PROJECT 3: NUMERICAL METHODS FOR CONSERVATION LAWS

In this project, we consider several schemes for nonlinear conservation laws. We shall use the following inviscid Burgers' equation as the model problem

$$u_t + uu_x = 0, \quad x \in (-1, 1), t > 0, \tag{1}$$

$$u(x, 0) = u_0(x). \tag{2}$$

1. Implement the natural extension of the unwinding scheme

$$u_k^{n+1} = u_k^n - R u_k^n \delta_- u_k^n$$

for the equation (1) along with the initial condition

$$u_0(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

and boundary condition

$$u(-1, t) = 1, \; u(1, t) = 0.$$

Plot your numerical solution for $\Delta x = 0.01, \Delta t = 0.005$ (and thus $R = \Delta t/\Delta x = 0.5$) at time $t = 0.5$.

2. Implement a version of Lax-Wendroff scheme

$$u_k^{n+1} = u_k^n - \frac{R}{2}\delta_0 F_k^n + \frac{R^2}{2}\delta_-(a_{k+1/2}^n \delta_+ F_k^n),$$

where the speed $a_{k+1/2}^n$ is defined as

$$a_{k+1/2}^n = \begin{cases} \delta_+ F_k^n / \delta_+ u_k^n & \text{if } \delta_+ u_k^n \neq 0 \\ F'(u_k^n) & \text{if } \delta_+ u_k^n = 0. \end{cases}$$

The initial condition is

$$u_0(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0.5 & \text{if } x \geq 0 \end{cases}$$

and boundary condition $u(-1, t) = 1$ and numerical boundary condition , $u_N^n = 0$.

Use $\delta = 0.01, \Delta t = 0.001$ and calculate the solution at $t = 0.5$. Compare with the exact vanishing viscosity solution

$$v(x, t) = \begin{cases} 1 & \text{if } x \leq 3t/4 \\ 0.5 & \text{if } x > 3t/4. \end{cases}$$

3. Implement the Lax-Friedrichs scheme

$$u_k^{n+1} = \frac{1}{2}(u_{k-1}^n + u_{k+1}^n) - \frac{R}{2}(F_{k+1}^n - F_{k-1}^n).$$

Apply to the Burgers' equation with initial and boundary condition in 2. Use $\delta = 0.01, \Delta t = 0.001$ and calculate the solution at $t = 0.5$.

4. Use Godunov scheme to solve the initial-boundary-value problems in 2.

5. Implement the flux limiter scheme with numerical flux

$$h^n_{k+1/2} = \frac{1}{2}(F^n_k + F^n_{k+1}) - \frac{1}{2R}\delta_+ u^n_k + \phi^n_k(\theta^n_k)\left[\frac{1}{2}(F^n_k + F^n_{k+1}) + \frac{1}{2}\delta_- F^n_k + \frac{1}{2R}\delta_+ u^n_k\right],$$

where

$$\theta^n_k = \begin{cases} \dfrac{\delta_- u^n_k}{\delta_+ u^n_k} & \text{if } a^n_{k+1/2} > 0 \\[2ex] \dfrac{\delta_+ u^n_{k+1}}{\delta_+ u^n_k} & \text{if } a^n_{k+1/2} < 0 \end{cases}$$

and $\phi(\theta)$ is given by the Superbee limiter

$$\phi(\theta) = \max\{0, \min[1, 2\theta], \min[\theta, 2]\}.$$

Use this scheme to solve the initial-boundary-value problems in 2.