Ch 2. Investigation of the Equation of Motion

System with one degree of freedom $\ddot{x} = f(x)$

Kinetic energy $T = \frac{1}{2}\dot{x}^2$

Potential energy $U(x) = -\int_{x_0}^{x} f(\xi)\, d\xi$

$\Big\}$ Total energy $E = T + U$

Conservation of energy. If $\ddot{x} = f(x)$, then $E(x(t), \dot{x}(t))$ is independent of t.

Phase flow. $\begin{cases} \dot{x} = y \\ \dot{y} = f(x) \end{cases}$ (2). Phase plane (x, y).

The vector field $(y, f(x))$: phase velocity vector field.

a motion $\varphi: \mathbb{R} \to \mathbb{R}^2$ solves (2). The image of φ: phase curve.

$(\varphi(t), \dot{\varphi}(t))$

Equilibrium position: $\varphi(t) = \text{const}$, $\dot{\varphi}(t) = \text{const}$

$(\varphi_0, 0)$

Energy level set: $E(x, y) = h$. By conservation of energy, each phase curve lies entirely in one energy level set.

Example 2. Equilibrium pt $\longleftrightarrow$ critical pt of U. Figures.

The ball cannot jump out of the poential well.

Phase flow. $M(t) = g^t M \quad g^t : \mathbb{R}^2 \to \mathbb{R}^2$

$M(t)$ solves the system (2) and assumed exists

g^t: one-parameter group of diffeomorphisms

the phase flow of system (2).

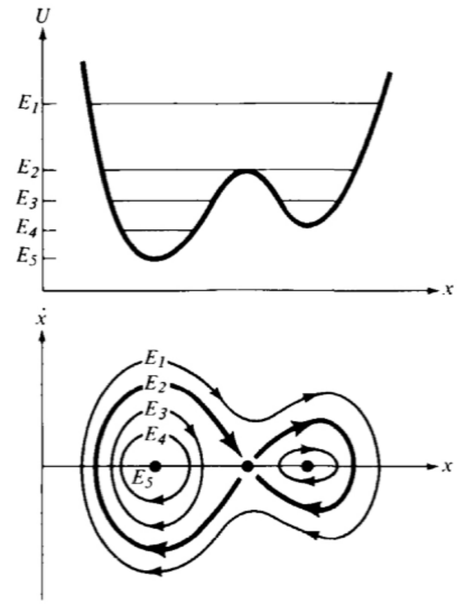


Figure 10 Potential energy and phase curves

🌟 Systems with two degrees of freedom

$\ddot{x} = f(x), \quad x \in E^2.$ Conservative system: $f = -\frac{\partial U}{\partial x}$

Th. The total energy of a conservative system is conserved.

For $x \in E^1$, $U = -\int^x f(\xi)\,d\xi$ is a poential. For $x \in E^d$, $d \geq 2$, there exists non-conservative system.

Similar as before, we have the phase space (x_1, x_2, y_1, y_2)

Phase flow: group of diffeomorphism of phase space.

Level set of energy Π_{E_0}. Then $g^t \Pi_{E_0} = \Pi_{E_0}$.

Visulization: project to (x_1, x_2) plane.

Example 1. Homework.

Example 2. Lissajous figures. $\begin{cases} \ddot{x}_1 = -x_1, \\ \ddot{x}_2 = -\omega^2 x_2. \end{cases}$ $U = \frac{1}{2}x_1^2 + \frac{1}{2}\omega^2 x_2^2$. $E = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + U(x_1, x_2)$

$$\begin{cases} x_1 = A_1 \sin(t+\varphi_1) \\ x_2 = A_2 \sin(\omega t + \varphi_2) \end{cases}$$

Figures.

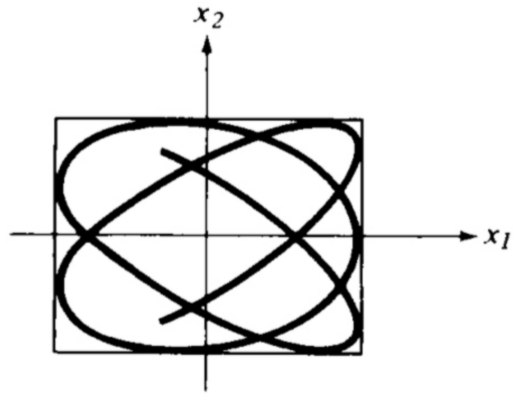


Figure 21 Lissajous figure with $\omega \approx 1$

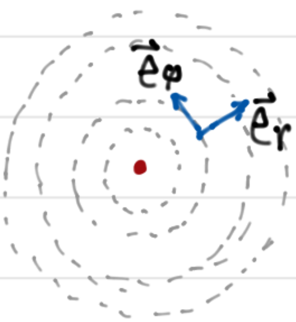
⭐ Conservative force fields

Work: $\int_l \vec{F} \cdot d\vec{s}$ line integral.

