

SOBOLEV SPACES AND ELLIPTIC EQUATIONS

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Sobolev spaces are fundamental in the study of partial differential equations and their numerical approximations. In this chapter, we shall give brief discussions on the Sobolev spaces and the regularity theory for elliptic boundary value problems.

1. SOBOLEV SPACES

We shall state and explain main results (without proofs) on Sobolev spaces. We refer to [1] for comprehensive treatment.

1.1. Preliminaries. We first set up the environment of our discussion: Lipschitz domains, multi-index notation for differentiation, and some basic functional spaces.

A domain (a connected open set) $\Omega \subset \mathbb{R}^n$ is called a Lipschitz domain if its boundary $\partial\Omega$ can be locally represented by Lipschitz continuous function; namely for any $x \in \partial\Omega$, there exists a neighborhood of x , $G \subset \mathbb{R}^n$, such that $G \cap \partial\Omega$ is the graph of a Lipschitz continuous function under a proper local coordinate system.

All the smooth domains are Lipschitz. A non-smooth example is that every polygonal domain in \mathbb{R}^2 or polyhedron in \mathbb{R}^3 is Lipschitz. Every convex domain in \mathbb{R}^n is Lipschitz. A simple example of non-Lipschitz domain is two polygons touching at one vertex only and a more interesting one is a domain with cusp points on the boundary.

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, where \mathbb{Z}_+ is the set of non-negative integers, be a vector of nonnegative integers. Denote by $|\alpha| = \sum_{i=1}^n \alpha_i$. For a smooth function v and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, denote

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, \quad \text{and} \quad x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}.$$

Several basic Banach spaces will often be used in this book. $C(\bar{\Omega})$ is the space of continuous functions on $\bar{\Omega}$ with the usual maximum norm

$$\|v\|_{C(\bar{\Omega})} = \max_{x \in \bar{\Omega}} |v(x)|.$$

$C_0^\infty(\Omega)$ denotes the space of infinitely differential functions in Ω that vanish in some neighborhood of $\partial\Omega$; namely $\text{supp}(v) \subset \Omega$, where

$$\text{supp}(v) = \text{closure of } \{x \in \Omega : v(x) \neq 0\}.$$

Let $E(\Omega)$ represent the equivalent class of Lebesgue integrable functions corresponding to the equivalence $u \sim v$ if $u = v$ almost everywhere. Given $1 \leq p < \infty$, we shall denote the usual Lebesgue space of p th power integrable functions by $L^p(\Omega) = \{v \in E(\Omega) : \int_\Omega |v|^p dx < \infty\}$ and the essentially bounded function space by $L^\infty(\Omega)$. The norm is defined by

$$\|u\|_{p,\Omega} = \left(\int_\Omega |u|^p dx \right)^{1/p} \quad \text{for } 1 \leq p < \infty,$$

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and

$$\|u\|_{\infty, \Omega} = \operatorname{ess\,sup}_{\Omega} |u|.$$

Exercise 1.1. Prove that $L^p(\Omega)$, $1 \leq p \leq \infty$ defined above are Banach spaces.

1.2. Definition of Sobolev spaces. The conventional way to understand a function is through its point-wise values which is not adequate. It is better to understand a function as a functional through its action on a space of unproblematic test functions (conventional and well-behaved functions). In this way, the concept of functions can be generalized to distributions. “Integration by parts” is used to extend the differentiation operators from classic differentiable functions to distributions by shifting the operators to the test functions.

Sobolev spaces will be first defined here for integer orders using the concept of distributions and their weak derivatives. The fractional order Sobolev spaces will be introduced by looking at the p th power integrable of quotient of difference. Definitions will also be given to Sobolev spaces satisfying certain zero boundary conditions.

Distributions and weak derivatives. We begin with the nice function space $C_0^\infty(\Omega)$. Obviously $C_0^\infty(\Omega)$ is a real vector space and can be turned into a topological vector space by a proper topology. The space $C_0^\infty(\Omega)$ equipped with the following topology is denoted by $\mathcal{D}(\Omega)$: a sequence of functions $\{\phi_k\} \subset C_0^\infty(\Omega)$ is said to be convergent to a function $\phi \in C_0^\infty(\Omega)$ in the space $\mathcal{D}(\Omega)$ if

- (1) there exists a compact set $K \subset \Omega$ such that for all k , $\operatorname{supp}(\phi_k - \phi) \subset K$, and
- (2) for every $\alpha \in \mathbb{Z}_+^n$, we have $\lim_{k \rightarrow \infty} \|D^\alpha(\phi_k - \phi)\|_\infty = 0$.

The space, denoted by $\mathcal{D}'(\Omega)$, of all continuous linear functionals on $\mathcal{D}(\Omega)$ is called the (Schwarz) *distribution space*. The space $\mathcal{D}'(\Omega)$ will be equipped with the weak star topology. Namely, in $\mathcal{D}'(\Omega)$, a sequence T_n converge to T in the distribution sense if and only if $\langle T_n, \phi \rangle \rightarrow \langle T, \phi \rangle$ for all $\phi \in \mathcal{D}(\Omega)$, where $\langle \cdot, \cdot \rangle : \mathcal{D}'(\Omega) \times \mathcal{D}(\Omega) \rightarrow \mathbb{R}$ is the duality pair. A function ϕ belonging to $\mathcal{D}(\Omega)$ is called a *test function* since the action of a distribution on ϕ can be thought as a test. The property of a distribution can be extracted by clever choice of test functions.

