

SUBSPACE CORRECTION METHOD AND AUXILIARY SPACE METHOD

LONG CHEN

We discuss the subspace correction method and the auxiliary space method developed by Xu [5, 7, 8] for solving the linear operator equation

$$(1) \quad Au = f,$$

posed on a finite-dimensional Hilbert space $\mathbb{V} \cong \mathbb{R}^N$ equipped with an inner product (\cdot, \cdot) . Here $A : \mathbb{V} \mapsto \mathbb{V}$ is a *symmetric and positive definite (SPD)* operator, $f \in \mathbb{V}$ is given, and we seek $u \in \mathbb{V}$ such that (1) holds.

1. SPACE DECOMPOSITION AND SUBSPACE CORRECTION METHODS

In the spirit of divide and conquer, we decompose the space \mathbb{V} as a sum of subspaces and correspondingly decompose problem (1) into subproblems of smaller size that are easier to solve. This approach was developed by Xu [5].

Let $\mathbb{V}_i \subset \mathbb{V}$, $i = 1, \dots, J$, be subspaces of \mathbb{V} . If

$$\mathbb{V} = \sum_{i=1}^J \mathbb{V}_i,$$

then $\{\mathbb{V}_i\}$ is called a *space decomposition* of \mathbb{V} . By definition, for any $u \in \mathbb{V}$, we can write

$$u = \sum_{i=1}^J u_i, \quad u_i \in \mathbb{V}_i, \quad i = 1, \dots, J.$$

Since the sum $\sum_{i=1}^J \mathbb{V}_i$ is not necessarily direct, such decompositions of u are in general not unique. We include background on sums and products of linear spaces in the appendix.

We introduce the following operators for $i = 1, \dots, J$:

- $I_i : \mathbb{V}_i \hookrightarrow \mathbb{V}$, the natural inclusion;
- $Q_i : \mathbb{V} \mapsto \mathbb{V}_i$, the projection in the inner product (\cdot, \cdot) ;
- $A_i : \mathbb{V}_i \mapsto \mathbb{V}_i$, the restriction of A to the subspace $\mathbb{V}_i \times \mathbb{V}_i$;
- $R_i : \mathbb{V}_i \mapsto \mathbb{V}_i$, an approximation of A_i^{-1} , often referred to as a smoother.

We explore relations between these operators. By definition,

$$(Q_i^\top v_i, v) = (v_i, Q_i v) = (v_i, v) = (I_i v_i, v), \quad \forall v_i \in \mathbb{V}_i, v \in \mathbb{V},$$

so Q_i^\top coincides with the natural inclusion I_i , or equivalently $I_i^\top = Q_i$. At the continuous level, I_i is simply the identity on \mathbb{V}_i , and the notation is often omitted. In implementation, the prolongation matrix is the representation of I_i with respect to chosen bases, and its transpose I_i^\top acts as the restriction matrix.

The operator A can be viewed through the bilinear form on $\mathbb{V} \times \mathbb{V}$ defined by

$$a(u, v) := (u, v)_A = (Au, v) = (u, Av), \quad u, v \in \mathbb{V}.$$

The restriction of $a(\cdot, \cdot)$ to $\mathbb{V}_i \times \mathbb{V}_i$ induces the operator A_i through

$$(A_i u_i, v_i) = (u_i, v_i)_{A_i} := a(u_i, v_i) = (u_i, v_i)_A, \quad u_i, v_i \in \mathbb{V}_i.$$

Thus A_i is the Galerkin projection of A , i.e.,

$$A_i = I_i^T A I_i.$$

The consistent notation for the smoother R_i would be B_i , the preconditioner of A_i . We keep R_i to avoid conflict with the notation B for the preconditioner of the full operator A .

We now describe the subspace correction method. For a given residual r , let $r_i = Q_i r = I_i^T r$ denote the restriction of the residual to the subspace \mathbb{V}_i . In each subspace, we consider the residual equation $A_i e_i^* = r_i$ and approximate its solution by $e_i = R_i r_i$. The diagram below summarizes the relation between the global and local problems:

$$\begin{array}{ccc} \mathbb{V} & \xleftarrow{A^{-1}} & \mathbb{V} \\ \uparrow I_i & & \downarrow I_i^T \\ \mathbb{V}_i & \xleftarrow{R_i} & \mathbb{V}_i \end{array} \quad \begin{array}{ccc} e & \xleftarrow{A^{-1}} & r \\ \uparrow I_i & & \downarrow I_i^T \\ e_i & \xleftarrow{R_i} & r_i \end{array}$$

The local corrections e_i are then combined to form a correction in the full space \mathbb{V} , which gives the *subspace correction method*.

There are two basic ways to assemble subspace corrections.

Parallel Subspace Correction (PSC). In this method, the corrections on all subspaces are carried out in parallel. In operator form, the iteration is

$$(2) \quad u_{k+1} = u_k + B_a(f - Au_k),$$

where

$$(3) \quad B_a = \sum_{i=1}^J I_i R_i I_i^T.$$

The correction on each subspace \mathbb{V}_i is

$$e_i = I_i R_i I_i^T (f - Au_k), \quad i = 1, \dots, J,$$

and the total correction in \mathbb{V} is $\sum_{i=1}^J e_i$.

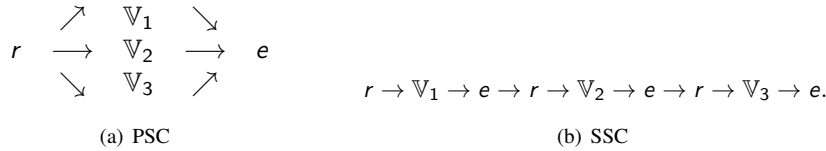


FIGURE 1. Illustration of PSC and SSC.

Successive Subspace Correction (SSC). In this method, the corrections are applied one after another. In operator form, the iteration is

$$(4) \quad v^1 = u_k, \quad v^{i+1} = v^i + I_i R_i I_i^T (f - Av^i), \quad i = 1, \dots, J, \quad u_{k+1} = v^{J+1}.$$

The iterator B_m associated with the SSC method does not have a simple closed form; it will be discussed in the next section.

