UNIFIED ERROR ANALYSIS

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CONTENTS

Let $\mathscr{X}, \mathscr{Y}$ be two Banach spaces and $L \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$ be a linear and bounded operator. We consider the problem of approximating an operator equation: given $y \in \mathscr{Y}$, find $x \in \mathscr{X}$ such that

$$Lx = y. \tag{1}$$

We assume (1) is well defined, i.e., for any $y \in \mathscr{Y}$, there exists a unique solution to (1). Then L is a one-to-one linear and bounded map and so is L^{-1} by the open mapping theorem. We shall study the convergence analysis in this notes and refer to *Inf-sup conditions for operator equations* for conditions on the well-posedness of (1).

1. LAX EQUIVALENCE THEOREM

We are interested in the approximation of L. Suppose $L_h \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$, for $h \in \mathcal{H}, h \to 0$, is a family of discretization of L. We consider the problem: given $y \in \mathscr{Y}$, find $x_h \in \mathscr{X}$, such that

$$L_h x_h = y. \tag{2}$$

We also assume (2) is well defined. So L_h^{-1} is a linear and bounded map. The norm $\|L_h^{-1}\|$, however, could depend on the parameter h. The uniform boundedness $\|L_h^{-1}\|$ will be called the *stability* of the discretization. Namely, there exists a constant C independent of h such that

$$\|L_h^{-1}\|_{\mathscr{Y} \to \mathscr{X}} \leq C, \quad \text{for all } h \in \mathcal{H}. \tag{3}$$

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The consistency measures the approximation of L_h to L. We call the discretization is *consistent* if, for all $x \in \mathscr{X}$,

$$\|Lx - L_h x\|_{\mathscr{Y}} \to 0 \quad \text{as } h \to 0. \tag{4}$$

The *convergence* of the discretization is defined as

$$\|x - x_h\|_{\mathscr{X}} \to 0 \quad \text{as } h \to 0, \tag{5}$$

where x_h and x are solutions to (2) and (1), respectively.

We follow the book [4] to present the Lax equivalence theorem.

Theorem 1.1 (Lax equivalence theorem)**.** *Suppose the discretization L_h of L is consistent, i.e., (4) holds for all $x \in \mathscr{X}$, then the stability (3) is equivalent to the convergence (5).*

Proof. We write the error as

$$x - x_h = L_h^{-1}(L_h x - L_h x_h) = L_h^{-1}(L_h x - Lx).$$

Here we use the fact $L_h x_h = Lx = y$. If the scheme is stable, then

$$\|x - x_h\|_{\mathscr{X}} \leq C\|Lx - L_h x\|_{\mathscr{Y}} \to 0 \quad \text{as } h \to 0.$$

On the other hand, for any $y \in \mathscr{Y}$, let $x_h = L_h^{-1}y$ and $x = L^{-1}y$. The convergence $x_h \to x$ means for any $y \in \mathscr{Y}$

$$L_h^{-1}y \to L^{-1}y \quad \text{as } h \to 0.$$

By the uniformly boundedness principle, we conclude $\|L_h^{-1}\|$ is uniformly bounded. $\square$

2. Abstract Error Analysis

The Lax equivalence theorem may not be ready to use for the error analysis of discretizations since discrete spaces $\mathscr{X}_h, \mathscr{Y}_h$ are used to approximate $\mathscr{X}, \mathscr{Y}$, respectively. The equation may not be solved exactly, i.e., $L_h x_h \neq Lx$. Instead, it solves approximately in $\mathscr{Y}_h$, i.e., $L_h x_h = y_h$ for some $y_h \in \mathscr{Y}_h$. In some cases $\mathscr{X}_h \subset \mathscr{X}$ but in general $\mathscr{X}_h$ and $\mathscr{X}$ are different spaces so that even $x - x_h$ does not make sense. The norm $\|\cdot\|_{\mathscr{X}_h}$ could be an approximation of $\|\cdot\|_{\mathscr{X}}$ etc.

We now refine the analysis to handle these cases. Here we follow closely Temam [3]. Let us introduce two linear operators $I_h : \mathscr{X} \to \mathscr{X}_h$, and $\Pi_h : \mathscr{Y} \to \mathscr{Y}_h$. We consider the discrete problem: given $y \in \mathscr{Y}$, find $x_h \in \mathscr{X}_h$ such that

$$L_h x_h = \Pi_h y \quad \text{in } \mathscr{Y}_h. \tag{6}$$

The following diagram is not commutative

$$\begin{array}{ccc}
\mathscr{X} & \xrightarrow{\ L\ } & \mathscr{Y} \\
\Big\downarrow I_h & & \Big\downarrow \Pi_h \\
\mathscr{X}_h & \xrightarrow{\ L_h\ } & \mathscr{Y}_h
\end{array}$$

and the difference is defined as the consistency error

$$\|\Pi_h Lx - L_h I_h x\|_{\mathscr{Y}_h}.$$

The (uniform) stability is the same as before

$$\|L_h^{-1}\|_{\mathscr{Y}_h \to \mathscr{X}_h} \leq C, \quad \text{for all } h \in \mathcal{H}.$$

On the convergence, even $x - x_h$ may not be well defined as in general $\mathscr{X}_h$ is not a subspace of $\mathscr{X}$. The error is revised to $I_h x - x_h$.

Remark 2.1. The operator Π_h is in fact part of the discretization and to reflect to this fact the discretization will be denoted by (L_h, Π_h), while the operator I_h is introduced for the error analysis. For a given Π_h, I_h is not unique and a suitable choice of I_h is the art of numerical analysis. □

We shall prove if the scheme is consistent and stable, then the discrete error $I_h x - x_h$ is convergent.

