Ch3 Variational Principles

Hamilton's principle of least action $\Rightarrow$ Newton's eqn.

12 Calculus of Variation

Function $\Phi(x) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))\,dt$

$t \to x(t) \longrightarrow \Phi(x)$

- Chain rule
- Integration by parts
- Change of variables

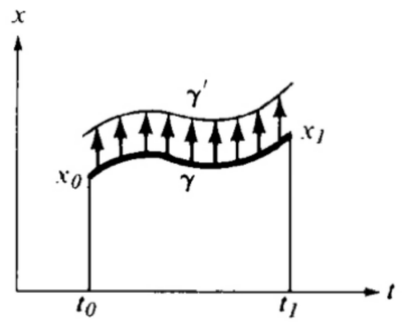


Figure 41 Variation of a curve

A Variations

Def. Φ is differentiable if $\Phi(x+h) - \Phi(x) = \langle \delta\Phi(x), h \rangle + R(x,h)$, where $\langle \delta\Phi(x), \cdot \rangle$ is linear and $R(x,h) = O(h^2)$ in the sense that for $h \in C^1(t_0, t_1)$, $\|h\|_{C^1} \leq \varepsilon$ (i.e. $|h(t)| \leq \varepsilon$, $|\dot{h}(t)| \leq \varepsilon$, $\forall t \in [t_0, t_1]$), then $|R| < C\varepsilon^2$. The linear part $\delta\Phi(x)$ is called the differential / variation.

Variational form $\langle \delta\Phi(x), h \rangle = \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial x} h + \frac{\partial L}{\partial \dot{x}} \dot{h}\right] dt$

Proof. Taylor series and skip the $O(h^2)$ part.

Strong form. $\langle \delta\Phi(x), h \rangle = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right) h\,dt + \left(\frac{\partial L}{\partial \dot{x}} h\right)\Big|_{t_0}^{t_1}$

Integration by parts from the variational form.

B Extremals

x is an extremal of Φ if $\langle \delta\Phi(x), h \rangle = 0 \quad \forall h$

Theorem. $x(t)$ is an extremal of $\Phi(x) = \int_{t_0}^{t_1} L(t, x, \dot{x})\, dt$ on the $M = \{ x \in C^1[t_0, t_1],\ x(t_0) = x_0,\ x(t_1) = x_1 \}$ if and only if

$$\text{(E-L)} \qquad \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0}$$

Pf. $x + h \in M \Rightarrow h(t_0) = h(t_1) = 0$. So the bd term is gone.

$\langle \delta\Phi(x), h \rangle = 0 \iff \int_{t_0}^{t_1} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} \right] h\, dt = 0, \ \forall h \in C_0^1 \Rightarrow$ E-L eqn. #

Example. $L = \sqrt{1 + \dot{x}^2}$, $x = c_1 t + c_2$. Straight lines .

Example. $L = \frac{1}{2} m \dot{x}^2 - U(x)$. $\quad m\ddot{x} = -\frac{\partial U}{\partial x}$.

C. The Euler-Lagrange Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \qquad \text{2nd order ODE} \qquad x(t_0) = x_0,\ x(t_1) = x_1.$$

D. An important remark

The condition for a curve to be an extremal of a functional does not depends on the choice of coordinate system.

13 Lagrange's Equations

A. Hamilton's principle of least action

Theorem. E-L eqn of $\Phi = \int_{t_0}^{t_1} L\,dt$, with $L = T - U$, is Newton's eqn.

Corollary. Let $\vec{q} = (q_1, \cdots, q_{3n})$ be any coordinates in the configuration space of a system of n mass points. Then $\vec{q} = \vec{q}(t)$ satisfies E-L eqn

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0, \quad \text{where } L = T - U.$$

Definition. $L(t, q, \dot{q})$: Lagrangian; $\vec{q}$: generalized coordinates

$\dot{\vec{q}}$: generalized velocity, $\vec{p} = \frac{\partial L}{\partial \dot{\vec{q}}}$: generalized momenta

$\frac{\partial L}{\partial q}$: generalized forces, $\int_{t_0}^{t_1} L(t, q, \dot{q})\,dt$: action

Theorem is called "Hamilton's form of the principle of least motion"

B. The simplest examples

Example 1. $L = T = \frac{1}{2} m \dot{\vec{r}}^2$. The extremals are straight lines.