We present algorithms of PSC and SSC in the following form to emphasis it is a procedure to solve the residual equation, i.e., given a residual r , return a correction e . One iteration of PSC or SSC can be used as a preconditioner in PCG.

```

function e = PSC(r)
% Solve the residual equation Ae = r by PSC method
for i = 1:J
    ri = Ii'*r;    % restrict the residual to subspace
    ei = Ri*ri;    % solve the residual equation in subspace
    e = e + Ii*ei; % prolongate the correction to the big space
end

function e = SSC(r)
% Solve the residual equation Ae = r by SSC method
rd = r;
for i = 1:J
    ri = Ii'*rd;    % restrict the residual to subspace
    ei = Ri*ri;    % solve the residual equation in subspace
    e = e + Ii*ei; % prolongate the correction to the big space
    rd = r - A*e;  % update residual
end

```

Comparing the PSC and SSC methods, we immediately see that in SSC the residual is updated after each subspace correction, while in PSC it is not. In terms of convergence, SSC is superior. On the other hand, PSC is embarrassingly parallel, whereas SSC is essentially a sequential method.

Example 1.1. Consider the matrix equation

$$\mathbf{A}\mathbf{u} = \mathbf{f},$$

where \mathbf{A} is an $N \times N$ SPD matrix. Let $\{\mathbf{e}_i\}_{i=1}^N$ be the canonical basis of \mathbb{R}^N . Take the trivial space decomposition $\mathbb{R}^N = \sum_{i=1}^N \mathbb{V}_i$ with $\mathbb{V}_i = \text{span}\{\mathbf{e}_i\}$. Then

$$I_i \mathbf{c} = c \mathbf{e}_i, \quad Q_i \mathbf{r} = r_i, \quad A_i = I_i^T \mathbf{A} I_i = a_{ii}.$$

- For $R_i = \alpha$, PSC reduces to the Richardson method.
- For $R_i = A_i^{-1} = a_{ii}^{-1}$, PSC becomes the Jacobi method.
- For $R_i = A_i^{-1} = a_{ii}^{-1}$, SSC becomes the Gauss–Seidel method.

The only nontrivial statement is that Gauss–Seidel is SSC applied to this trivial space decomposition. We now verify this relation.

Introduce the dynamically updated vector

$$\mathbf{v}^i = (u_{k+1}^1, \dots, u_{k+1}^i, u_k^{i+1}, \dots, u_k^N),$$

so $\mathbf{v}^0 = \mathbf{u}_k$, $\mathbf{v}^N = \mathbf{u}_{k+1}$, and

$$\mathbf{v}^i - \mathbf{v}^{i-1} = I_i(u_{k+1}^i - u_k^i).$$

The componentwise Gauss–Seidel update is

$$\begin{aligned}
 u_{k+1}^i &= a_{ii}^{-1} \left(f_i - \sum_{j=1}^{i-1} a_{ij} u_{k+1}^j - \sum_{j=i+1}^N a_{ij} u_k^j \right) \\
 &= u_k^i + a_{ii}^{-1} \left(f_i - \sum_{j=1}^{i-1} a_{ij} u_{k+1}^j - \sum_{j=i}^N a_{ij} u_k^j \right) \\
 &= u_k^i + a_{ii}^{-1} Q_i(f - A\mathbf{v}^{i-1}).
 \end{aligned}$$

Hence the Gauss–Seidel iteration can be written as

$$\mathbf{v}^i = \mathbf{v}^{i-1} + I_i a_{ii}^{-1} Q_i(f - A\mathbf{v}^{i-1}), \quad i = 1, \dots, N,$$

which is exactly SSC applied to $\mathbb{R}^N = \sum_{i=1}^N \mathbb{V}_i$.

Later we will also view PSC as a Jacobi method and SSC as a Gauss–Seidel method for a larger system formed in the product space of the subspaces. \square

Example 1.2. Multigrid methods can be interpreted as the successive subspace correction (SSC) method applied to a nested space decomposition

$$\mathbb{V}_1 \subset \mathbb{V}_2 \subset \cdots \subset \mathbb{V}_J = \mathbb{V}.$$

See *Programming of Multigrid Methods* for more details. \square

We introduce two additional operators from \mathbb{V} to \mathbb{V}_i to express the error operators and to analyze the convergence of PSC and SSC. Define

- $P_i : \mathbb{V} \rightarrow \mathbb{V}_i$, the projection with respect to $(\cdot, \cdot)_A$;
- $T_i : \mathbb{V} \rightarrow \mathbb{V}_i$, defined by $T_i = R_i Q_i A = R_i A_i P_i$.

From the definition,

$$A_i P_i = Q_i A,$$

so the diagram on the left is commutative:

$$\begin{array}{ccc} \mathbb{V} & \xrightarrow{A} & \mathbb{V} \\ \downarrow P_i & & \downarrow Q_i \\ \mathbb{V}_i & \xrightarrow{A_i} & \mathbb{V}_i \end{array} \quad \begin{array}{ccc} \mathbb{V} & \xrightarrow{A} & \mathbb{V} \\ \downarrow T_i & & \downarrow Q_i \\ \mathbb{V}_i & \xleftarrow{R_i} & \mathbb{V}_i \end{array}$$

The operator $T_i = R_i Q_i A = R_i A_i P_i$ is defined so that the diagram on the right is commutative. When $R_i = A_i^{-1}$, we obtain $T_i = P_i = A_i^{-1} Q_i A$, so T_i can be viewed as an approximation of P_i . Restricted to \mathbb{V}_i , the projection P_i becomes the identity, and thus

$$T_i|_{\mathbb{V}_i} = T_i I_i^\top = R_i A_i.$$

With a slight abuse of notation, we write $T_i^{-1} = (T_i|_{\mathbb{V}_i})^{-1}$. The actions of T_i and T_i^{-1} satisfy

$$(T_i u_i, u_i)_A = (R_i A_i u_i, A_i u_i), \quad (T_i^{-1} u, u)_A = (R_i^{-1} u, u).$$

With the prolongation operator I_i , the composite operator $I_i T_i : \mathbb{V} \rightarrow \mathbb{V}$ is often used. In many contexts, we still write it as T_i since I_i acts as the identity on elements of \mathbb{V}_i when embedded in \mathbb{V} . All three operators

$$T_i : \mathbb{V} \rightarrow \mathbb{V}_i, \quad T_i I_i : \mathbb{V}_i \rightarrow \mathbb{V}_i, \quad I_i T_i : \mathbb{V} \rightarrow \mathbb{V}$$

are usually denoted by T_i , and the intended meaning is clear from the context.