Theorem 2.2. *Suppose the scheme (L_h, Π_h) is consistent with order r, i.e., there exists an operator I_h such that*

$$\|\Pi_h L x - L_h I_h x\|_{\mathscr{Y}_h} \leq C h^r,$$

and stable

$$\|L_h^{-1}\|_{\mathscr{Y}_h \to \mathscr{X}_h} \leq C \quad \text{for all } h \in \mathcal{H},$$

then we have the discrete convergence with order r:

$$\|I_h x - x_h\|_{\mathscr{X}_h} \leq C h^r.$$

Proof. The proof is straightforward using the definition, assumptions, and the identity:

$$I_h x - x_h = L_h^{-1} L_h (I_h x - x_h) = L_h^{-1}(L_h I_h x - \Pi_h L x).$$

□

If we really want to control $x - x_h$, one way is to define another linear operator $P_h : \mathscr{X}_h \to \mathscr{X}$ to put a discrete function x_h into $\mathscr{X}$ in a stable way, i.e., $\|P_h x_h\|_{\mathscr{X}} \leq C\|x_h\|_{\mathscr{X}_h}$. Then by the triangle inequality, we have

$$\begin{aligned}
\|x - P_h x_h\|_{\mathscr{X}} &\leq \|x - P_h I_h x\|_{\mathscr{X}} + \|P_h(I_h x - x_h)\|_{\mathscr{X}_h} \\
&\leq \|x - P_h I_h x\|_{\mathscr{X}} + C\|I_h x - x_h\|_{\mathscr{X}_h}.
\end{aligned}$$

The additional error term $\|x - P_h I_h x\|_{\mathscr{X}}$ measures the approximation property of $\mathscr{X}$ by the subspace $P_h \mathscr{X}_h$ and will be called approximability.

Theorem 2.3. *Assume*

- *consistency:* $\|\Pi_h L x - L_h I_h x\|_{\mathscr{Y}_h} \leq C_1 h^r$,
- *stability:* $\|L_h^{-1}\|_{\mathscr{Y}_h \to \mathscr{X}_h} \leq C_2 \quad \forall h \in \mathcal{H}$,
- *approximability:* $\|x - P_h I_h x\|_{\mathscr{X}} \leq C_3 h^s$,

then we have convergence

$$\|x - P_h x_h\|_{\mathscr{X}} \leq C_2(C_1 h^r + C_3 h^s).$$

Remark 2.4. As the stability of L^{-1} is not used, the domain of the operator L could be just a subspace of $\mathscr{X}$. Strictly speaking we should write

$$L : \mathrm{dom}(L) \cap \mathscr{X} \to \mathscr{Y}.$$

We leave a bigger space $\mathscr{X}$ in the analysis such that the embedding $P : \mathscr{X}_h \to \mathscr{X}$ is easier to construct. And with operators I_h and Π_h, the space $\mathscr{X}$ and $\mathscr{Y}$ do not have to be Banach spaces. □

Another way is to embed both $\mathscr{X}$ and $\mathscr{X}_h$ into a larger space, e.g. $\mathscr{X} + \mathscr{X}_h$ and define a new norm. Similar assumption is made for $\mathscr{Y}$ and $\mathscr{Y}_h$. Also assume L_h can be defined on $\mathscr{X} + \mathscr{X}_h$. We can break the consistence error $\Pi_h L x - L_h I_h x$ into three parts

- perturbation of the data: $\Pi_h L x - L x = L_h x_h - L x = y_h - y$
- consistence error: $L x - L_h x$
- approximation error: $L_h(x - I_h x)$

The first term measures the approximation of the data in the subspace $\mathscr{Y}_h$ and the third one measures the approximation of the function. The middle one is the traditional consistence error introduced in Section 1.

3. Application: Finite Difference Methods

Consider the finite difference method for solving the Poisson equation $-\Delta u = f$ with homogenous Dirichlet boundary condition $u|_{\partial\Omega} = 0$. We assume Ω can be partitioned into equal size squares with side h. The number of interior nodes is N. The setting is

- $(\mathscr{X}, \|\cdot\|_{\mathscr{X}}) = (\mathscr{Y}, \|\cdot\|_{\mathscr{Y}}) = (C(\bar{\Omega}), \|\cdot\|_{\infty,\Omega})$;
- $(\mathscr{X}_h, \|\cdot\|_{\mathscr{X}_h}) = (\mathscr{Y}_h, \|\cdot\|_{\mathscr{Y}_h}) = (\mathbb{R}^N, \|\cdot\|_{l_\infty})$;
- $I_h = \Pi_h$ as the nodal interpolation;
- $P_h : \mathscr{X}_h \to \mathscr{X}$ can be defined using the bilinear finite element space on the uniform grid or the linear finite element space on the uniform triangular grid;
- $L = -\Delta$ and $L_h = -\Delta_h$ is the 5-point stencil matrix. Note that the domain of L is a subspace of $\mathscr{X}$.

The consistency error is

$$\|(\Delta u)_I - \Delta_h u_I\|_{l_\infty} = \max_{1\leq i\leq N} |(\Delta u)_i - (\Delta_h u_I)_i| \leq C(u)h^2,$$

which can be easily analyzed by the Taylor expansion. Here $C(u) = \|\sum_{i=1}^d D_i^4 u\|_{\infty,\Omega}$.

The stability $\|\Delta_h^{-1}\|_{l_\infty\to l_\infty}$ may be proved using the discrete maximum principle; see *Finite Difference Methods*.

The approximation property is given by the interpolation error estimate

$$\|u - u_I\|_{\infty,\Omega} \leq Ch^2\|D^2 u\|_{\infty,\Omega}.$$

Here follow the convention, we use the same notation u_I for $P_h u_I$. The readers are encouraged to nail down the difference and the connection of the vector u_I and its corresponding finite element function $P_h u_I$.

The maximum norm is too strong. Later on we shall show a 'correct' space and stability for the finite difference method.