Theorem. $\vec{F}$ is conservative $\Leftrightarrow$ $\int_l \vec{F} \cdot d\vec{s}$ is independent of path.

Definition. A vector field is called central with center at 0, if it is invariant w.r.t. the group of motions of the plane which fix 0.

$$\vec{F}(\vec{r}) = \Phi(r)\, \vec{e}_r$$



$\vec{r} = (r, \phi)$, $\quad \vec{e}_r = \dfrac{\vec{r}}{r} = \dfrac{\vec{r}}{|\vec{r}|}$ is the unit vector

$$\begin{cases} x = r\cos\phi, \\ y = r\sin\phi. \end{cases}$$

$\vec{e}_\varphi \perp \vec{e}_r$ and $\vec{e}_r, \vec{e}_\varphi$ form a RH coor.

Theorem. Every central field is conservative and $U(\vec{r}) = \int_0^{\vec{r}} \Phi(r)\, dr$.

Angular momentum

Motion in a central field $\boxed{\ddot{r} = \Phi(r)\, e_r}$ (1)

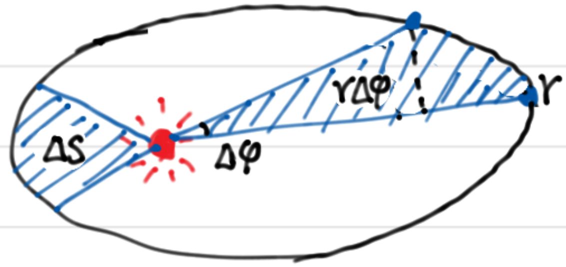
The angular momentum: $M = r \times \dot{r}$

Conservation of the angular momentum

THEOREM. If r is determined by (1), then $M = r \times \dot{r}$ is a constant.

Pf. $M = r \times \dot{r}$. Then $\dot{M} = \dot{r} \times \dot{r} + r \times \ddot{r} = 0 + r \times \Phi(r)\, r = 0$. #.

Kepler's law $\frac{dS}{dt} = \text{const}$

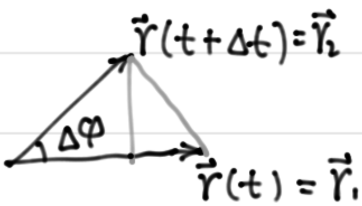


is the law of conservation of angular momentum.

$$\Delta S = \frac{1}{2} r (r \Delta\varphi) + o(\Delta t) = \frac{1}{2} r^2 \dot{\varphi} \Delta t + o(\Delta t).$$

$$M = \vec{r} \times \dot{\vec{r}} = \vec{r} \times (\dot{r}\vec{e}_r + r\dot{\varphi}\vec{e}_\varphi) = r^2 \dot{\varphi} (\vec{e}_r \times \vec{e}_\varphi).$$

} $r^2 \dot{\varphi}$ is const.



$$\vec{r}_2 = (r_2 \cos\Delta\varphi)\, \vec{e}_{r_1} + (r_2 \sin\Delta\varphi)\, \vec{e}_\varphi$$

$$\vec{r}_1 = r_1 \vec{e}_{r_1}$$

$$\dot{\vec{r}} = \dot{r}\vec{e}_r + r\dot{\varphi}\vec{e}_\varphi$$

$$\begin{cases} \dot{\vec{e}}_r = \dot{\varphi}\vec{e}_\varphi \\ \dot{\vec{e}}_\varphi = -\dot{\varphi} e_r \end{cases} \quad \text{rotation}$$

$$\frac{\vec{r}_2 - \vec{r}_1}{\Delta t} = \underbrace{\frac{r_2 \cos\Delta\varphi - r_1}{\Delta t}}_{\dot{r}_1} \vec{e}_{r_1} + \underbrace{\frac{r_2 \sin\Delta\varphi}{\Delta t}}_{r_1\dot{\varphi}} \vec{e}_\varphi$$

$$r_2 = r_1 + \dot{r}_1 \Delta t + o(\Delta t), \quad \cos\Delta\varphi = 1 + o(\Delta\varphi), \quad \sin\Delta\varphi = \Delta\varphi + o(\Delta\varphi)$$

⭐ Investigation of motion in a central field

Solve $\ddot{\vec{r}} = -\frac{\partial U}{\partial \vec{r}}$, $U = U(r)$ depends only on $r = |\vec{r}|$.