We now state the error formulas for PSC and SSC:

- *Parallel Subspace Correction (PSC):*

$$u - u_{k+1} = \left[I - \sum_{i=1}^J T_i \right] (u - u_k).$$

- *Successive Subspace Correction (SSC):*

$$u - u_{k+1} = \left[\prod_{i=1}^J (I - T_i) \right] (u - u_k).$$

Thus PSC is also called the *additive* method, while SSC is the *multiplicative* method. In the notation $\prod_{i=1}^J a_i$, we assume the following ordering:

$$\prod_{i=1}^J a_i = a_J \cdots a_2 a_1.$$

2. AUXILIARY SPACE METHODS

In this section, we present a variation of the fictitious space method of Nepomnyaschikh [4] and the auxiliary space method of Xu [6]. We follow the presentation in [1].

Let $\tilde{\mathbb{V}}$ and \mathbb{V} be two Hilbert spaces and let $\Pi : \tilde{\mathbb{V}} \rightarrow \mathbb{V}$ be a surjective linear map. Denote by $\Pi^\top : \mathbb{V} \rightarrow \tilde{\mathbb{V}}$ the adjoint of Π with respect to the underlying inner products, i.e.,

$$(\Pi^\top u, \tilde{v}) = (u, \Pi \tilde{v}) \quad \forall u \in \mathbb{V}, \tilde{v} \in \tilde{\mathbb{V}}.$$

For notational simplicity we use (\cdot, \cdot) for the inner products on both \mathbb{V} and $\tilde{\mathbb{V}}$. Since Π is surjective, its adjoint Π^\top is injective.

Given an SPD operator $A : \mathbb{V} \rightarrow \mathbb{V}$, define its lifted operator $\tilde{A} = \Pi^\top A \Pi : \tilde{\mathbb{V}} \rightarrow \tilde{\mathbb{V}}$. Note that \tilde{A} is possibly singular with $\ker(\tilde{A}) = \ker(\Pi)$. To construct a preconditioner of A , we find a preconditioner of \tilde{A} and then project it back. Let $\tilde{B} : \tilde{\mathbb{V}} \rightarrow \tilde{\mathbb{V}}$ be SPD and define

$$B := \Pi \tilde{B} \Pi^\top : \mathbb{V} \rightarrow \mathbb{V}.$$

As Π^\top is injective, B is also SPD. The relationships among these operators are summarized in the following diagram:

$$\begin{array}{ccc} \tilde{\mathbb{V}} & \xrightleftharpoons[\tilde{B}]{\tilde{A}} & \tilde{\mathbb{V}} \\ \downarrow \Pi & & \uparrow \Pi^\top \\ \mathbb{V} & \xrightleftharpoons[B]{A} & \mathbb{V} \end{array}$$

Theorem 2.1. *Let $\tilde{\mathbb{V}}$ and \mathbb{V} be two Hilbert spaces, and let $\Pi : \tilde{\mathbb{V}} \rightarrow \mathbb{V}$ be a surjective map. Let $\tilde{B} : \tilde{\mathbb{V}} \rightarrow \tilde{\mathbb{V}}$ be a symmetric and positive definite operator. Then the operator*

$$B := \Pi \tilde{B} \Pi^\top : \mathbb{V} \rightarrow \mathbb{V}$$

is also symmetric and positive definite. Moreover,

$$(5) \quad (B^{-1}v, v) = \inf_{\Pi \tilde{v} = v} (\tilde{B}^{-1}\tilde{v}, \tilde{v}).$$

Proof. To enforce the constraint $\Pi \tilde{v} = v$, we introduce a Lagrange multiplier $\lambda \in \mathbb{V}$ and consider the Lagrangian

$$\mathcal{L}(\tilde{v}, \lambda) = (\tilde{B}^{-1}\tilde{v}, \tilde{v}) - 2(\lambda, \Pi \tilde{v} - v).$$

The optimality conditions $\partial_{\tilde{v}} \mathcal{L} = 0, \partial_{\lambda} \mathcal{L} = 0$ give the saddle-point system

$$\begin{cases} \tilde{B}^{-1}\tilde{v} - \Pi^\top \lambda = 0, \\ \Pi \tilde{v} = v. \end{cases}$$

The first equation yields

$$\tilde{v} = \tilde{B} \Pi^\top \lambda.$$

Substituting into the constraint gives

$$\lambda = B^{-1}v, \quad B = \Pi \tilde{B} \Pi^\top.$$

Hence the minimizer is

$$\tilde{v}^* = \tilde{B} \Pi^\top B^{-1} v.$$

The minimal value of the functional follows as

$$(\tilde{B}^{-1} \tilde{v}^*, \tilde{v}^*) = (\tilde{B}^{-1} \tilde{B} \Pi^\top B^{-1} v, \tilde{v}^*) = (\Pi^\top B^{-1} v, \tilde{v}^*) = (B^{-1} v, v).$$

□

The symmetric positive definite operator B can be used as a preconditioner for solving $Au = f$ using PCG. To estimate the condition number $\kappa(BA)$, we only need to compare B^{-1} and A .

Lemma 2.2. *Let A and B be SPD operators. If*

$$c_0(Av, v) \leq (B^{-1}v, v) \leq c_1(Av, v) \quad \text{for all } v \in \mathbb{V},$$

then $\kappa(BA) \leq c_1/c_0$.