4. Application: Finite Element Methods

Consider the finite element method for solving the Poisson equation $-\Delta u = f$ with homogenous Dirichlet boundary condition $u|_{\partial\Omega} = 0$. The setting is

- $(\mathscr{X}, \|\cdot\|_{\mathscr{X}}) = (H_0^1(\Omega), |\cdot|_1)$, and $(\mathscr{Y}, \|\cdot\|_{\mathscr{Y}}) = (H^{-1}(\Omega), \|\cdot\|_{-1})$;
- $(\mathscr{X}_h, \|\cdot\|_{\mathscr{X}}) = (\mathbb{V}_h, |\cdot|_1)$, and $(\mathscr{Y}_h, \|\cdot\|_{\mathscr{Y}_h}) = (\mathbb{V}_h', \|\cdot\|_{-1,h})$. Here the dual norm
$$\|f_h\|_{-1,h} = \sup_{v_h\in\mathbb{V}_h} \frac{\langle f_h, v_h\rangle}{|v_h|_1}, \quad \text{for } f_h \in \mathscr{Y}_h.$$
- $I_h : H_0^1(\Omega) \to \mathbb{V}_h$ is arbitrary;
- $\Pi_h = Q_h : H^{-1}(\Omega) = (H_0^1(\Omega))' \to \mathbb{V}_h'$, i.e.,
$$\langle Q_h f, v_h\rangle := \langle f, v_h\rangle, \quad \forall v_h \in \mathbb{V}_h.$$
When Q_h is restricted to $L^2(\Omega)$, it is the L^2 projection. Note that $(H_0^1(\Omega))' \subset \mathbb{V}_h'$, the operator Q_h can be also thought as the natural embedding of the dual space and thus is usually omit in the notation.
- $P_h : \mathbb{V}_h \to H_0^1(\Omega)$ is the natural embedding since now $\mathbb{V}_h \subset H_0^1(\Omega)$;

- $L = -\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ and $L_h = Q_h L P_h : \mathbb{V}_h \to \mathbb{V}_h'$. When Q_h restricted to $L^2(\Omega)$, it is the adjoint of P_h in the L^2-inner product and $L_h = Q_h L Q_h^{\intercal}$.

Using these notation, we now formulate the finite element methods for solving the Poisson equation with homogenous Dirichlet boundary condition. Given an $f \in H^{-1}(\Omega)$, find $u_h \in \mathbb{V}_h$ such that

$$L_h u_h = Q_h f \quad \text{in } \mathbb{V}_h'.$$

The *correct* (comparing with finite difference method) stability for L_h^{-1} is $\|L_h^{-1}\|_{\|\cdot\|_{-1,h}\to|\cdot|_1}$:

$$|u_h|_1 \le \|Q_h f\|_{-1,h}, \tag{7}$$

and can be proved as follows

$$|u_h|_1^2 = \langle L_h u_h, u_h\rangle = \langle Q_h f, u_h\rangle \le |Q_h f|_{-1,h}|u_h|_1.$$

Comparing with the finite difference method, now the consistency error is measured in a much weaker norm.

We can control the weaker dual norm $\|Q_h f\|_{-1,h}$ by a stronger one $\|f\|_{-1}$. Indeed

$$(Q_h f, u_h) = \langle f, u_h\rangle \le \|f\|_{-1}|u_h|_1,$$

implies

$$\|Q_h f\|_{-1,h} \le \|f\|_{-1}. \tag{8}$$

Let us denoted by $v_h = I_h u$. Using the fact $L_h = Q_h L P_h$ and inequality (8), the consistency error is

$$\|Q_h L u - L_h v_h\|_{-1,h} = \|Q_h L(u - v_h)\|_{-1,h} \le \|L(u - v_h)\|_{-1} \le |u - v_h|_1.$$

By the stability result (7), we obtain

$$|u_h - v_h|_1 \le |u - v_h|_1.$$

Since v_h is arbitrary, by the triangle inequality, we obtain the Céa lemma

$$|u - u_h|_1 \le 2 \inf_{v_h\in\mathbb{V}_h} |u - v_h|_1. \tag{9}$$

The approximation property can be proved using Bramble-Hilbert lemma. For example, for the linear finite element method and $u \in H^2(\Omega)$,

$$\inf_{v_h\in\mathbb{V}_h} |u - v_h|_1 \le Ch|u|_2.$$

Then Theorem 2.3 will give optimal error estimate for finite element methods.

Remark 4.1. Here we use only the stability and consistency defined for the Banach spaces. If we use the inner product structure and the orthogonality (or more general the variational formulation), we could improve the constant in (9) to 1. □

5. Application: Conforming Discretization of Variational Problems

We now generalize the analysis for Poisson equation to general elliptic equations and show the connection with the traditional error analysis of variational problems.

The operator $L : \mathbb{V} \to \mathbb{V}'$ is defined through a bilinear form: for $u, v \in \mathbb{V}$

$$\langle Lu, v\rangle := a(u, v).$$

We consider the conforming discretization by choosing $\mathbb{V}_h \subset \mathbb{V}$ and the discrete operator $L_h : \mathbb{V}_h \to \mathbb{V}' \hookrightarrow \mathbb{V}_h'$ being the restriction of the bilinear form: for $u_h, v_h \in \mathbb{V}_h$

$$\langle L_h u_h, v\rangle = a(u_h, v_h).$$

If we denoted by $P_h : \mathbb{V}_h \to \mathbb{V}$ and $Q_h : \mathbb{V}' \to \mathbb{V}_h'$ as the natural embedding. Then by definition $L_h = Q_h L P_h : \mathbb{V}_h \to \mathbb{V}_h'$.

The well known Lax-Milgram theorem (Lax again!) says if the bilinear form satisfies:

- coercivity:
$$\alpha \|u\|_{\mathbb{V}}^2 \leq a(u,u).$$
- continuity:
$$a(u,v) \leq \beta \|u\|_{\mathbb{V}} \|v\|_{\mathbb{V}}.$$

Then there exists a unique solution to the variational problem: given $f \in \mathbb{V}'$ find $u \in \mathbb{V}$ such that

$$a(u,v) = \langle f, v\rangle \quad \text{for all } v \in \mathbb{V}. \tag{10}$$

Since $\mathbb{V}_h \subset \mathbb{V}$, it also implies the existence and uniqueness of the solution $u_h \in \mathbb{V}_h$

$$a(u_h,v_h) = \langle f, v_h\rangle \quad \text{for all } v_h \in \mathbb{V}_h. \tag{11}$$

This establishes the well posedness of (10) and (11). The coercivity condition can be relaxed to the so-called inf-sup condition; see *Inf-sup conditions for operator equations*.