Although $\vec{r} \in E^2$, (2 d.o.f.) central field reduces it to 1-D problem.

$$\boxed{\ddot{r} = -\frac{\partial V}{\partial r}}$$

$V(r) = U(r) + \frac{M^2}{2r^2}$ effective potential

Pf. $\dot{\vec{r}} = \dot{r}\vec{e}_r + r\dot{\varphi}\vec{e}_\varphi$, $\dot{\vec{e}}_r = \dot{\varphi}\vec{e}_\varphi$, $\dot{e}_\varphi = -\dot{\varphi}\vec{e}_r$

$$\ddot{\vec{r}} = \ddot{r}\vec{e}_r + \dot{r}\dot{\vec{e}}_r + \dot{r}\dot{\varphi}\vec{e}_\varphi + r\ddot{\varphi}\vec{e}_\varphi + r\dot{\varphi}\dot{\vec{e}}_\varphi$$

$$= (\ddot{r} - r\dot{\varphi}^2)\vec{e}_r + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\vec{e}_\varphi$$

$$\frac{\partial U}{\partial \vec{r}} = \frac{\partial U}{\partial r}\vec{e}_r$$

$$\left.\begin{aligned} &\ddot{r} - r\dot{\varphi}^2 = -\frac{\partial U}{\partial r} \\ &2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0 \end{aligned}\right.$$

$$d(r^2\dot{\varphi}) = 0$$

angular momentum

$\dot{\varphi} = \frac{M}{r^2}$ substitute back to get the eqn. #.

Try to solve ODE: $\boxed{\ddot{r} = -\frac{\partial U}{\partial r} + \frac{M^2}{r^3}}$.

Use the method: conservation of energy. $E = \frac{1}{2}\dot{r}^2 + V(r) = \text{const}$

$$\dot{r} = \sqrt{2(E - V(r))}, \quad \dot{\varphi} = \frac{M}{r^2}$$

$$\frac{dr}{dt} \qquad \frac{d\varphi}{dt}$$

$$\frac{d\varphi}{dr} = \frac{M}{r^2\sqrt{2(E-V(r))}}$$

$$\varphi = \int \frac{M}{r^2\sqrt{2(E-V(r))}}\,dr$$

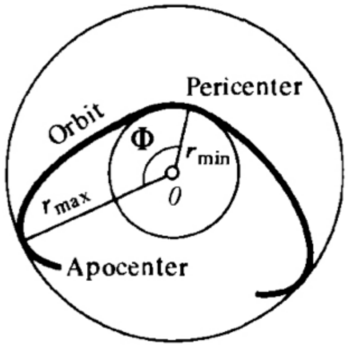
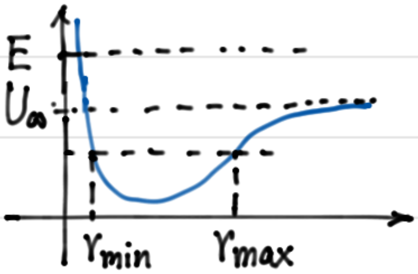


Figure 32 Orbit of a point in a central field

$$\Phi = \int_{r_{min}}^{r_{max}} \frac{M}{r^2\sqrt{2(E-V(r))}}\,dr$$

if $\Phi = 2\pi\left(\frac{m}{n}\right)$, then the orbit is closed. Otherwise it is dense in the annulus.



$E = \frac{1}{2}\dot{r}^2 + V(r)$ is constant

if $\lim_{r\to+\infty} V(r) = \lim_{r\to+\infty} U(r) = U_\infty$, then $\dot{r}_\infty = \sqrt{2(E-V_\infty)}$

The point goes to infinity with speed $\dot{r}_\infty$

Kepler's problem: $U = -\frac{k}{r}$, $V(r) = -\frac{k}{r} + \frac{M^2}{2r^2}$

gravity poential

Integrate to get $r = \frac{p}{1+e\cos\varphi}$, p: parameter; e: eccentricity.

The motion of a point in three-space.

Skip. Almost identical.