Example 1.2. By the definition, an element in $\mathcal{D}'(\Omega)$ is uniquely determined by its action. The action could be very general and abstract as long as it is linear and continuous. As an example, let us introduce the Dirac delta distribution $\delta \in \mathcal{D}'(\Omega)$ with $0 \in \Omega \subseteq \mathbb{R}^n$ defined as

$$\langle \delta, \phi \rangle = \phi(0) \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

One important class of distributions is to use the integration as the action. A function is called *locally integrable* if it is Lebesgue integrable over every compact subset of Ω . We define the space $L_{\text{loc}}^1(\Omega)$ as the space containing all locally integrable functions. We can embed $L_{\text{loc}}^1(\Omega)$ into $\mathcal{D}'(\Omega)$ using the integration as the duality action. For a function $u \in L_{\text{loc}}^1(\Omega)$, define $T_u \in \mathcal{D}'(\Omega)$ as

$$\langle T_u, \phi \rangle := \int_{\Omega} u \phi \, dx \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

We shall still denote T_u by u . The correspondence $u \mapsto T_u$ is often used to identify an “ordinary” function as a distribution.

A distribution is often also known as a *generalized function* as the concept of distribution is a more general than the concept of the classic function. One of the basic distribution which is not an “ordinary” function is the Dirac δ -distribution introduced in Example

1.2. Indeed, one motivation of the invention of distribution space is to include Dirac delta “function”.

Exercise 1.3. Prove that it is not possible to represent delta distribution by a locally integrable function.

If u is a smooth function, it follows from integration by parts that, for any $\alpha \in \mathbb{Z}_+^n$

$$\int_{\Omega} D^{\alpha} u(x) \phi(x) \, dx = (-1)^{|\alpha|} \int_{\Omega} u(x) D^{\alpha} \phi(x) \, dx \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

There are no boundary terms, since ϕ has compact support in Ω and thus ϕ , together with its derivatives, vanishes near $\partial\Omega$. The above identity is the basis for defining derivatives for a distribution. If $T \in \mathcal{D}'(\Omega)$, then for any $\alpha \in \mathbb{Z}_+^n$, we define *weak derivative* $D^{\alpha}T$ as the distribution given by

$$\langle D^{\alpha}T, \phi \rangle = (-1)^{|\alpha|} \langle T, D^{\alpha}\phi \rangle \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

It is easy to see for a differentiable function, its weak derivative coincides with its classical derivative. But in general, the weak derivative is much weaker than the classical one such that the differential operator can be extended from differential functions to a much larger space – the space of distributions. For example, we can even talk about the derivative of a discontinuous function.

Example 1.4. The Heaviside step function is defined as $S(x) = 1$ for $x > 0$ and $S(x) = 0$ for $x < 0$. By the definition

$$\int_{\mathbb{R}} S' \phi \, dx = - \int_{\mathbb{R}} S \phi' \, dx = - \int_0^{\infty} \phi' \, dx = \phi(0).$$

Therefore $S' = \delta$ in the distribution sense but δ is not a function in $L_{\text{loc}}^1(\Omega)$ (Exercise 1.3). Roughly speaking, any distribution is locally a (multiple) derivative of a continuous function. A precise version of this result can be found at Rudin [10].

Integer order Sobolev spaces. The Sobolev space of index (k, p) , where k is a nonnegative integer and $p \geq 1$, is defined by

$$W^{k,p}(\Omega) := \{v \in L^p(\Omega) : D^{\alpha}v \in L^p(\Omega) \text{ for all } |\alpha| \leq k\},$$

with a norm $\|\cdot\|_{k,p,\Omega}$ given by

$$(1) \quad \|v\|_{k,p,\Omega}^p := \sum_{|\alpha| \leq k} \|D^{\alpha}v\|_{0,p(\Omega)}^p.$$

We will have occasions to use the seminorm $|\cdot|_{k,p,\Omega}$ given by

$$|v|_{k,p,\Omega}^p := \sum_{|\alpha|=k} \|D^{\alpha}v\|_{0,p,\Omega}^p.$$

For $p = 2$, it is customary to write $H^k(\Omega) := W^{k,2}(\Omega)$ which is a Hilbert space together with an inner product as follows

$$(u, v) = \sum_{|\alpha| \leq k} (D^{\alpha}u, D^{\alpha}v)$$

and the corresponding norm is denoted by $\|v\|_{k,\Omega} = \|v\|_{k,2,\Omega}$. The majority of our use of Sobolev space will be in this case.

We also define $W_{\text{loc}}^{k,p}(\Omega) = \{u \in E(\Omega) : \text{for any } U \subset\subset \Omega, u \in W^{k,p}(U)\}$ and $H_{\text{loc}}^k(\Omega) = W_{\text{loc}}^{k,2}(\Omega)$.

Exercise 1.5. Prove that $W^{k,p}(\Omega)$ is a Banach space.

For a function in $L^p(\Omega)$, treating it as a distribution, its weak derivatives always exists as distributions. But the weak derivative may not be in the space $L^p(\Omega)$. Therefore an element in $W^{k,p}(\Omega)$ possesses certain smoothness.

Example 1.6. We consider the Heaviside function restricted to $(-1, 1)$ and still denote by S . The weak derivative of S is Delta distribution which is not integrable. Therefore $S \notin H^1(-1, 1)$.

Example 1.7. Let $u(x) = |x|$ for $x \in (-1, 1)$ be an anti-derivative of $2(S - 1/2)$. Obviously $u \in L^2(-1, 1)$ and $u' \in L^2(-1, 1)$. Therefore $u \in H^1(-1, 1)$.

Example 1.6 and 1.7 explain that for a piecewise smooth function u to be in $H^1(\Omega)$ requires more global smoothness of u in Ω . See Exercise 1.18.