The continuity of the bilinear form $a(\cdot,\cdot)$ implies the continuity of L and L_h. The coercivity can be used to prove the stability of L_h^{-1} in a straightforward way:

$$\|u_h\|_{\mathbb{V}_h}^2 \lesssim a(u_h,u_h) = \langle f, u_h\rangle \lesssim \|f\|_{\mathbb{V}_h'} \|u_h\|_{\mathbb{V}_h}.$$

For the conforming discretization, we have the 'orthogonality' (borrowed the name when $a(\cdot,\cdot)$ is an inner product but in general $a(\cdot,\cdot)$ may not be even symmetric)

$$a(u-u_h, v_h) = 0 \quad \text{for all } v_h \in \mathbb{V}_h.$$

i.e.,

$$Lu = L_h u_h \quad \text{ in } \mathbb{V}_h',$$

which can be interpreted as consistency result: $Lu = f_h$ in $\mathbb{V}_h'$ or in a less precise notation

$$L_h u = Lu \quad \text{ in } \mathbb{V}_h'$$

with $L_h u$ understood as $Q_h L u$. Note that the consistency is measured in the weak norm $\mathbb{V}_h'$ but not in $\mathbb{V}'$. The residual $Lu - L_h u_h = 0$ only in $\mathbb{V}_h'$ but $\neq 0$ in $\mathbb{V}'$.

In our setting, the consistence means: for any $v_h \in \mathbb{V}_h$

$$Lv_h = L_h v_h \quad \text{ in } \mathbb{V}_h'.$$

In the operator form, it is simply $L_h = Q_h L P_h$.

The traditional error analysis is as follows

$$\begin{aligned}
\|u-u_h\|_{\mathbb{V}}^2 &\leq \frac{1}{\alpha} a(u-u_h, u-u_h) \quad \text{(coercivity)}\\
&= \frac{1}{\alpha} a(u-u_h, u-v_h) \quad \text{(orthogonality)}\\
&\leq \frac{\beta}{\alpha} \|u-u_h\|_{\mathbb{V}} \|u-v_h\|_{\mathbb{V}} \quad \text{(continuity)}
\end{aligned}$$

which implies the Céa lemma

$$\|u-u_h\|_{\mathbb{V}} \leq \frac{\beta}{\alpha} \inf_{v_h \in \mathbb{V}_h} \|u-v_h\|_{\mathbb{V}}.$$

Using our framework of stability and consistency, the estimate is like

$$
\begin{aligned}
\|u_h - v_h\|_{\mathbb{V}_h} &\le \frac{1}{\alpha}\|L_h(u_h - v_h)\|_{\mathbb{V}_h'} \quad \text{(stability)}\\
&= \frac{1}{\alpha}\|Lu - L_h v_h\|_{\mathbb{V}_h'} \quad \text{(orthogonality)}\\
&= \frac{1}{\alpha}\|L(u - v_h)\|_{\mathbb{V}_h'} \quad \text{(consistency)}\\
&\le \frac{\beta}{\alpha}\|u - v_h\|_{\mathbb{V}_h} \quad \text{(continuity)}.
\end{aligned}
$$

Combined with the triangle inequality, we get

$$\|u - u_h\|_{\mathbb{V}} \le \left(1 + \frac{\beta}{\alpha}\right) \inf_{v_h \in \mathbb{V}_h} \|u - v_h\|_{\mathbb{V}}.$$

6. Application: Perturbed Discretizations

We continue the study of the conforming discretization but with perturbation in the data and/or the bilinear form. We first consider the quadrature of the right hand side, i.e., $\langle f, v_h\rangle \approx \langle f_h, v_h\rangle$. The setting is as before except $\Pi_h : \mathbb{V}' \to \mathbb{V}_h'$ defined as

$$\langle \Pi_h f, v_h\rangle = \langle f_h, v_h\rangle,$$

and solve the equation

$$L_h u_h = \Pi_h f \quad \text{in } \mathbb{V}_h'.$$

The stability of L_h^{-1} is unchanged. We only need to estimate the consistency error $\|\Pi_h L_h u_h - L_h v_h\|_{\mathbb{V}_h'} = \|\Pi_h f - L_h v_h\|_{\mathbb{V}_h'}$. By the triangle inequality, we have

$$
\begin{aligned}
\|\Pi_h f - L_h v_h\|_{\mathbb{V}_h'} &\le \|Q_h f - L_h v_h\|_{\mathbb{V}_h'} + \|\Pi_h f - Q_h f\|_{\mathbb{V}_h'}\\
&\lesssim \|u - v_h\|_{\mathbb{V}_h} + \|\Pi_h f - Q_h f\|_{\mathbb{V}_h'}.
\end{aligned}
$$

By the definition

$$\|\Pi_h f - Q_h f\|_{\mathbb{V}_h'} = \sup_{w_h \in \mathbb{V}_h} \frac{|\langle \Pi_h f - Q_h f, w_h\rangle|}{\|w_h\|} = \sup_{w_h \in \mathbb{V}_h} \frac{|\langle f_h, w_h\rangle - \langle f, w_h\rangle|}{\|w_h\|}.$$

We then end up with a version of the first Strang lemma

$$\|u - u_h\|_{\mathbb{V}} \lesssim \inf_{v_h \in \mathbb{V}_h} \|u - v_h\|_{\mathbb{V}} + \sup_{w_h \in \mathbb{V}_h} \frac{|\langle f_h, w_h\rangle - \langle f, w_h\rangle|}{\|w_h\|_{\mathbb{V}_h}}.$$

That is, a perturbation of the data in $\|\cdot\|_{\mathbb{V}_h'}$ is included.

We then consider the perturbation of the bilinear form which includes the case of using numerical quadrature to compute the bilinear form $a(\cdot,\cdot)$ or a non-conforming discretization of L. We are solving the equation

$$\tilde{L}_h u_h = \Pi_h f \quad \text{in } \mathbb{V}_h'.$$

To get similar estimate, we need to assume $\tilde{L}_h$ is defined on $\mathbb{V} + \mathbb{V}_h$. The space $\mathbb{V}_h$ is now endowed by a possibly different norm $\|\cdot\|_h$. We assume the stability and continuity of L_h with respect to the norm $\|\cdot\|_h$.

- Coercivity: for all $v_h \in \mathbb{V}_h$

$$\langle \tilde{L}_h v_h, v_h\rangle \ge \alpha \|v_h\|_h^2.$$

- Continuity: for all $u \in \mathbb{V} + \mathbb{V}_h, v_h \in \mathbb{V}_h$

$$\langle \tilde{L}_h u, v_h\rangle \le \beta \|u\|_h \|v_h\|_h.$$

Then it is straightforward to prove the stability of $\tilde{L}_h : (\mathbb{V}_h', \|\cdot\|_{\mathbb{V}_h'}) \to (\mathbb{V}_h, \|\cdot\|_h)$.

