, there exists a constant $s \in (1/2, 1]$ depending only on Ω such that*

$$X \hookrightarrow \boldsymbol{H}^s(\Omega)$$

and

$$\|\boldsymbol{v}\|_s \lesssim \|\boldsymbol{v}\|_{\operatorname{curl};\Omega}.$$

Consequently, X is compactly embedded in $\boldsymbol{L}^2(\Omega)$. When Ω is convex, $s = 1$.

With the compact embedding, we can adapt the proof for an H^1-type Poincaré inequality to obtain (14). Here is a sketch.

Proof of Lemma 4.3 using Lemma 4.4. Assume (14) does not hold. Then we can find a sequence $\{\boldsymbol{v}_n\} \subset X$ such that $\|\boldsymbol{v}_n\| = 1$ and $\|\operatorname{curl} \boldsymbol{v}_n\| \le \frac{1}{n} \to 0$ as $n \to +\infty$. Since $X \hookrightarrow \boldsymbol{L}^2(\Omega)$ is compact, we can find an L^2-convergent subsequence $\{\boldsymbol{v}_{n_k}\}$ that converges to an element $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$. Then, by the definition of weak derivatives and convergence in L^2, we can show that $\operatorname{curl} \boldsymbol{v} = 0$, $(\boldsymbol{v}, \nabla\phi) = 0$ for all $\phi \in H_0^1(\Omega)$, and $\|\boldsymbol{v}\| = 1$. Since γ_τ is continuous, we also have $\gamma_\tau \boldsymbol{v} = 0$, which implies $\boldsymbol{v} \in X$.

Then, there exists a scalar potential $p \in H_0^1(\Omega)$ such that $\boldsymbol{v} = \nabla p$. Taking $\phi = p$ in $(\boldsymbol{v}, \nabla\phi) = 0$, we obtain $\|\nabla p\| = 0$, and thus $p = 0$ and $\boldsymbol{v} = 0$. This contradicts the condition $\|\boldsymbol{v}\| = 1$. $\square$

We summarize the well-posedness as follows:

Theorem 4.5. *Let Ω be a Lipschitz polyhedron domain that is topologically trivial. Then there exists a unique solution $(\boldsymbol{u}, p)$ to the saddle point system (11) such that*

$$\|\boldsymbol{u}\| + \|\alpha^{1/2}\nabla \times \boldsymbol{u}\| + \|\beta^{1/2}\nabla p\| \lesssim \|\boldsymbol{f}\|_{V'}.$$

Furthermore, if $\operatorname{div} \boldsymbol{f} = 0$, then the Lagrange multiplier $p = 0$.

Proof. The well-posedness follows from Brezzi's theory. When $\operatorname{div} \boldsymbol{f} = 0$, choose the test function $\boldsymbol{v} = \nabla p$ in (11a) to obtain $\|\beta^{1/2}\nabla p\| = 0$, which implies $p = 0$ since $p \in H_0^1(\Omega)$. □

We now revisit the stability of the weak formulation (10), with an additional requirement that $\operatorname{div} \boldsymbol{f} = 0$. We consider the stability in the space X, where we can apply the Poincaré inequality (14) to ensure coercivity even when β is near 0.

Theorem 4.6. *Let Ω be a Lipschitz polyhedron domain, and let β be a positive constant. For a given $\boldsymbol{f} \in V'$ with $\operatorname{div} \boldsymbol{f} = 0$, there exists a unique solution $\boldsymbol{u}$ to the symmetric and positive definite problem* (10)*, and*

$$\|\boldsymbol{u}\|_{\mathrm{curl}} \lesssim \frac{1}{\alpha_{\min}} \|\boldsymbol{f}\|_{V'},$$

with a stability constant that is independent of β.

Proof. Since the bilinear form is equivalent to the inner product of $H(\mathrm{curl}\,;\Omega)$, the existence and uniqueness of the solution $\boldsymbol{u}$ to (10) can be derived from the Riesz representation theorem. Given that $\beta > 0$ and $\operatorname{div} \boldsymbol{f} = 0$, we select $\boldsymbol{v} = \nabla p$ in (10) to deduce that $\operatorname{div}^w \boldsymbol{u} = 0$, which implies $\boldsymbol{u} \in X$. We can then apply the Poincaré inequality (14) to establish coercivity:

$$\alpha_{\min}(\|\boldsymbol{u}\|^2 + \|\nabla \times \boldsymbol{u}\|^2) \lesssim a(\boldsymbol{u}, \boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{u}) \lesssim \|\boldsymbol{f}\|_{V'} \|\boldsymbol{u}\|_{\mathrm{curl}}\,,$$

from which the desired stability result follows. □

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