Fractional order Sobolev spaces. In the definition of classic derivatives, it takes the point-wise limit of the quotient of difference. For functions in Sobolev space, we shall use the p th power integrability of the quotient difference to characterize the differentiability.

For $0 < \theta < 1$ and $1 \leq p < \infty$, we define

$$(2) \quad W^{\theta,p}(\Omega) = \left\{ v \in L^p(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|v(x) - v(y)|^p}{|x - y|^{n+\theta p}} dx dy < \infty \right\}$$

and

$$H^{\theta}(\Omega) = W^{\theta,2}(\Omega).$$

In $W^{\theta,p}(\Omega)$, we define the following semi-norm

$$(3) \quad |v|_{\theta,p,\Omega}^p := \int_{\Omega} \int_{\Omega} \frac{|v(x) - v(y)|^p}{|x - y|^{n+\theta p}} dx dy,$$

and norm

$$(4) \quad \|v\|_{\theta,p,\Omega}^p := \|v\|_{0,p,\Omega}^p + |v|_{\theta,p,\Omega}^p.$$

In the definition of the fractional Sobolev space, the index of the dominator in the quotient of difference depends on the dimension of the space. This can be seen more clearly if we use polar coordinate in the integration. The fractional derivative seems weird. We now give a concrete example.

Example 1.8. For the Heavisde function S restricted to $(-1, 1)$, we look at the integral (3) in the skewed coordinate $\tilde{x} = x - y$, $\tilde{y} = x + y$. To be in the space $W^{\theta,p}(\Omega)$, it is essential to have the integral

$$\int_0^1 \frac{1}{\tilde{x}^{\theta p}} d\tilde{x} < \infty.$$

That is to require $\theta > 1/p$. In particular, we conclude $S \in H^{1/2-\epsilon}(-1, 1)$ for any $0 < \epsilon \leq 1/2$ but $S \notin H^{1/2}(-1, 1)$.

Given $s = k + \theta$ with a real number $\theta \in (0, 1)$ and an integer $k \geq 0$, define

$$W^{s,p}(\Omega) := \{v \in W^{k,p}(\Omega) : D^{\alpha}v \in W^{\theta,p}(\Omega), |\alpha| \leq k\}.$$

In $W^{s,p}(\Omega)$, we define the following semi-norm and norm

$$|v|_{s,p,\Omega} = \left(\sum_{|\alpha|=k} |D^{\alpha}v|_{\theta,p,\Omega}^p \right)^{1/p}, \quad \|v\|_{s,p,\Omega} = \left(\sum_{|\alpha| \leq k} \|D^{\alpha}v\|_{\theta,p,\Omega}^p \right)^{1/p}.$$

Negative order Sobolev spaces. $W^{k,p}(\Omega)$ is a Banach space, i.e., it is complete in the topology induced by the norm $\|\cdot\|_{k,p,\Omega}$. Indeed $W^{k,p}(\Omega)$ is the closure of $C^\infty(\Omega)$ with respect to $\|\cdot\|_{k,p,\Omega}$. The closure of $C_0^\infty(\Omega)$ with respect to the same topology is denoted by $W_0^{k,p}(\Omega)$. For $p = 2$, we usually write $H_0^k(\Omega) = W_0^{k,2}(\Omega)$. Roughly speaking for $u \in H_0^1(\Omega)$, $u|_{\partial\Omega} = 0$ in an appropriate sense. Except $k = 0$ or $\Omega = \mathbb{R}^n$, $W_0^{k,p}(\Omega)$ is a proper subspace of $W^{k,p}(\Omega)$.

For $k \in \mathbb{N}$, $W^{-k,p}(\Omega)$ is defined as the dual space of $W_0^{k,p'}(\Omega)$, where p' is the conjugate of p , i.e., $1/p + 1/p' = 1$. In particular $H^{-k}(\Omega) = (H_0^k(\Omega))'$, and for $f \in H^{-1}(\Omega)$

$$\|f\|_{-1,\Omega} = \sup_{v \in H_0^1(\Omega)} \frac{\langle f, v \rangle}{\|v\|_{1,\Omega}}.$$

Since $C_0^\infty(\Omega)$ is dense in $W_0^{k,p}(\Omega)$, $W^{-k,p}(\Omega)$ can be thought as a subspace of $\mathcal{D}'(\Omega)$. Furthermore, we have the following characterization of $W^{-k,p}(\Omega)$ [10] which says, roughly speaking, $D^{-k}v \in L^p$ if we can define D^{-k} in an appropriate way.

Theorem 1.9. *Let $v \in \mathcal{D}'(\Omega)$. Then $v \in W^{-k,p}(\Omega)$ if and only if*

$$v = \sum_{|\alpha| \leq k} D^\alpha v_\alpha, \quad \text{for some } v_\alpha \in L^p(\Omega).$$

Note that $C_0^\infty(\Omega)$ is not dense in $W^{k,p}(\Omega)$ for $k \in \mathbb{N}$. Therefore the dual space of $W^{k,p}(\Omega)$ cannot be embedded as a subspace of the distribution space.

Example 1.10. Since the Heavside function $S \in L^2(-1, 1)$, by Theorem 1.9, we have $\delta = S' \in H^{-1}(-1, 1)$. But this is only true in one dimension; see Example 1.11.