Proof. Since BA is A -symmetric, we have

$$\begin{aligned} \lambda_{\min}^{-1}(BA) &= \lambda_{\max}((BA)^{-1}) = \sup_{u \in \mathbb{V} \setminus \{0\}} \frac{((BA)^{-1}u, u)_A}{(u, u)_A} \\ &= \sup_{u \in \mathbb{V} \setminus \{0\}} \frac{(B^{-1}u, u)}{(Au, u)}. \end{aligned}$$

The upper bound $(B^{-1}v, v) \leq c_1(Av, v)$ implies $\lambda_{\min}(BA) \geq c_1^{-1}$. Similarly, the lower bound $(B^{-1}v, v) \geq c_0(Av, v)$ implies $\lambda_{\max}(BA) \leq c_0^{-1}$. The estimate of $\kappa(BA)$ then follows. □

We use the partial ordering of symmetric operators to simplify the above argument. Recall that for symmetric operators X and Y we write $X \geq Y$ if $(Xu, u) \geq (Yu, u)$ for all $u \in \mathbb{V}$. With this notation, the assumptions of the lemma can be written as

$$c_0 A \leq B^{-1} \leq c_1 A.$$

Equivalently,

$$c_0 I \leq A^{-1/2} B^{-1} A^{-1/2} \leq c_1 I,$$

Taking inverses gives

$$c_1^{-1} I \leq A^{1/2} B A^{1/2} \leq c_0^{-1} I,$$

which shows that the eigenvalues of BA lie in $[c_1^{-1}, c_0^{-1}]$ and yields $\kappa(BA) \leq c_1/c_0$.

Theorem 2.3. *Let $\tilde{\mathbb{V}}$ and \mathbb{V} be two Hilbert spaces and let $\Pi : \tilde{\mathbb{V}} \rightarrow \mathbb{V}$ be a surjective map. Let $\tilde{B} : \tilde{\mathbb{V}} \rightarrow \tilde{\mathbb{V}}$ be a symmetric and positive definite operator and set $B = \Pi \tilde{B} \Pi^\top$. If*

$$(6) \quad c_0(Av, v) \leq \inf_{\Pi \tilde{v} = v} (\tilde{B}^{-1} \tilde{v}, \tilde{v}) \leq c_1(Av, v) \quad \text{for all } v \in \mathbb{V},$$

then

$$\kappa(BA) \leq c_1/c_0.$$

Remark 2.4. In the literature, for example in the fictitious space lemma of [4], the condition (6) is usually decomposed into the following two conditions:

- (1) For any $v \in \mathbb{V}$, there exists $\tilde{v} \in \tilde{\mathbb{V}}$ such that $\Pi \tilde{v} = v$ and

$$\|\tilde{v}\|_{\tilde{B}^{-1}}^2 \leq c_1 \|v\|_A^2.$$

- (2) For any $\tilde{v} \in \tilde{\mathbb{V}}$,

$$\|\Pi \tilde{v}\|_A^2 \leq c_0^{-1} \|\tilde{v}\|_{\tilde{B}^{-1}}^2.$$

3. AN AUXILIARY SPACE OF PRODUCT TYPE

Given a space decomposition

$$\mathbb{V} = \sum_{i=1}^J \mathbb{V}_i, \quad \mathbb{V}_i \subseteq \mathbb{V},$$

we construct the auxiliary product space

$$\tilde{\mathbb{V}} = \mathbb{V}_1 \times \mathbb{V}_2 \times \cdots \times \mathbb{V}_J,$$

equipped with the standard product inner product $(\tilde{u}, \tilde{v}) := \sum_{i=1}^J (u_i, v_i)$.

Define the operator $\Pi : \tilde{\mathbb{V}} \rightarrow \mathbb{V}$ by $\Pi \tilde{u} = \sum_{i=1}^J u_i$. In operator form, Π is the row vector

$$\Pi = (I_1, I_2, \dots, I_J),$$

when $\tilde{u} = (u_1, \dots, u_J)^\top$ is viewed as a column vector. Since $\mathbb{V} = \sum_{i=1}^J \mathbb{V}_i$, the operator Π is surjective.

Let $\tilde{A} = \Pi^\top A \Pi$ and $\tilde{f} = \Pi^\top f$. A direct computation gives

$$(7) \quad \tilde{a}_{ij} = Q_i A I_j = A_i P_i I_j,$$

and in particular $\tilde{a}_{ii} = A_i$. The operator \tilde{A} is symmetric, and may be singular because $\ker(\Pi)$ may be nontrivial, but all diagonal blocks A_i are nonsingular.

We obtain the lifted system

$$(8) \quad \tilde{A} \tilde{u} = \tilde{f}.$$

Since \tilde{A} can be singular, (8) may admit multiple solutions. If \tilde{u} satisfies (8), then by definition

$$\Pi^\top A (\Pi \tilde{u}) = \Pi^\top f.$$

As Π^\top is injective, $u = \Pi \tilde{u}$ solves $Au = f$. The non-uniqueness from the $\ker(\Pi)$ is gone after the projection Π .

We derive PSC and SSC by applying classical iterative methods to the lifted system (8). Let $R_i : \mathbb{V}_i \rightarrow \mathbb{V}_i$ be nonsingular operators (smoothers) that approximate A_i^{-1} . Define the block-diagonal operator

$$\tilde{R} = \text{diag}(R_0, R_1, \dots, R_J) : \tilde{\mathbb{V}} \rightarrow \tilde{\mathbb{V}},$$

which is also nonsingular. We split \tilde{A} as

$$\tilde{A} = \tilde{D} + \tilde{L} + \tilde{U},$$

where $\tilde{D} = \text{diag}(A_0, A_1, \dots, A_J)$, \tilde{L} and \tilde{U} are the strictly lower and strictly upper triangular blocks, and $\tilde{L}^\top = \tilde{U}$.

Considering the iteration

$$(9) \quad \tilde{u}_{k+1} = \tilde{u}_k + \tilde{R}(\tilde{f} - \tilde{A} \tilde{u}_k).$$

Let $u_k = \Pi \tilde{u}_k$. Applying Π to (9) and noting that

$$\tilde{f} = \Pi^\top f, \quad \text{and} \quad \tilde{A} \tilde{u}_k = \Pi^\top A u_k,$$

we obtain the PSC method

$$u_{k+1} = u_k + \sum_{i=1}^J I_i R_i Q_i (f - Au_k),$$

The multiplicative method is more subtle. Following [3], we shall view the SSC for solving $Au = f$ as a Gauss-Seidel type method for $\tilde{A}\tilde{u} = \tilde{f}$.