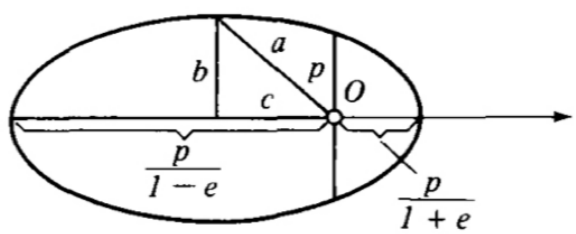


Figure 35 Keplerian ellipse

Motions of a System of n points

Newton's equations $m_i\ddot{\vec{r}}_i = \vec{F}_i$, $i = 1, 2, \cdots, n$.

forces of interaction: $\vec{F}_{ij} = -\vec{F}_{ji} = f_{ij}\vec{e}_{ij}$

closed system: all forces are interaction $\vec{F}_i = \sum_j \vec{F}_{ij}$

Kepler's Problem

① · **First law**: the planets describe ellipse, with the sun at one focus.

② · **Second law**: in equal times the radius vector sweeps out equal areas, so that the sectorial velocity is constant

③ · **Third law**: the period of revolution around an elliptical orbit depends only on the size of the major semi-axes. The squares of the revolution periods of two planets on different elliptical orbits have the same ratio as the cubes of their major semi-axes: $T^2 \sim a^3$

Answer or Proofs. ② second law is the conservation of angular momentum which holds for all central field motion.

①, ③ can be derived by solving $\ddot{\vec{r}} = -\frac{\partial U}{\partial \vec{r}}$, for $U(\vec{r}) = -\frac{k}{\vec{r}}$.

Use moving frame $(\vec{e}_r, \vec{e}_\varphi)$, the eqn $\ddot{\vec{r}} = -\frac{\partial U}{\partial \vec{r}}$ can be reduced to two

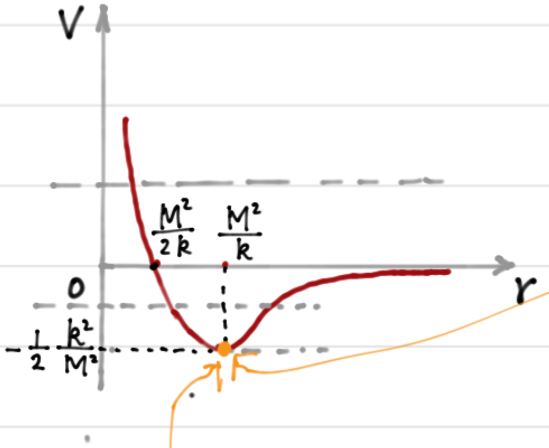
$$\begin{cases} \ddot{r} = -\frac{\partial V}{\partial r}, & \text{with effective poential } V(r) = U(r) + \frac{M^2}{2r^2} \\ M := r^2 \dot{\varphi} = \text{const} & \text{conservation of angular momentum} \end{cases}$$

Use conservation of energy, $E = \frac{1}{2}\dot{r}^2 + V(r) = \text{const}$, we can solve $\dot{r} = \sqrt{2(E-V)}$. Together with $\dot{\varphi} = \frac{M}{r^2}$, we get

$$\frac{d\varphi}{dr} = \frac{M/r^2}{\sqrt{2(E-V(r))}}. \quad \varphi = \int \frac{M/r^2}{\sqrt{2(E-V(r))}} dr.$$

initial location and velocity $(\vec{r}_0, \dot{\vec{r}}_0) \to (E, M) \to \varphi = \varphi(r)$.

For gravational field, $U(r) = -\frac{k}{r}$, $V(r) = -\frac{k}{r} + \frac{M^2}{2r^2}$



For $r_0 = \frac{M^2}{k}$, $\dot{r}_0 = 0$, the trajectory is a circle $r = r_0$. Note that only the norm remains the same. The vector $\vec{r}(t)$ is changing as $\dot{\varphi} = \frac{M}{r^2}$ is const.

When $E > 0$, $r(t) \to +\infty$ as $t \to +\infty$ as $V_\infty = 0$, $\dot{r} = \sqrt{2E} > 0$.

For $U(r) = -\frac{k}{r}$, it is integrable and the solution can be written as

$$r = \frac{p}{1 + e\cos\varphi}, \quad p = \frac{M^2}{k}, \quad e = \sqrt{1 + \frac{2EM^2}{k^2}} \quad \text{eccentricity}$$

When $E = -\frac{1}{2}\frac{k^2}{M^2}$, $e = 0$. It reduces to $r = p$ which is a circle.

For planets orbits, $e \ll 1$. It is almost a circle.