Characterization of Sobolev spaces using Fourier transform. When $\Omega = \mathbb{R}^n$, we can characterize the Sobolev space $H^k(\mathbb{R}^n)$ using the Fourier transform. Given a $v \in H^k(\mathbb{R}^n)$, by the obvious identity $\widehat{D^\alpha v} = (i\xi)^\alpha \hat{v}$ and the Plancherel's theorem, we deduce that

$$\|D^\alpha v\|_{0,\mathbb{R}^n} = \|\xi^\alpha \hat{v}\|_{0,\mathbb{R}^n}.$$

Thus, by definition

$$\|v\|_{k,\mathbb{R}^n}^2 = \int_{\mathbb{R}^n} \left(\sum_{|\alpha| \leq k} \xi^{2\alpha} \right) |\hat{v}(\xi)|^2 d\xi.$$

Using the elementary inequalities

$$(5) \quad (1 + |\xi|^2)^k \lesssim \sum_{|\alpha| \leq k} (|\xi|^2)^\alpha \lesssim (1 + |\xi|^2)^k,$$

we conclude that

$$\|v\|_{k,\mathbb{R}^n} \approx \|(1 + |\cdot|^2)^{k/2} \hat{v}\|_{0,\mathbb{R}^n}.$$

This relation shows that the Sobolev space $H^k(\mathbb{R}^n)$ may be equivalently defined by

$$H^k(\mathbb{R}^n) = \{v \in L^2(\mathbb{R}^n), (1 + |\xi|^2)^{k/2} \hat{v} \in L^2(\mathbb{R}^n)\}.$$

This alternative definition can be used to characterize Sobolev spaces with real index. Namely, for any given $s \in [0, \infty)$, the Sobolev space $H^s(\mathbb{R}^n)$ can be defined as follows:

$$(6) \quad H^s(\mathbb{R}^n) := \{v \in L^2(\mathbb{R}^n), (1 + |\xi|^2)^{s/2} \hat{v} \in L^2(\mathbb{R}^n)\} \quad \text{for all } s \in [0, \infty)$$

with a norm defined by

$$(7) \quad \|v\|_{s,\mathbb{R}^n} := \|(1 + |\cdot|^2)^{s/2} \hat{v}\|_{0,\mathbb{R}^n}.$$

Example 1.11. In \mathbb{R}^n , the Fourier transform of δ -distribution is one. By looking at the integration in the polar coordinate, we see $\delta \in H^{-s}(\mathbb{R}^n)$ for $s > n/2$ only. Therefore for $n \geq 2$, the delta distribution is not in $H^{-1}(\Omega)$.

1.3. Extension theorems. The extension theorem presented below is a fundamental result for Sobolev spaces. Fourier transform is a powerful tool. But unfortunately it only works for functions defined in the entire space \mathbb{R}^n . To generalize results proved on the whole \mathbb{R}^n to a bounded domain Ω , we can try to extend the function defined in $W^{k,p}(\Omega)$ to $W^{k,p}(\mathbb{R}^n)$. The extension of a function $u \in L^p(\Omega)$ is trivial. For example, we can simply set $u(x) = 0$ when $x \notin \Omega$ which is called zero extension. But such extension will create a bad discontinuity along the boundary and thus cannot control the norm of derivatives especially the boundary is non-smooth. The extension of Sobolev space $W^{k,p}(\Omega)$ is subtle. We only present the result here.

Theorem 1.12. *For any bounded Lipschitz domain Ω , for any $s \geq 0$ and $1 \leq p \leq \infty$, there exists a linear operator $E : W^{s,p}(\Omega) \rightarrow W^{s,p}(\mathbb{R}^n)$ such that*

- (1) $E u|_{\Omega} = u$, and
- (2) E is continuous. Namely there exists a constant $C(s, \Omega)$ which is increasing with respect to $s \geq 0$ such that, for all $1 \leq p \leq \infty$,

$$\|E v\|_{s,p,\mathbb{R}^n} \leq C(s, \Omega) \|v\|_{s,p,\Omega} \quad \text{for all } v \in W^{s,p}(\Omega).$$

Theorem 1.12 is well-known for integer order Sobolev spaces defined on smooth domains and the corresponding proof can be found in most text books on Sobolev spaces. But for Lipschitz domains and especially for fractional order spaces Theorem 1.12 is less well-known and the the proof of the theorem for these cases is quite complicated. For integer order Sobolev spaces on Lipschitz domains, we refer to Calderón [3] or Stein [11]. For fractional order Sobolev spaces on Lipschitz domains, we refer to the book by McLean [9].

1.4. Embedding theorems. Embedding theorems of Sobolev spaces are what make the Sobolev spaces interesting and important. The Sobolev spaces $W^{k,p}(\Omega)$ are defined using weak derivatives. The smoothness using weak derivatives is weaker than that using classic derivatives.

Example 1.13. In two dimensions, consider the function $u(x) = \ln |\ln |x||$ when $|x| < 1/e$ and $u(x) = 0$ when $|x| \geq 1/e$. It is easy to verify that $u \in H^1(\mathbb{R}^2)$. But u is unbounded, i.e., $u \notin C(\mathbb{R}^2)$.

Sobolev embedding theorem connects ideas of smoothness using “weak” and “classic” derivatives. Roughly speaking, it says that if a function is weakly smooth enough, then it implies certain classic smoothness with less order of smoothness.