Lemma 3.1. *Let $\tilde{A} = \tilde{D} + \tilde{L} + \tilde{U}$ and $\tilde{B} = (\tilde{R}^{-1} + \tilde{L})^{-1}$. Then SSC for $Au = f$ with smoother R_i is equivalent to the Gauss-Seidel type method for solving $\tilde{A}\tilde{u} = \tilde{f}$:*

$$(10) \quad \tilde{u}_{k+1} = \tilde{u}_k + \tilde{B}(\tilde{f} - \tilde{A}\tilde{u}_k).$$

Proof. By multiplying $\tilde{R}^{-1} + \tilde{L}$ to (10) and rearranging the terms, we have

$$\tilde{R}^{-1}\tilde{u}_{k+1} = \tilde{R}^{-1}\tilde{u}_k + \tilde{f} - \tilde{L}\tilde{u}_{k+1} - (\tilde{D} + \tilde{U})\tilde{u}_k.$$

Multiplying \tilde{R} , we obtain

$$\tilde{u}_{k+1} = \tilde{u}_k + \tilde{R} \left(\tilde{f} - \tilde{L}\tilde{u}_{k+1} - (\tilde{D} + \tilde{U})\tilde{u}_k \right),$$

and its component-wise formula, for $i = 1, \dots, J$

$$\begin{aligned} u_{k+1}^i &= u_k^i + R_i \left(f_i - \sum_{j=1}^{i-1} \tilde{a}_{ij} u_{k+1}^j - \sum_{j=i}^J \tilde{a}_{ij} u_k^j \right) \\ &= u_k^i + R_i Q_i \left(f - A \sum_{j=1}^{i-1} u_{k+1}^j - A \sum_{j=i}^J u_k^j \right). \end{aligned}$$

Let the dynamic update

$$v^i = \Pi(u_{k+1}^1, \dots, u_{k+1}^i, u_k^{i+1}, \dots, u_k^J)^\top = \sum_{j=1}^i I_j u_{k+1}^j + \sum_{j=i+1}^J I_j u_k^j.$$

Noting that $v^i - v^{i-1} = I_i(u_{k+1}^i - u_k^i)$, we then get, for $i = 1, \dots, J+1$

$$v^i = v^{i-1} + I_i R_i Q_i (f - Av^{i-1}),$$

which is exactly the correction in the subspace \mathbb{V}_i ; see (4). □

Let B_m be the preconditioner defined by

$$I - B_m A = (I - R_J Q_J A)(I - R_{J-1} Q_{J-1} A) \cdots (I - R_1 Q_1 A).$$

We derive a representation of B_m from the auxiliary space formulation. Define

$$\tilde{B}_m = (\tilde{R}^{-1} + \tilde{L})^{-1},$$

and its symmetrization

$$(11) \quad \overline{\tilde{B}}_m = \tilde{B}_m^\top + \tilde{B}_m - \tilde{B}_m^\top \tilde{A} \tilde{B}_m = \tilde{B}_m^\top (\tilde{B}_m^{-\top} + \tilde{B}_m^{-1} - \tilde{A}) \tilde{B}_m.$$

Lemma 3.2. *For the SSC method,*

$$B_m = \Pi \tilde{B}_m \Pi^\top, \quad \overline{B}_m = \Pi \overline{\tilde{B}}_m \Pi^\top.$$

Proof. Let $u_k = \Pi \tilde{u}_k$. Applying Π to the lifted iteration (10) and using

$$\tilde{f} = \Pi^\top f, \quad \tilde{A} \tilde{u}_k = \Pi^\top A u_k,$$

we obtain

$$u_{k+1} = u_k + \Pi \tilde{B}_m \Pi^\top (f - A u_k).$$

This gives $B_m = \Pi \tilde{B}_m \Pi^\top$. The identity for \bar{B}_m follows from the same argument applied to the symmetrized operator. \square

4. IDENTITIES FOR ADDITIVE AND MULTIPLICATIVE METHODS

The operator for the additive method PSC is

$$(12) \quad B_a = \Pi \tilde{R} \Pi^\top = \sum_{i=1}^J I_i R_i I_i^\top.$$

Applying Theorem 2.1 yields the following identity for the preconditioner B_a .

Theorem 4.1. *If each R_i is SPD on \mathbb{V}_i for $i = 1, \dots, J$, then B_a defined by (12) is SPD on \mathbb{V} . Moreover,*

$$(13) \quad (B_a^{-1} v, v) = \inf_{\sum_{i=1}^J v_i = v} \sum_{i=1}^J (R_i^{-1} v_i, v_i) = \inf_{\sum_{i=1}^J v_i = v} \sum_{i=1}^J (T_i^{-1} v_i, v_i)_A.$$

When $R_i = A_i^{-1}$, we have $\tilde{B}_m = (\tilde{D} + \tilde{L})^{-1}$ and

$$(14) \quad \bar{\tilde{B}}_m = (\tilde{D} + \tilde{U})^{-1} \tilde{D} (\tilde{D} + \tilde{L})^{-1},$$

as well as

$$(15) \quad \bar{\tilde{B}}_m^{-1} = \tilde{A} + \tilde{L} \tilde{D}^{-1} \tilde{U}.$$

The formulas (14) and (15) are the standard identities for the Gauss–Seidel method; see *Classical Iterative Methods*.

In the general case, set

$$\mathcal{D} = \tilde{R}^{-\top} + \tilde{R}^{-1} - \tilde{D}, \quad \mathcal{U} = \tilde{D} - \tilde{R}^{-1} + \tilde{U}, \quad \mathcal{L} = \mathcal{U}^\top.$$

Then

$$\tilde{A} = \mathcal{D} + \mathcal{L} + \mathcal{U}, \quad \tilde{B}_m = (\tilde{R}^{-1} + \tilde{L})^{-1} = (\mathcal{D} + \mathcal{L})^{-1}.$$

In other words, we obtain a new splitting of \tilde{A} for which \tilde{B}_m is the Gauss–Seidel iterator associated with the decomposition $\tilde{A} = \mathcal{D} + \mathcal{L} + \mathcal{U}$. Symbolically, in (14) and (15) we can replace \tilde{X} by \mathcal{X} for $X = D, L, U$.