For the consistency error,

$$\|\Pi_h f - \tilde{L}_h v_h\|_{\mathbb{V}_h'} \leq \|\Pi_h f - \tilde{L}_h u\|_{\mathbb{V}_h'} + \|\tilde{L}_h(u - v_h)\|_{\mathbb{V}_h'}$$
$$\leq \sup_{w_h \in \mathbb{V}_h} \frac{\left|\langle f_h, w_h\rangle - \langle \tilde{L}_h u, w_h\rangle\right|}{\|w_h\|} + \|u - v_h\|_h.$$

Therefore we end up with the second Strang lemma

$$\|u - u_h\|_h \lesssim \inf_{v_h \in \mathbb{V}_h} \|u - v_h\|_h + C \sup_{w_h \in \mathbb{V}_h} \frac{\left|\langle f_h, w_h\rangle - \langle \tilde{L}_h u, w_h\rangle\right|}{\|w_h\|_h}.$$

7. Application: Nonconforming Finite Element Methods

We apply the analysis to the nonconforming finite element discretization, i.e., $\mathbb{V}_h \not\subset \mathbb{V} = H_0^1$, of the model Poisson equations using the Crouzeix-Raviart (CR) element as an example.

Given a shape regular mesh $\mathcal{T}_h$, denote all the edges in Ω and on $\partial\Omega$ by $\mathcal{E}_h$ and $\mathcal{E}_h^{\partial}$, respectively. On each $e \in \mathcal{E}_h$, let $[\cdot]$ denote the jump across the edge, i.e. for each piecewise H^1 function v,

$$[v] = v|_{\tau_1} - v|_{\tau_2},$$

where τ_1 and τ_2 are the triangles sharing the edge e. For the boundary edge $e \in \mathcal{E}_h^{\partial}$, $[v] = v|_e$. We then define the Crouzeix-Raviart element as

$$\mathbb{V}_h^{\rm CR} = \left\{ v \mid v \in L^2(\Omega), v|_\tau \in \mathcal{P}_1 \quad \forall \tau \in \mathcal{T}_h, \text{and } \int_e [v] = 0 \; \forall e \in \mathcal{E}_h \cup \mathcal{E}_h^{\partial} \right\}.$$

It can be verified that a function $v \in \mathbb{V}_h^{\rm CR}$ is uniquely determined by its average on each edge $\int_e v \,\mathrm{d}s$ which is known as the degree of freedom and thus the dimension of the space $\mathbb{V}_h^{\rm CR}$, denoted by NE, is the number of the interior edges of the triangulation $\mathcal{T}_h$ and

As before, the operator $L : \mathbb{V} \to \mathbb{V}'$ is defined through the bilinear form:

$$\langle Lu, v\rangle := a(u, v) := (\nabla u, \nabla v).$$

Since the finite element space $\mathbb{V}_h^{\rm CR}$ is not a subspace of $\mathbb{V}$, we shall use a modified bilinear form $a_h(\cdot, \cdot)$, that is

$$a_h(u_h, v_h) = (\nabla_h u_h, \nabla_h v_h) := \sum_{\tau \in \mathcal{T}_h} \int_\tau \nabla u_h \cdot \nabla v_h dx \quad \forall u_h, v_h \in \mathbb{V}_h^{\rm CR}.$$

Namely the gradient operator is applied element-wise. The corresponding discrete operator $\tilde{L}_h : \mathbb{V}_h^{\rm CR} \to (\mathbb{V}_h^{\rm CR})'$ is defined as: for all $u_h, v_h \in \mathbb{V}_h$ that

$$\langle \tilde{L}_h u_h, v_h\rangle := a_h(u_h, v_h).$$

Note that the bilinear form $a_h(\cdot, \cdot)$ is also well defined on $\mathbb{V} \times \mathbb{V}$ and indeed $a_h(u, v) = a(u, v)$ when $u, v \in H^1(\Omega)$. Therefore the operator $\tilde{L}_h$ can be also defined from $\mathbb{V} \to (\mathbb{V})'$.

On the space $\mathbb{V}_h^{\rm CR} + \mathbb{V}$, we use the norm induced by the bilinear form $a_h(\cdot, \cdot)$:

$$\|v\|_{1,h} = a_h^{1/2}(v, v) \quad \forall v \in \mathbb{V}_h^{\rm CR} + \mathbb{V}.$$

One can easily obtain coercivity and continuity of the bilinear form by showing $\|\cdot\|_{1,h}$ is indeed a norm on $\mathbb{V}_h$ (but not on $\mathbb{V}$). Then for any $v, u \in \mathbb{V}_h + \mathbb{V}$, it holds that

$$\|v\|_{1,h}^2 = a_h(v,v),$$
$$a_h(u,v) \le \|u\|_{1,h}\|v\|_{1,h}.$$

Consequently the discrete operator $\tilde{L}_h$ and $\tilde{L}_h^{-1}$ is uniformly bounded.

Given an $f \in L^2(\Omega)$, the nonconforming discrete problem is to find a $u_h \in \mathbb{V}_h^{\rm CR}$ such that

$$\tilde{L}_h u_h = Q_h f.$$

We emphasize that as $\mathbb{V}_h \not\subset \mathbb{V}$, different from the conforming element method, there is no orthogonality

$$a_h(u - u_h, v_h) \neq 0 \quad \forall v_h \in \mathbb{V}_h^{\rm CR},$$

or equivalently in the operator form

$$\tilde{L}_h u \neq \tilde{L}_h u_h \quad \text{in } (\mathbb{V}_h^{\rm CR})'.$$

It is inconsistent. But the consistency error can be controlled as follows.