Exercise 1.14. *Prove that if $u \in W^{1,1}(0,1)$, then $u \in L^\infty(0,1)$.*

We now present the general embedding theorem. For two Banach spaces B_1, B_0 , we say B_1 is continuously embedded into B_0 , denoted by $B_1 \hookrightarrow B_0$, if for any $u \in B_1$, it is also in B_0 and the embedding map is continuous, i.e., for all $u \in B_1$

$$\|u\|_{B_0} \lesssim \|u\|_{B_1}.$$

Theorem 1.15 (General Sobolev embedding). *Let $1 \leq p \leq \infty, k \in \mathbb{Z}_+$ and Ω be a bounded Lipschitz domain in \mathbb{R}^n .*

Case 1. $kp > n$

$$W^{k,p}(\Omega) \hookrightarrow C(\bar{\Omega}).$$

Case 2. $kp = n$

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \text{for all } q \in [1, \infty).$$

Furthermore

$$W^{n,1}(\Omega) \hookrightarrow C(\bar{\Omega}).$$

Case 3. $kp < n$

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \text{with } \frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$$

Exercise 1.16. Use the Fourier transform to prove that if $u \in H^s(\mathbb{R}^n)$ for $s > n/2$, then $u \in L^\infty(\mathbb{R}^n)$. Then use the extension theorem to prove similar results for Lipschitz domains Ω .

We take the following visualization of Sobolev spaces from DeVore [5] (page 93). This will give us a simple way to keep track of various results. We shall do this by using points in the upper right quadrant of the plane. The x -axis will correspond to the L^p spaces except that L^p is identified with $x = 1/p$ not with $x = p$. The y -axis will correspond to the order of smoothness. For example, the point $(1/p, k)$ represents the Sobolev space $W^{k,p}(\Omega)$. The line with slope n (the dimension of Euclidean spaces) passing through $(1/p, 0)$ is the demarcation line for embeddings of Sobolev spaces into $L^p(\Omega)$ (see Figure 1). Any Sobolev space with indices corresponding to a point above that line is embedded into $L^p(\Omega)$ and space below that line cannot be embedded into $L^p(\Omega)$.

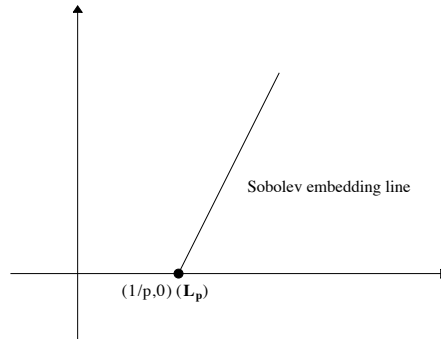


FIGURE 1. Sobolev embedding line.

To be quickly determine if a point lies above the demarcation line or not, we introduce the *Sobolev number*

$$\text{sob}_n(k, p) = k - \frac{n}{p}.$$

If $\text{sob}_n(k, p) > 0$, functions from $W^{k,p}(\Omega)$ are continuous (or more precisely can find a continuous representative in its equivalent class). In general

$$W^{k,p}(\Omega) \hookrightarrow W^{l,q}(\Omega) \quad \text{if } k > l \text{ and } \text{sob}_n(k, p) > \text{sob}_n(l, q).$$

Sobolev spaces corresponding to points on the demarcation line may or may not be embedded in $L^p(\Omega)$. For example, for $n \geq 1$

$$W^{n,1}(\Omega) \hookrightarrow L^\infty(\Omega) \quad \text{but } W^{1,n}(\Omega) \not\hookrightarrow L^\infty(\Omega)$$

Note that $W^{1,n}(\Omega) \hookrightarrow L^p(\Omega)$, for all $1 \leq p < \infty$. Furthermore there exists a constant $C(n, \Omega)$ depending on the dimension of Euclidean space and the Lipschitz domain such that

$$(8) \quad \|v\|_{0,p,\Omega} \leq C(n, \Omega) p^{1-1/n} \|v\|_{1,n,\Omega} \quad \text{for all } 1 \leq p < \infty.$$

Example 1.17. By the embedding theorem, we have

- (1) $H^1(\Omega) \hookrightarrow C(\bar{\Omega})$ in one dimension (Exercise 1.14)
- (2) $H^1(\Omega) \not\hookrightarrow C(\bar{\Omega})$ for $n \geq 2$ (Example 1.13). Namely there is no continuous representation of a function in $H^1(\Omega)$ for $n \geq 2$ and thus its point-wise value is not well defined.
- (3) In 2-D, $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for any $1 \leq p < \infty$; in 3-D, $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for $1 \leq p \leq 6$.

For a piecewise smooth function which is also in $H^1(\Omega)$, then it is globally continuous.

Exercise 1.18. Let Ω, Ω_1 and Ω_2 are three open and bounded domains in \mathbb{R}^n . Suppose $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. Let u be a function defined in Ω such that $u|_{\Omega_1} \in C^1(\Omega_1) \cap H^1(\Omega_1)$ and $u|_{\Omega_2} \in C^1(\Omega_2) \cap H^1(\Omega_2)$. Prove that $u \in H^1(\Omega)$ if and only if $u \in C(\Omega)$.

1.5. Trace Theorem. The trace theorem is to define function values on the boundary. If $u \in C(\bar{\Omega})$, then $u(x)$ is well defined for $x \in \partial\Omega$. But for $u \in W^{k,p}(\Omega)$, the function is indeed defined as an equivalent class of Lebesgue integrable functions, i.e., $u \sim v$ if and only if $u = v$ almost everywhere. The boundary $\partial\Omega$ is a measure zero set (in n th dimensional Lebesgue measure) and thus the point wise value of $u|_{\partial\Omega}$ is not well defined for functions in Sobolev spaces.