To simplify the notation for \mathcal{D} , introduce the block-diagonal operator

$$(16) \quad \tilde{\bar{R}} = \text{diag}(\bar{R}_1, \bar{R}_2, \dots, \bar{R}_J), \quad \bar{R}_i = R_i^\top (R_i^{-\top} + R_i^{-1} - A_i) R_i.$$

Then

$$\mathcal{D} = \tilde{R}^{-\top} \tilde{\bar{R}} \tilde{R}^{-1}, \quad \tilde{R}^{-1} (\tilde{R}^{-1} + \tilde{L})^{-1} = (\tilde{I} + \tilde{L} \tilde{R})^{-1}.$$

We conclude with the following identities for $\bar{\tilde{B}}_m$.

Lemma 4.2. *For $\tilde{B}_m = (\tilde{R}^{-1} + \tilde{L})^{-1}$, the symmetrized operator $\bar{\tilde{B}}_m$ satisfies*

$$(17) \quad \bar{\tilde{B}}_m = (\mathcal{D} + \mathcal{U})^{-1} \mathcal{D} (\mathcal{D} + \mathcal{L})^{-1} = (\tilde{I} + \tilde{R}^\top \tilde{U})^{-1} \tilde{\bar{R}} (\tilde{I} + \tilde{L} \tilde{R})^{-1},$$

and

$$(18) \quad \bar{\tilde{B}}_m^{-1} = \tilde{A} + \mathcal{L} \mathcal{D}^{-1} \mathcal{U} = \tilde{A} + \left[\tilde{R}^\top (\tilde{D} + \tilde{U} - \tilde{R}^{-1}) \right]^\top \tilde{\bar{R}}^{-1} \left[\tilde{R}^\top (\tilde{D} + \tilde{U} - \tilde{R}^{-1}) \right].$$

It is clear from (18) that $\tilde{\bar{B}}_m$ is symmetric. For it to be positive definite, it suffices that $\tilde{\bar{R}}$, or equivalently each \bar{R}_i , is positive definite. This holds when the local iterator R_i is a contraction in the A_i -norm.

$$(C) \quad \|I - R_i A_i\|_{A_i} < 1, \quad i = 1, \dots, J.$$

Theorem 4.3. Assume (C) holds. Then $\bar{B}_m = \Pi \tilde{\bar{B}}_m \Pi^\top$ is SPD, and

$$(19) \quad (\bar{B}_m^{-1} v, v) = \inf_{\sum_{i=1}^J v_i = v} \sum_{i=1}^J \left\| v_i + R_i^\top A_i P_i \sum_{j>i}^J v_j \right\|_{\bar{R}_i^{-1}}^2,$$

$$(20) \quad (\bar{B}_m^{-1} v, v) = \|v\|_A^2 + \inf_{\sum_{i=1}^J v_i = v} \sum_{i=1}^J \left\| R_i^\top (A_i P_i \sum_{j=i}^J v_j - R_i^{-1} v_i) \right\|_{\bar{R}_i^{-1}}^2.$$

In particular, if $R_i = A_i^{-1}$, then

$$(21) \quad (\bar{B}_m^{-1} v, v) = \|v\|_A^2 + \inf_{\sum_{i=1}^J v_i = v} \sum_{i=1}^J \left\| P_i \sum_{j=i+1}^J v_j \right\|_A^2.$$

Proof. From (18), we have

$$\tilde{\bar{B}}_m^{-1} = (\tilde{I} + \tilde{R}^\top \tilde{U})^\top \tilde{R}^{-1} (\tilde{I} + \tilde{R}^\top \tilde{U}).$$

Using the componentwise identity and the formula (7) for \tilde{a}_{ij} ,

$$(\tilde{U} \tilde{v})_i = \sum_{j=i+1}^J \tilde{a}_{ij} v_j = \sum_{j=i+1}^J A_i P_i v_j,$$

and applying Theorem 2.1, we obtain (19).

To derive (20), we first present the argument for the special case $R_i = A_i^{-1}$ to make the main idea clear. The assumption (C) holds for the exact local solver, hence

$$(\bar{B}_m^{-1} v, v) = \inf_{\Pi \tilde{v} = v} (\tilde{\bar{B}}_m^{-1} \tilde{v}, \tilde{v}) = (\tilde{A} \tilde{v}, \tilde{v}) + \inf_{\Pi \tilde{v} = v} (\tilde{D}^{-1} \tilde{U} \tilde{v}, \tilde{U} \tilde{v}).$$

For any $\tilde{v} \in \tilde{\mathbb{V}}$, with $v = \Pi \tilde{v}$, we compute

$$(\tilde{A} \tilde{v}, \tilde{v}) = (\Pi^\top A \Pi \tilde{v}, \tilde{v}) = (Av, v) = \|v\|_A^2.$$

Moreover,

$$(\tilde{D}^{-1} \tilde{U} \tilde{v}, \tilde{U} \tilde{v}) = \sum_{i=1}^J (A_i^{-1} \sum_{j=i+1}^J A_i P_i v_j, \sum_{j=i+1}^J A_i P_i v_j) = \sum_{i=1}^J \left\| P_i \sum_{j=i+1}^J v_j \right\|_A^2.$$

This gives (21). The general identity (20) follows from the same componentwise expansion applied to (18). \square

When the vector lies in the subspace \mathbb{V}_i , we have

$$T_i = R_i A_i : \mathbb{V}_i \rightarrow \mathbb{V}_i, \quad T_i^* = R_i^\top A_i, \quad \bar{T}_i := T_i + T_i^* - T_i^* T_i = \bar{R}_i A_i,$$

and, under assumption (C), \bar{T}_i^{-1} is well defined and

$$(\bar{R}_i^{-1} u_i, u_i) = (\bar{T}_i^{-1} u_i, u_i)_A.$$

The identity (20) can be written in the original form of [9] as

$$(\bar{B}_m^{-1}v, v) = \|v\|_A^2 + \inf_{\sum_{i=1}^J v_i = v} \sum_{i=1}^J (\bar{T}_i^{-1} T_i^* w_i, T_i^* w_i)_A,$$

with $w_i = P_i \sum_{j=i}^J v_j - T_i^{-1} v_i$. Similarly, the identity (19) becomes the formula in [2]

$$(\bar{B}_m^{-1}v, v) = \inf_{\sum_{i=1}^J v_i = v} \sum_{i=1}^J (\bar{T}_i^{-1} (v_i + T_i^* P_i \sum_{k=i+1}^J v_k), v_i + T_i^* P_i \sum_{k=i+1}^J v_k)_A.$$

Combining Lemma 3.2 and Theorem 4.3, we obtain the X–Z identity [9] for multiplicative methods.