Lemma 7.1. *Assume that the triangulation $\mathcal{T}_h$ is quasi-uniform, and the solution u has H^2 regularity. Then, it holds for any $w_h \in \mathbb{V}_h^{\rm CR}$ that*

$$(f, w_h) - \langle \tilde{L}_h u, w_h\rangle \le Ch|u|_{H^2}\|w_h\|_{1,h}.$$

Proof. By integration by parts, we derive that

$$(f, w_h) - \langle \tilde{L}_h u, w_h\rangle = \sum_{e\in\mathcal{E}_h}\int_e \nabla u\cdot\boldsymbol{n}\,[w_h]ds.$$

Let $\overline{\nabla u\cdot\boldsymbol{n}}^e$ and $\overline{w}_h^e$ be the average of $\nabla u\cdot\boldsymbol{n}$ and w_h on e respectively. Then,

$$\begin{aligned}\int_e \nabla u\cdot\boldsymbol{n}[w_h]ds &= \int_e (\nabla u\cdot\boldsymbol{n} - \overline{\nabla u\cdot\boldsymbol{n}}^e)[w_h - \overline{w}_h^e]ds \\ &\le \|\nabla u\cdot\boldsymbol{n} - \overline{\nabla u\cdot\boldsymbol{n}}^e\|_{L^2(e)}\|[w_h - \overline{w}_h^e]\|_{L^2(e)}.\end{aligned}$$

For each e, we denote τ_e be the union of the two triangles sharing e, and have the trace theorem with scaling

$$\|v\|_{L^2(e)}^2 \le C(h^{-1}\|v\|_{L^2(\tau_e)}^2 + h|v|_{H^1(\tau_e)}^2) \quad \forall v \in H^1(\tau_e).$$

which yields

$$\int_e \nabla u\cdot\boldsymbol{n}[w_h]ds \le Ch|u|_{H^2(\tau_e)}|w_h|_{H^1(\tau_e)}.$$

Sum over all edges to complete the proof. □

The approximation error can be established using the following canonical interpolation $I_h : \mathbb{V} \to \mathbb{V}_h^{\rm CR}$ using the d.o.f.

$$\int_e I_h v\,\mathrm{d}s = \int_e v\,\mathrm{d}s \quad \forall e \in \mathcal{E}_h.$$

Then there exists a constant independent of the mesh size such that for any $\tau \in \mathcal{T}_h$,

$$|v - I_h v|_{H^1(\tau)} \le Ch^{m-1}|v|_{H^m(\tau)} \quad \forall v \in H^m(\tau), m = 1, 2.$$

The inequality follows from the scaling argument, the trace theorem, and the fact that the operator I_h preserve linear function on τ.

Combining the approximation and consistency, together with the second Strang lemma, we have the optimal order convergence for CR non-conforming approximation u_h to the solution of Poisson equation

$$\|u - u_h\|_{1,h} \leq Ch|u|_{H^2}.$$

8. Application: Finite Volume Methods

We consider the vertex-centered finite volume method and refer to *Finite Volume Methods* for a detailed description and proof of corresponding results. Simply speaking, if we choose the dual mesh by connecting an interior point to middle points of edges in each triangle, the stiffness matrix is identical to that from the linear finite element method. So we shall use the setting for the finite element method and treat the vertex-centered linear finite volume method as a perturbation.

The only difference is the right hand side, i.e., $\Pi_h : L^2(\Omega) \to \mathbb{V}_h'$. Here we need to shrink the space $\mathscr{Y}$ from $H^{-1}(\Omega)$ to $L^2(\Omega)$ to have the volume integral $\int_b f \, \mathrm{d}x$ well defined. To define such Π_h, let us introduce the piecewise constant space on the dual mesh $\mathcal{B}$ and denoted by $\mathbb{V}_{0,\mathcal{B}}$. We rewrite the linear finite element space as $\mathbb{V}_{1,\mathcal{T}}$. Through the point values at vertices, we define the following mapping

$$\Pi^* : \mathbb{V}_{1,\mathcal{T}} \to \mathbb{V}_{0,\mathcal{B}} \quad \text{as } \Pi^* v_h = \sum_{i=1}^{N} v_h(x_i)\chi_{b_i}.$$

For any $f \in L^2(\Omega)$, we then define $\Pi_h f \in \mathbb{V}_h'$ as

$$\langle \Pi_h f, v_h \rangle = (f, \Pi^* v_h), \quad \text{for all } v_h \in \mathbb{V}_h.$$

The analysis below follows closely to Hackbusch [2]. Denoted by u_h^G as the standard Galerkin (finite element) approximation and u_h^B is the box (finite volume) approximation. The equivalence of the stiffness matrices means

$$L_h u_h^G = Q_h f, \quad L_h u_h^B = \Pi_h f.$$

Therefore by the stability of L_h^{-1}, we have

$$|u_h^G - u_h^B|_1 \leq \|Q_h f - \Pi_h f\|_{-1,h}.$$

By the definition

$$\langle Q_h f - \Pi_h f, v_h \rangle = (f, v_h - \Pi^* v_h).$$

Denote the support of the hat basis function at vertex x_i as ω_i. Note that $b_i \subset \omega_i$ and the operator $I - \Pi^*$ preserve constant function in the patch ω_i and thus

$$(f, v_h - \Pi^* v_h)_{b_i} \leq \|f\|_{b_i} \|v_h - \Pi^* v_h\|_{\omega_i} \leq Ch\|f\|_{b_i} |v_h|_{1,\omega_i}.$$

Summing up and using the Cauchy Schwarz inequality, we get the first order convergence

$$|u_h^G - u_h^B|_1 \leq Ch\|f\|.$$

Furthermore, if the dual mesh is symmetric in the sense that we use barcenters as a vertex of control volumes. Then we have

$$\int_\tau v_h = \int_\tau \Pi^* v_h,$$

and thus, let $\bar{f}_h$ be the L^2 projection f to the piecewise constant function in each triangle,

$$(f, v_h - \Pi^* v_h) = (f - \bar{f}_h, v_h - \Pi^* v_h) \leq Ch^2 \|f\|_1 |v_h|_1.$$