Theorem 1.19. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth or Lipschitz boundary $\Gamma = \partial\Omega$. Then the trace operator $\gamma : C^1(\bar{\Omega}) \mapsto C(\Gamma)$ can be continuously extended to $\gamma : H^1(\Omega) \mapsto H^{1/2}(\Gamma)$, i.e., the trace inequality holds:

$$(9) \quad \|\gamma(u)\|_{1/2,\Gamma} \lesssim \|u\|_{1,\Omega}, \quad \text{for all } u \in H^1(\Omega).$$

Furthermore the trace operator is surjective and has a continuous right inverse.

More general, if Ω is a Lipschitz domain, then the trace operator $\gamma : H^s(\Omega) \rightarrow H^{s-1/2}(\Gamma)$ is bounded for $1/2 < s < 3/2$. See McLean [9] (pages 100–106).

1.6. Norm Equivalence Theorem. The definition of norm $\|\cdot\|_{k+1,p}$ involves all the i th derivatives for $i \leq k+1$. But the highest one is the critical one. That is we can prove

$$\|u\|_{k+1,p} \approx \|u\|_{0,p} + |u|_{k+1,p} \quad \text{for all } u \in W^{k+1,p}(\Omega).$$

Considering this fact for the decomposition

$$W^{k+1,p}(\Omega) = \mathcal{P}_k(\Omega) \oplus (W^{k+1,p}(\Omega)/\mathcal{P}_k(\Omega)),$$

on the quotient space $W^{k+1,p}(\Omega)/\mathcal{P}_k(\Omega)$, the seminorm $|u|_{k+1,p}$ is indeed a norm.

Lemma 1.20 (Bramble-Hilbert). For any $u \in W^{k+1,p}(\Omega)$, we have

$$\inf_{p \in \mathcal{P}_k(\Omega)} \|u + p\|_{k+1,p,\Omega} \approx |u|_{k+1,p,\Omega}.$$

On the polynomial space $\mathcal{P}_k(\Omega)$, since it is finite dimensional, the norm $\|\cdot\|_{0,p}$ can be replaced by any other norms defined on $\mathcal{P}_k(\Omega)$. In general, we can consider a semi-norm $F(\cdot)$ on $W^{k+1,p}(\Omega)$, i.e., $F : W^{k+1,p}(\Omega) \rightarrow \mathbb{R}^+$ and

$$F(u+v) \leq F(u) + F(v), \quad \text{and} \quad F(\alpha u) \leq |\alpha|F(u).$$

The difference of a norm and a semi-norm is the null space $\ker(F)$ of a semi-norm is non-trivial. For a semi-norm F of $W^{k+1,p}(\Omega)$, with the condition: for $p \in \mathcal{P}_k(\Omega)$, $F(p) = 0$ if and only if $p = 0$, it will define a norm on the subspace $\mathcal{P}_k(\Omega)$.

The functional F do not needs to be continuous but *lower-semi-continuous* is enough, i.e., if $u_k \rightarrow u$ in $W^{k+1,p}$ topology, then

$$F(u) \leq \liminf_{k \rightarrow \infty} F(u_k).$$

Theorem 1.21 (Sobolev norm equivalence). *If F is a lower-semi-continuous seminorm on $W^{k+1,p}(\Omega)$ such that for $p \in \mathcal{P}_k(\Omega)$, $F(p) = 0$ if and only if $p = 0$, then*

$$(10) \quad \|v\|_{k+1,p} \approx F(v) + |v|_{k+1,p} \quad \text{for all } v \in W^{k+1,p}(\Omega).$$

As a special case of the Sobolev norm equivalence theorem, we present the following variants of Poincaré or Friedrichs inequalities. Assume that Γ is a measurable subset of $\partial\Omega$ with positive measure (in $n-1$ dimensional Lebesgue measure). Choosing $F(v) = |\int_{\Gamma} v ds|$ in (10), we obtain the Friedrichs inequality

$$\|v\|_{1,p,\Omega} \lesssim |v|_{1,p,\Omega} + \left| \int_{\Gamma} v ds \right| \quad \text{for all } v \in W^{1,p}(\Omega).$$

Consequently, we have the Poincaré inequality

$$\|v\|_{0,p} \lesssim |v|_{1,p,\Omega} \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

Choosing $F(v) = \int_{\Omega} v dx$, we obtain the Poincaré-Friedrichs inequality

$$\|v\|_{1,p,\Omega} \lesssim |v|_{1,p,\Omega} + \left| \int_{\Omega} v dx \right| \quad \text{for all } v \in W^{1,p}(\Omega).$$

Consequently, let $\bar{v} = \int_{\Omega} v dx / |\Omega|$ denote the average of v over Ω , we get the averaged Poincaré inequality

$$\|v - \bar{v}\|_{0,p,\Omega} \lesssim |v|_{1,p,\Omega} \quad \text{for all } v \in W^{1,p}(\Omega).$$

All constants hidden in the notation \lesssim depends on the size and shape of the domain Ω . This is important when apply them to one simplex with diameter h in finite element methods.

2. ELLIPTIC BOUNDARY VALUE PROBLEMS AND REGULARITY

2.1. Weak formulation. Let us take the Poisson equation with homogenous Dirichlet boundary condition

$$(11) \quad -\Delta u = f \text{ in } \Omega, \quad \text{and} \quad u|_{\partial\Omega} = 0,$$

as an example to illustrate the main idea. If there exists a function $u \in C^2(\Omega) \cap C_0(\Omega)$ satisfying the Poisson equation, we call u a *classic solution*. The smoothness of u excludes many interesting solutions for physical problems. We need to seek a weak solution in more broader spaces – Sobolev spaces. Here “weak” means the smoothness is imposed by weak derivatives.