Theorem 4.4 (X–Z identity). *Suppose assumption (C) holds. Then*

$$(22) \quad \|I - B_m A\|_A^2 = \|I - \bar{B}_m A\|_A = 1 - \frac{1}{K},$$

where

$$K = \sup_{\|v\|_A=1} \inf_{\sum_{i=1}^J v_i = v} \sum_{i=1}^J \|v_i + R_i^\top A_i P_i \sum_{j>i}^J v_j\|_{R_i^{-1}}^2.$$

Moreover,

$$(23) \quad \|I - B_m A\|_A^2 = \|I - \bar{B}_m A\|_A = 1 - \frac{1}{1 + c_0},$$

where

$$c_0 = \sup_{\|v\|_A=1} \inf_{\sum_{i=1}^J v_i = v} \sum_{i=1}^J \|R_i^\top (A_i P_i \sum_{j=i}^J v_j - R_i^{-1} v_i)\|_{R_i^{-1}}^2.$$

In particular, for $R_i = A_i^{-1}$,

$$(24) \quad \|(I - P_J)(I - P_{J-1}) \cdots (I - P_1)\|_A^2 = 1 - \frac{1}{1 + c_0},$$

where

$$c_0 = \sup_{\|v\|_A=1} \inf_{\sum_{i=1}^J v_i = v} \sum_{i=1}^J \|P_i \sum_{j=i+1}^J v_j\|_A^2.$$

5. ORTHOGONAL TELESCOPE DECOMPOSITION

We assume the subspaces \mathbb{V}_i are nested, i.e., $\mathbb{V}_1 \subset \mathbb{V}_2 \subset \cdots \subset \mathbb{V}_J = \mathbb{V}$. Then SSC applied to $\sum \mathbb{V}_i$ is equivalent to a multigrid V-cycle.

Recall that $P_k = I_k A_k^{-1} Q_k A$ is the A -orthogonal projection, and we use I_k to view P_k as an operator from \mathbb{V} to \mathbb{V} . Since $Q_k = I_k^\top$, the operator P_k is A -symmetric. Because $A_k = Q_k A I_k$, we also have $P_k^2 = P_k$. We will choose the orthogonal telescoping decomposition $u_k = (P_k - P_{k-1})u$, $k = 0, \dots, J$, in the X–Z identity to obtain a direct convergence analysis of multigrid.

Lemma 5.1. *Let $\mathbb{V}_1 \subset \mathbb{V}_2 \subset \cdots \subset \mathbb{V}_J = \mathbb{V}$ be nested subspaces of a Hilbert space. For each k , let $P_k : \mathbb{V} \rightarrow \mathbb{V}$ be the A -orthogonal projection onto \mathbb{V}_k . Then the following properties hold:*

- (P1) $P_k P_\ell = P_k = P_\ell P_k$ for $\ell \geq k$;
- (P2) $P_k (P_\ell - P_{\ell-1}) = 0$ for $\ell > k$;
- (P3) $(P_k - P_{k-1})(P_\ell - P_{\ell-1}) = 0$ for $\ell > k$, and $(P_k - P_{k-1})^2 = P_k - P_{k-1}$.

Proof. (P1). Let $\ell \geq k$. For any $v \in \mathbb{V}$ we can write $v = P_\ell v + (I - P_\ell)v$, where $(I - P_\ell)v \perp_A \mathbb{V}_\ell \supseteq \mathbb{V}_k$. Thus $P_k(I - P_\ell)v = 0$, and hence

$$P_k v = P_k P_\ell v,$$

which shows $P_k P_\ell = P_k$. Since $P_k v \in \mathbb{V}_k \subset \mathbb{V}_\ell$, we also have $P_\ell P_k v = P_k v$, and therefore $P_\ell P_k = P_k$.

(P2). If $\ell > k$, then $\ell \geq k$ and $\ell - 1 \geq k$. Applying (P1) to both indices gives

$$P_k(P_\ell - P_{\ell-1}) = P_k - P_k = 0.$$

(P3). For $\ell > k$, orthogonality follows directly from (P2) applied to k and $k - 1$. For idempotence, note that

$$(P_k - P_{k-1})^2 = P_k - P_k P_{k-1} - P_{k-1} P_k + P_{k-1} = P_k - P_{k-1},$$

because (P1) implies $P_k P_{k-1} = P_{k-1}$ and $P_{k-1} P_k = P_{k-1}$. \square

A direct consequence is the orthogonal decomposition identity

$$(25) \quad \sum_{k=0}^J \|u_k\|_A^2 = \|u\|_A^2, \quad u_k = (P_k - P_{k-1})u, \quad k = 0, \dots, J.$$

For such decomposition, using (P2), the first X-Z identity (19) simplifies to

$$(26) \quad \sum_{k=0}^J \left\| u_k + R_k^\top A_k P_k \left(\sum_{\ell > k} u_\ell \right) \right\|_{\bar{R}_k^{-1}}^2 = \sum_{k=0}^J \|u_k\|_{\bar{R}_k^{-1}}^2.$$

A natural smoothing property assumption is:

(\bar{S}_P) The smoother \bar{R}_k reduces the high-frequency error in $(P_k - P_{k-1})\mathbb{V}$. That is, there exists $c_R \geq 0$ such that

$$(27) \quad (\bar{R}_k^{-1} u_k, u_k) \leq c_R (A u_k, u_k), \quad \text{for all } u_k \in (P_k - P_{k-1})\mathbb{V}, \quad k = 0, \dots, J.$$

The convergence is a direct consequence of the X-Z identity (19) and assumption (\bar{S}_P) .