We then obtain the superconvergence or the supercloseness of u_h^G and u_h^B:

$$|u_h^G - u_h^B|_1 \le Ch^2\|f\|_1. \tag{12}$$

With such relation, we can obtain optimal L^2 error estimate for u_h^B and quasi-optimal L^∞ error estimate in two dimensions. By the triangle inequality, Poincaré inequality, and the optimal order convergence of $\|u - u_h^G\|$,

$$\|u - u_h^B\| \le \|u - u_h^G\| + \|u_h^G - u_h^B\| \le Ch^2\|u\|_2 + |u_h^G - u_h^B|_1 \le Ch^2(\|u\|_2 + \|f\|_1),$$

and similarly with discrete embedding $\|v_h\|_\infty \le C|\log h||v_h|_1$,

$$\begin{aligned}\|u - u_h^B\|_\infty &\le \|u - u_h^G\|_\infty + \|u_h^G - u_h^B\|_\infty \le Ch^2\|u\|_2 + C|\log h||u_h^G - u_h^B|_1\\ &\le C|\log h|h^2(\|u\|_2 + \|f\|_1).\end{aligned}$$

9. Application: Finite Difference Methods (revisited)

We revisit the finite difference method but treat it as a perturbation of the linear finite element method. So we will use the setting for finite element methods in Section §4. The relation between the linear finite element and 5-point stencil on the uniform grids is

$$\langle L_h u_h, v_h\rangle = \boldsymbol{u}_h^{\intercal}\boldsymbol{A}_h\boldsymbol{v}_h = -h^2\boldsymbol{u}_h^{\intercal}\boldsymbol{\Delta}_h\boldsymbol{v}_h.$$

Here we use boldface small letters to denoted the vector formed by the function values at vertices. More precisely for $u_h \in \mathbb{V}_h$, we define $\boldsymbol{u}_h \in \mathbb{R}^N$ such that $(\boldsymbol{u}_h)_i = u_h(x_i)$.We use boldface capital letters for matrices. For example, $-\boldsymbol{\Delta}_h$ is the stencil matrix and $\boldsymbol{A}_h$ is the matrix for P1 finite element method. We rescale the stiffness matrix $\boldsymbol{A}_h$ from the linear finite element method to get the five-point stencil matrix $-\boldsymbol{\Delta}_h$. Here uniform grids we mean the three directional triangulation obtained by using diagonals with the same direction of uniform rectangular grids; see Fig. 1.

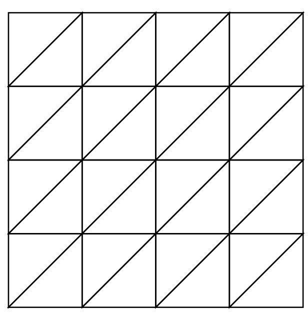

FIGURE 1. A uniform mesh for a square

We choose I_h as the nodal interpolation and define $\Pi_h : C(\Omega) \to \mathbb{V}_h'$ as

$$\langle \Pi_h f, v_h\rangle = h^2\boldsymbol{f}_I^{\intercal}\boldsymbol{v}_h = \sum_{i=1}^N h^2 f(x_i)v_h(x_i).$$

It can be interpret as using three vertices as the numerical quadrature. The equation for FDM can be rewritten as

$$L_h u_h = \Pi_h f.$$

By the first Strang lemma, we only need to estimate the perturbation $\|\Pi_h f - Q_h f\|_{-1,h}$ which is left as an exercise, cf. Exercise 1 in Section §11. The approximation error $|u - u_I|_1 \lesssim h|u|_2$ is the interpolation error studied in *Introduction to Finite Element Methods*.

In summary, the first order error estimate

$$|u_I - u_h|_1 \lesssim h(|u|_2 + |f|_1)$$

can be easily proved without checking the point-wise truncation error and the smoothness of the solution u can be relaxed to $u \in H^2(\Omega)$ only.

If we assume that the truncation error in l_∞ norm is of second order, i.e.,

$$|(\Delta u)_I - \boldsymbol{\Delta}_h \boldsymbol{u}_I|_{l_\infty} \leq Ch^2,$$

then we can prove a second order convergence by examining the consistency error $\|\Pi_h f - L_h u_I\|_{-1,h}$ more carefully:

$$\begin{aligned} |\langle \Pi_h f - L_h u_I, v_h\rangle| &= |h^2(\boldsymbol{f}_I + \boldsymbol{\Delta}_h \boldsymbol{u}_I)\cdot \boldsymbol{v}_h| \\ &\leq \|\boldsymbol{f}_I + \boldsymbol{\Delta}_h \boldsymbol{u}_I\|_\infty \sum_{i=1}^N h^2 |v_h(x_i)| \\ &\leq C(u) h^2 |v_h|_1. \end{aligned}$$

In the last step, we have used the bound

$$\sum_{i=1}^N h^2 |v_h(x_i)| \leq C\|v_h\|_{L^1} \leq C\|v_h\|_{L^2} \leq C|v_h|_1.$$

Therefore $\|\Pi_h f - L_h u_I\|_{-1,h} \leq C(u)h^2$ and consequently

$$|u_I - u_h|_1 \lesssim h^2 |u|_{4,\infty}.$$

In 2-D, using the discrete embedding theorem, we can obtain the nearly optimal maximum norm estimate

$$\|u_I - u_h\|_\infty \lesssim (1 + |\log h|)|u_I - u_h|_1 \leq C(u,f)(1 + |\log h|)h^2.$$

Here we consider Dirichlet boundary condition and assume the truncation error for interior nodes is second order. For Neumann boundary condition or cell centered finite difference methods, the boundary or near boundary stencil is modified and the truncation error could be only first order $\mathcal{O}(h)$; see *Finite Difference Methods*. We shall show that the boundary stencil can be just first order while the scheme is still second order measure in H^1-norm.