Recall the basic idea of Sobolev space is to treat function as functional. Let us try to understand the equation (11) in the distribution sense. We seek a solution $u \in \mathcal{D}'(\Omega)$ such that for any $\phi \in C_0^\infty(\Omega)$,

$$\langle -\nabla \cdot \nabla u, \phi \rangle := \langle \nabla u, \nabla \phi \rangle = \langle f, \phi \rangle,$$

which suggests a weak formulation: find $u \in H_0^1(\Omega)$ such that

$$(12) \quad \int_{\Omega} \nabla u \cdot \nabla \phi \, dx = \int_{\Omega} f \phi \, dx, \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

But we do not discretize (12) directly since it is impossible to construct a finite dimensional subspace of $C_0^\infty(\Omega)$.

We first extend the action of ∇u on $C_0^\infty(\Omega)$ to a broader space. Let us define a bilinear form on $H^1(\Omega) \times C_0^\infty(\Omega)$:

$$a(u, \phi) = \int_{\Omega} \nabla u \cdot \nabla \phi \, dx.$$

By Cauchy-Schwarz inequality,

$$a(u, \phi) \leq \|\nabla u\| \|\nabla \phi\|.$$

Thus $a(u, \cdot)$ is continuous in the H^1 topology. Thanks to the fact $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, the bilinear form $a(\cdot, \cdot)$ can be continuously extend to $U \times V := H^1(\Omega) \times H_0^1(\Omega)$. Here the space U is the one we seek a solution and thus called *trial space* and V is still called *test space*. In (12) the right-hand side $f \in L^2(\Omega) \cong L^2(\Omega)'$. After $C_0^\infty(\Omega)$ is extend to $H_0^1(\Omega)$, we can take $f \in H^{-1}(\Omega) := (H_0^1(\Omega))' = V'$.

The boundary condition can be imposed by choosing different trial and test spaces. For example:

- homogenous Dirichlet boundary condition:

$$(13) \quad U = H_0^1(\Omega), \quad V = H_0^1(\Omega)$$

- homogenous Neumann boundary condition:

$$(14) \quad U = H^1(\Omega)/\mathbb{R}, \quad V = H^1(\Omega).$$

We are in the position to present the variational (or so-called weak) formulation of the Poisson equation: given an $f \in V'$, find a solution $u \in U$ such that

$$(15) \quad a(u, v) = \langle f, v \rangle \quad \text{for all } v \in V.$$

Exercise 2.1. Prove the existence and uniqueness of the weak solution for the Poisson equation with homogenous Dirichlet or Neumann boundary conditions.

2.2. Regularity theory. The weak solution u can be proved to be a solution of (11) in a more classic sense if u is smooth enough such that we can integration by parts back. The theory for proving the smoothness of the weak solution is called *regularity* theory, which is the bridge to connect classical and weak solutions.

First we assume right hand side $f \in L^2$ which will imply $\Delta u \in L^2(\Omega)$.

Theorem 2.2. Let u be the solution of (15). If $f \in L^2(\Omega)$, then $-\Delta u = f$ in $L^2(\Omega)$, i.e.

$$(16) \quad \int_{\Omega} -\Delta u v \, dx = \int_{\Omega} f v \, dx, \quad \forall v \in L^2(\Omega).$$

Furthermore for the choice of (13), u satisfies the boundary condition $u = 0$ on $\partial\Omega$, and for the choice of (14), $\nabla u \cdot \mathbf{n} = 0$ on $\partial\Omega$ in the trace sense.

Proof. When $f \in L^2(\Omega)$, $\langle f, v \rangle = (f, v)$. Choosing $v \in C_0^\infty(\Omega) \subset V$ in (15) implies that

$$\langle -\Delta u, v \rangle = (f, v), \quad \forall v \in C_0^\infty(\Omega).$$

That is $-\Delta u$ as a distribution is equal to $f \in L^2(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$ and both sides are continuous in L^2 topology, we conclude (16).

When $\Delta u \in L^2(\Omega)$, we are allowed to use a generalized Gauss theorem, for any $v \in H^1(\Omega)$:

$$(17) \quad \int_{\Omega} \nabla u \cdot \nabla v \, dx = - \int_{\Omega} \Delta u v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} v \, dS.$$

For the choice of (13), the boundary condition is built into the space. Now we prove the choice of (14) will lead to the Neumann boundary condition. We have already proved (16) and thus from (17) and (15), we conclude

$$\int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} v \, dS = 0.$$

Using the fact that $\gamma : H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega)$ is onto, we conclude $\nabla u \cdot \mathbf{n} = 0$ in $H^{-1/2}(\partial\Omega)$. \square

Next we will show $u \in H^2(\Omega)$ if $\Delta u \in L^2(\Omega)$ provided the domain is smooth. We first give a formal derivation for the model problem $-\Delta u = f$ in \mathbb{R}^n and assume u is smooth and vanishes sufficiently rapidly as $|x| \rightarrow \infty$ to justify the following integration by parts:

$$\begin{aligned} \int_{\mathbb{R}^n} f^2 \, dx &= \int_{\mathbb{R}^n} (\Delta u)^2 \, dx = \sum_{i,j=1}^n \int_{\mathbb{R}^n} \partial_{ii} u \partial_{jj} u \, dx \\ &= - \sum_{i,j=1}^n \int_{\mathbb{R}^n} \partial_{iij} u \partial_j u \, dx = \sum_{i,j=1}^n \int_{\mathbb{R}^n} \partial_{ij} u \partial_{ij} u \, dx = \int_{\mathbb{R}^n} |D^2 u|^2 \, dx. \end{aligned}$$