Theorem 5.2. Suppose the smoother R_k satisfies assumptions (C) and (\bar{S}_P) . Then the corresponding $V(1, 1)$ -cycle converges uniformly with rate $1 - 1/c_R$, i.e.,

$$\|I - \bar{B}_m A\|_A \leq 1 - \frac{1}{c_R}.$$

When the smoother is symmetric, we may use the smoothing property of R_k , which is typically easier to verify for symmetric smoothers, instead of that of \bar{R}_k .

(S_P) The smoother R_k reduces the high-frequency error in $(P_k - P_{k-1})\mathbb{V}$. That is, there exists $c_R > 0$ such that

$$(28) \quad (R_k^{-1} u_k, u_k) \leq c_R (A u_k, u_k), \quad \text{for all } u_k \in (P_k - P_{k-1})\mathbb{V}, \quad k = 0, \dots, J.$$

Example 5.3. Consider the Richardson smoother $R_k = \lambda_{\max}^{-1}(A_k)I_k = c h_k^2$. To simplify notation, write \mathbb{V} and \mathbb{V}_H for two consecutive subspaces \mathbb{V}_k and \mathbb{V}_{k-1} , and write $I - P_H$ for $P_k - P_{k-1}$. Then (S_P) becomes

$$(29) \quad \|(I - P_H)u\|^2 \lesssim h^2 \|(I - P_H)u\|_A^2.$$

In the finite element setting for elliptic equations, (29) follows from a duality argument under the H^2 -regularity assumption. \square

We now connect the smoothing properties (S_P) and (\bar{S}_P) .

Lemma 5.4. *Let R be an SPD smoother that satisfies (S_P) , and let $\omega = \lambda_{\max}(RA) < 2$. Then R also satisfies (\bar{S}_P) with constant $c_R/(2 - \omega)$.*

Proof. Let $T = RA$ and $\bar{T} = \bar{R}A$. Since R is symmetric, both T and \bar{T} are A -symmetric and

$$\bar{T} = 2T - T^2 = (2 - T)T \geq_A (2 - \omega)T.$$

Hence

$$(\bar{R}^{-1}u, u) = (\bar{T}^{-1}u, u)_A \leq \frac{1}{2 - \omega} (T^{-1}u, u)_A = \frac{1}{2 - \omega} (R^{-1}u, u),$$

which gives (\bar{S}_P) . \square

Combining the above results, we obtain the convergence of multigrid methods.

Corollary 5.5. *Assume the H^2 -regularity holds. Then the multigrid V-cycle with the Richardson smoother converges uniformly for solving the linear system arising from finite element discretizations of Poisson equation on nested meshes.*

We *do not* form the subspace $(P_k - P_{k-1})\mathbb{V}$. If we did, as in the case of Fourier bases, the corresponding matrix would be diagonal. Such well-structured bases are hard to build in general, for example for variable coefficients, complex domains, or unstructured triangulations. In multigrid methods, however, we avoid forming the frequency decomposition explicitly and instead relax on a larger collection of basis functions that are much easier to construct. This redundancy is useful and improves the smoothing effect.

APPENDIX: SUM AND PRODUCT OF TWO LINEAR SPACES

Given two linear spaces V_1, V_2 and assume they are subspaces of a larger linear space V . We have the following operations of these two spaces

- $V_1 + V_2 = \{v_1 + v_2 : v_1 \in V_1, v_2 \in V_2\}$;
- $V_1 \oplus V_2 = V_1 + V_2$ and $V_1 \cap V_2 = \{0\}$;
- $V_1 \times V_2 = \{(v_1, v_2) : v_1 \in V_1, v_2 \in V_2\}$;
- $V_1 \otimes V_2 = \text{span}\{v_1 \otimes v_2 : v_1 \in V_1, v_2 \in V_2\}$.

The tensor product $v_1 \otimes v_2$ is a bilinear mapping on the dual space $V_1' \times V_2'$. A natural product topology can be defined for $V_1 \times V_2$ component-wise.

The relation of dimensions are

- $\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2) \leq \dim(V_1) + \dim(V_2)$;
- $\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2)$;
- $\dim(V_1 \times V_2) = \dim(V_1) + \dim(V_2)$;
- $\dim(V_1 \otimes V_2) = \dim(V_1) \times \dim(V_2)$.

We emphasize the sum $V_1 + V_2$ may not enlarge the space. For example, when $V_1 \subset V_2$ (a line on a plane), $V_1 + V_2 = V_2$.

REFERENCES

- [1] L. Chen. Deriving the X-Z Identity from Auxiliary Space Method. In Y. Huang, R. Kornhuber, O. Widlund, and J. Xu, editors, *Domain Decomposition Methods in Science and Engineering XIX*, pages 309–316. Springer Berlin Heidelberg, 2010.
- [2] D. Cho, J. Xu, and L. Zikatanov. New estimates for the rate of convergence of the method of subspace corrections. *Numer. Math. Theor. Meth. Appl.*, 1:44–56, 2008.
- [3] M. Griebel and P. Oswald. On the abstract theory of additive and multiplicative Schwarz methods. *Numer. Math.*, 70:163–180, 1995.
- [4] S. V. Nepomnyaschikh. Decomposition and fictitious domains methods for elliptic boundary value problems. In *Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations (Norfolk, VA, 1991)*, pages 62–72. SIAM, Philadelphia, PA, 1992.
- [5] J. Xu. Iterative methods by space decomposition and subspace correction. *SIAM Rev.*, 34:581–613, 1992.
- [6] J. Xu. The auxiliary space method and optimal multigrid preconditioning techniques for unstructured meshes. *Computing*, 56:215–235, 1996.
- [7] J. Xu. An introduction to multilevel methods. 1997. published by Oxford University Press, New York.
- [8] J. Xu. *Multilevel Finite Element Methods*. Lecutre Notes, 2004.
- [9] J. Xu and L. Zikatanov. The method of alternating projections and the method of subspace corrections in Hilbert space. *J. Amer. Math. Soc.*, 15:573–597, 2002.