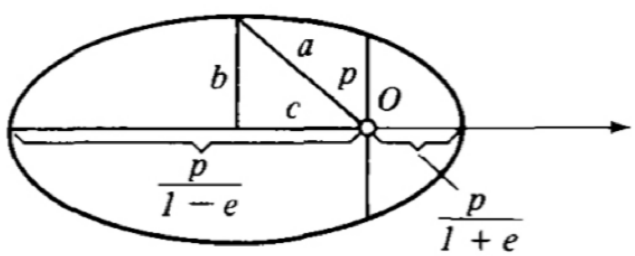
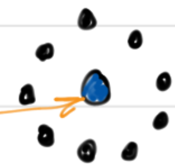


Figure 35 Keplerian ellipse

$$a = \frac{p}{1-e^2}, \quad b = a\sqrt{1-e^2} \approx a\left(1 - \frac{1}{2}e^2\right), \quad c = ae$$

external force: F_i'

Momentum: $P = \sum_{i=1}^{n} m_i \dot{\vec{r}}_i$, $\quad \frac{dP}{dt} = \sum_{i=1}^{n} F_i'$



Center of mass: $\vec{r} = \frac{\sum m_i \vec{r}_i}{\sum m_i}$

Equation for the center of mass: $(\sum m_i)\ddot{\vec{r}} = \sum \vec{F}_i$ $\quad$ 1-D problem

Angular momentum of a system: $M = \sum_{i=1}^{n} \vec{r}_i \times m_i \dot{\vec{r}}_i$.

Equation: $\frac{dM}{dt} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i'$ $\quad$ only external force.

$$\vec{r}_i \times \vec{F}_{ij} + \vec{r}_j \times \vec{F}_{ji} = (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij} = 0$$

Kinetic energy $\quad T = \sum_{i=1}^{n} \frac{m_i \dot{\vec{r}}_i^{\,2}}{2}$

Equation: $\frac{dT}{dt} = \sum_{i=1}^{n} (\dot{\vec{r}}_i, \vec{F}_i)$

$\vec{r} = (\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n)$

$\vec{F} = (\vec{F}_1, \vec{F}_2, \cdots, \vec{F}_n)$

Conservative system: $\int_{M_1}^{M_2} \vec{F} \cdot d\vec{r} = \Phi(M_1, M_2)$ $\quad$ independent of path

$\Leftrightarrow \quad \vec{F} = -\frac{\partial U}{\partial \vec{r}}$

For a conservative system $E = T + U$ is preserved.

Interaction force $\quad f_{ij} = f_{ij}(|\vec{r}_i - \vec{r}_j|)$, then $\quad U_{ij}(\vec{r}) = \int_{r_0}^{r} f_{ij}(\rho)\, d\rho$

$U(\vec{r}) = \sum_{i>j} U_{ij}(|\vec{r}_i - \vec{r}_j|)$ is the poential.

Pf. verify $\quad -\frac{\partial U_{ij}(|\vec{r}_i - \vec{r}_j|)}{\partial \vec{r}_i} = -f_{ij} \frac{\partial |\vec{r}_i - \vec{r}_j|}{\partial \vec{r}_i} = f_{ij} \vec{e}_{ij}$

The two-body problem.

$$\begin{cases} m_1 \ddot{\vec{r}}_1 = \vec{F}_1, & \vec{F}_i = -\dfrac{\partial U}{\partial \vec{r}_i} \qquad \vec{F}_1 = -\vec{F}_2 \\ m_2 \ddot{\vec{r}}_2 = \vec{F}_2, & U = U(|\vec{r}_1 - \vec{r}_2|) \quad \text{conservative} \end{cases}$$

So $m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2 = \vec{P}$ is constant. $\vec{r}_0 = \frac{m_1}{m_1+m_2} \vec{r}_1 + \frac{m_2}{m_1+m_2} \vec{r}_2$

Then $\dot{\vec{r}}_0 = \frac{1}{m_1+m_2} \vec{P}$. So the center of mass moves uniformly and linearly.

$$m_1 m_2 (\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2) = -(m_1+m_2) \frac{\partial U}{\partial (\vec{r}_1 - \vec{r}_2)}$$

For planets, the center's eqn is $\dot{\vec{r}}_0 = 0$. Not moving. It becomes the foci of the ellipse.

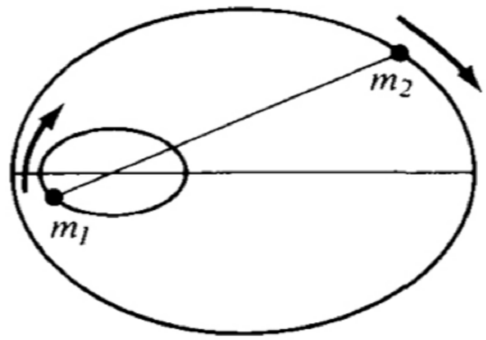


Figure 40 The two body problem