Let us label the boundary (or near boundary) nodes from $1 : N_b$ and interior nodes from $N_b + 1 : N$ and assume the truncation error for boundary nodes is only $\mathcal{O}(h)$ and for interior nodes is still $\mathcal{O}(h^2)$. Then

$$\begin{aligned} |\langle \Pi_h f - L_h u_I, v_h\rangle| &= |h^2(\boldsymbol{f}_I + \boldsymbol{\Delta}_h \boldsymbol{u}_I)\cdot \boldsymbol{v}_h| \leq \|\boldsymbol{f}_I + \boldsymbol{\Delta}_h \boldsymbol{u}_I\|_\infty \sum_{i=1}^N h^2 |v_h(x_i)| \\ &= h^2 \left(\sum_{i=1}^{N_b} h|v_h(x_i)| + \sum_{i=N_b+1}^{N} h^2 |v_h(x_i)| \right) \\ &\leq C(u) h^2 \left(\|v_h\|_{L^1(\partial\Omega)} + \|v_h\|_{L^1(\Omega)} \right) \\ &\leq C(u) h^2 |v_h|_1. \end{aligned}$$

In the last step, we have used the following inequalities

$$\|v_h\|_{L^1(\partial\Omega)} \lesssim \|v_h\|_{L^2(\partial\Omega)} \lesssim \|v_h\|_{1,\Omega} \lesssim |v_h|_1.$$

The trick is to bound the boundary part by the L^1 norm of the trace and then use the trace theorem to change to H^1-norm of the whole domain.

By the same line of the above proof, to get the first order convergence i.e. $|u_I - u_h|_1 \lesssim h$, the truncation error of the stencil on the boundary (or an interface) can be only $\mathcal{O}(h)$. Namely the scheme is not consistent point-wisely but still converges!

10. Application: Superconvergence of the linear finite element method

We shall revisit the linear finite element method and provide a refined analysis on the consistency error. In some cases we may have a sharper estimate of the consistency error in the weak norm $\|\cdot\|_{-1,h}$. For example, for 1-D Poisson problem, using integration by parts and noting that $(u-u_I)|_{\partial\tau} = 0$, we get, for any $v_h \in \mathbb{V}_h$

$$(Q_h Lu - L_h u_I, v_h) = (u' - u_I', v_h') = \sum_{\tau\in\mathcal{T}_h} (u - u_I, v_h')_\tau = 0,$$

which implies the discrete error is zero. Namely

$$u_h(x_i) = u(x_i) \text{ at each grid points } x_i.$$

In high dimensions, when the mesh is locally symmetric and the function is smooth enough, we may establish the following strengthened Cauchy-Schwarz inequality

$$\langle Q_h Lu - L_h u_I, v_h\rangle = (\nabla u - \nabla u_I, \nabla v_h) \le Ch^2 \|u\|_3 |v_h|_1. \tag{13}$$

Then consequently we obtain the superconvergence result

$$|u_h - u_I|_1 \le \|Q_h Lu - L_h u_I\|_{-1,h} \le Ch^2\|u\|_3.$$

Roughly speaking, the symmetry of the mesh will bring more cancelation when measuring the consistency error in a weak norm. For example, (13) holds when the two triangles sharing an interior edge forms an $\mathcal{O}(h^2)$ approximate parallelogram for most edges in the triangulation; See [1] for detailed condition on triangulation and a proof of (13).

We shall give a simple proof of superconvergence result on uniform grids by using the relation to the finite difference method. Rewrite $Q_h Lu = Q_h f$ and divide the consistency error into two parts

$$\|Q_h f - L_h u_I\|_{-1,h} \le \|Q_h f - \Pi_h f\|_{-1,h} + \|\Pi_h f - L_h u_I\|_{-1,h} = \mathrm{I}_1 + \mathrm{I}_2.$$

To estimate I_1, we note $\langle Q_h f, v_h\rangle = \sum_{i=1}^N (\int_{\omega_i} f\phi_i) v_h(x_i)$ and $h^2 f(x_i)$ is a quadrature for the integral $\int_{\omega_i} f\phi_i$. It is only exact for constant f restricted to one simplex. But for uniform grids, due to the configuration of ω_i, the support of the hat function ϕ_i, the linear functional

$$E_i(f) = \left| \int_{\omega_i} f\phi_i \, dx - h^2 f(x_i) \right|$$

preserve linear polynomials in $\mathcal{P}_1(\omega_i)$. Namely $E_i(f_h) = 0$ for $f_h \in \mathcal{P}_1(\omega_i)$. Therefore

$$|E_i(f)| = |E_i(f - f_h)| \le \|f - f_h\|_\infty h^2 \lesssim h^{2+2}\|f\|_{2,\infty}.$$

Remark 10.1. For finite volume approximations, the corresponding difference is

$$E_i(f) = \left| \int_{b_i} f \, dx - h^2 f(x_i) \right|.$$

For uniform grids, we can chose b_i as the dual grid which contains the vertex as the center and $|b_i| = h^2$ and thus $E_i(f)$ can preserve linear polynomial in $\mathcal{P}_1(b_i)$. $\Box$

Therefore we can estimate the I_1 as

$$|\langle Q_h f - \Pi_h f, v_h\rangle| = |\sum_{i=1}^{N} E_i(f) v_h(x_i)| \leq C_f h^2 \sum_{i=1}^{N} h^2 |v_h(x_i)| \leq C_f h^2 |v_h|_1.$$

And consequently

$$\|Q_h f - \Pi_h f\|_{-1,h} \leq C(f) h^2.$$

The second order estimate of part I_2, i.e., $\|\Pi_h f - L_h u_I\|_{-1,h} \leq C(u)h^2$ has been analyzed in Section §9.

In conclusion, we have proved the superconvergence for the linear finite element method on uniform grids

$$|u_I - u_h|_1 \leq C(u,f) h^2,$$

by estimating the consistency error in a weaker norm. In our simpler proof we need stronger smoothness assumption

$$C(u,f) = \|D^4 u\|_\infty + \|D^2 f\|_\infty,$$

which can be relaxed to $u \in H^3(\Omega)$; see [1].

11. Exercise

1. Use three vertices quadrature rule, i.e. for a triangle τ formed by three vertices $x_i, i = 1,2,3$,
$$\int_\tau g(x)dx \approx \frac{1}{3}\sum_{i=1}^{3} g(x_i)|\tau|,$$
to compute the right hand side (f, ϕ_i) and show it can be simply written as $\boldsymbol{M} f_I$, where $\boldsymbol{M}$ is a diagonal matrix. Prove the optimal first order approximation in H^1-norm for the linear finite element method of Poisson equation using this quadrature.
2. Show $\|\cdot\|_{1,h}$ defined in Section §7 is a norm defined on $\mathbb{V}_h^{\rm CR}$.
3. Present error estimate for the cell-centered finite difference method (Section 4 in *Finite Difference Methods*) with different treatment of the Dirichlet boundary condition.

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