We can also see it from Fourier transform for the Laplacian operator on the whole space. Given a distribution v defined on \mathbb{R}^n such that $\Delta v \in L^2$, by the properties of Fourier transform, we have

$$\widehat{D^\alpha v}(\xi) = (i\xi)^\alpha \widehat{v}(\xi) = -(i\xi)^\alpha |\xi|^{-2} \widehat{\Delta v}(\xi).$$

The function $(i\xi)^\alpha |\xi|^{-2}$ is bounded by 1 if $|\alpha| = 2$, hence

$$\|\widehat{D^\alpha v}\|_{0, \mathbb{R}^n} \leq \|\widehat{\Delta v}\|_{0, \mathbb{R}^n}.$$

By Planchel identity, we have

$$\|D^\alpha v\|_{0, \mathbb{R}^n} \leq \|\Delta v\|_{0, \mathbb{R}^n} \quad \text{for all } |\alpha| = 2.$$

The above inequality illustrates an important fact that if v is a function such that $\Delta v \in L^2$, then all its second order derivatives are also in L^2 . This is a rather significant fact since Δv is a very special combination of the second order derivatives of v .

Smooth or bounded domains. The properties of elliptic operators, described above can be extended to bounded domains with smooth boundary, but such an extension is not trivial. The following theorem is well-known and it can be found in most of the text books on elliptic boundary value problems; see, e.g., [6].

Theorem 2.3. *Let Ω be a smooth and bounded domain of \mathbb{R}^n . Then for each $f \in L^2(\Omega)$, there exists a unique $u \in H^2(\Omega)$, the solution of (15), that satisfies*

$$\|u\|_{2,\Omega} \leq C\|f\|_{0,\Omega}$$

where C is a positive constant depending on Ω .

Here is an outline of the proof. For general domains, when $f \in L^2(\Omega)$, the solution $u \in H_{\text{loc}}^2(\Omega)$. Namely for each domain $U \subset\subset \Omega$, $u \in H^2(U)$. The local result can be proved by using the difference quotient to define a discrete approximation of D^2u . The condition $U \subset\subset \Omega$ enables us to choose a smooth cut-off function away from the boundary. By means of a suitable change of coordinates, the global regularity result can also be achieved, provided that the boundary of Ω is smooth enough. Locally it can be changed to \mathbb{R}_+^n . All second derivatives in x_1, \dots, x_{n-1} directions can be proved as before. The normal direction, i.e., $\partial_{x_n}^2 u$ can be proved using the equation: $\partial_{x_n}^2 u = -f - \sum_{i=1}^{n-1} \partial_{x_i}^2 u$.

Lipschitz domains. Theorem 2.3, however, does not hold on general Lipschitz domains. The requirements of Ω is necessary. When $\partial\Omega$ is only Lipschitz continuous, we may not be able to glue boundary pieces on the corner points. This is not only a technical difficulty. We now give an example in which the regularity of u depends on the shape of Ω .

Example 2.4. Let us give a simple counter example. Given $\beta \in (0, 1)$, consider the following nonconvex domain

$$\Omega = \{(r, \theta) : 0 < r < 1, 0 < \theta < \pi/\beta\}.$$

Let $v = r^\beta \sin(\beta\theta)$. Being the imaginary part of the complex analytic function z^β , v is harmonic in Ω . Define $u = (1 - r^2)v$. A direct calculation shows that

$$-\Delta u = 4(1 + \beta)v \text{ in } \Omega, \quad \text{and } u|_{\partial\Omega} = 0.$$

Note that $4(1 + \beta)v \in L^\infty(\Omega) \subset L^2(\Omega)$, but $u \notin H^2(\Omega)$.

Nevertheless, a slightly weaker result does hold for general Lipschitz domains.

Theorem 2.5. *Assume that Ω is a bounded Lipschitz domain. Then there exists a constant $\alpha \in (0, 1]$ such that*

$$(18) \quad \|u\|_{1+\alpha} \leq C\|f\|_{\alpha-1},$$

for the solution u of (15), where C is a constant depending on the domain Ω .

Again the proof for the above theorem is quite complicated, we refer to [8, 4, 2].

Convex domains. A remarkable fact is that we can take $\alpha = 1$ in the above theorem for convex domains. This means that Theorem 2.3 can be extended to convex domains.

Theorem 2.6. *Let Ω be a convex, bounded domain of \mathbb{R}^n . Then for each $f \in L^2(\Omega)$, there exists a unique $u \in H^2(\Omega)$ to the solution of (15) that satisfies*

$$\|u\|_{2,\Omega} \leq C\|f\|_{0,\Omega}$$

where C is a positive constant depending only on the diameter of Ω and the coefficients of \mathcal{L} .

Concave polygons. When Ω is a concave polygon in \mathbb{R}^2 , we do not have full regularity, i.e., $\alpha < 1$. There are at least two ways to obtain analogous (but weaker) results for concave domain. The first one (see [8, 4, 2]) is to use fractional Sobolev spaces: namely $-\Delta : H_0^1(\Omega) \cap H^{1+s}(\Omega) \mapsto H^{-1+s}(\Omega)$ which holds for any $s \in [0, s_0)$ for some $s_0 \in (0, 1]$ depending on Ω .

Another approach is to use L^p space, instead of L^2 . It can be shown that (c.f. [7])

$$(19) \quad -\Delta : H_0^1(\Omega) \cap W^{2,p}(\Omega) \mapsto L^p(\Omega) \text{ is an isomorphism}$$

which holds for any $p \in (1, p_0)$ for some $p_0 > 1$ that depends on the domain Ω . We shall discuss more in the discussion of adaptive finite element methods.

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