Example 2. Planar motion in a central field.

$q_1 = r$, $q_2 = \varphi$. $\vec{q} = (r, \varphi)$. $T = \frac{1}{2} m \dot{\vec{r}}^2 = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2)$, $\dot{\vec{r}} = \dot{r}\vec{e}_r + r\dot{\varphi}\vec{e}_\varphi$

$p_1 = \frac{\partial L}{\partial \dot{q}_1} = m\dot{r}$, $p_2 = m r^2 \dot{\varphi}$. $L = T - U$, $U = U(r) = U(q_1)$.

$\dot{p}_1 = \frac{\partial L}{\partial q_1}$ becomes $m\ddot{r} = mr\dot{\varphi}^2 - \frac{\partial U}{\partial r}$

$\dot{p}_2 = \frac{\partial L}{\partial q_2}$ becomes $\dot{p}_2 = 0$, $p_2 = \text{const}$. Conservation of angular momentum

Def. q_i is cyclic if $\frac{\partial L}{\partial q_i} = 0$. angle φ in the above example.

Theorem. For a cyclic coordinate q_i, the generalized momentum is conserved i.e. $p_i = \text{const}$.

14 Legendre Transformation

$f(x)$ is a convex function.

Define $F(p, x) = px - f(x)$. The Legendre transform of $f(x)$ is $g(p)$

$$(1) \qquad g(p) = \max_x F(p, x) = \max_x \{px - f(x)\}.$$

Anlternate definition. Let $p = f'(x)$ and solve $x = x(p)$. Then define

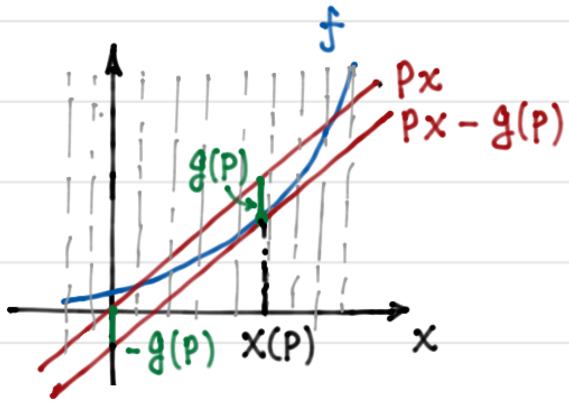
$$(2) \qquad g(p) = p\,x(p) - f(x(p)).$$

When $f \in C^1$ and convex, (1) & (2) are equivalent. Indeed the maximum is achieved at $\frac{\partial F}{\partial x} = 0$, i.e. $f'(x) = p$.

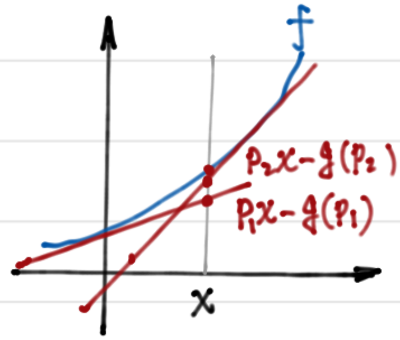
But (1) can be applied even f is not differentiable.

Example. $f(\dot{x}) = \frac{1}{2} m \dot{x}^2$, $p = m\dot{x}$, $g(p) = \frac{1}{2}\frac{p^2}{m}$. $\langle p, \dot{x}(p) \rangle = 2T$

Geometric meaning.



fix p, find x(p)



fix x, chang p.

C Involutivity

Theorem. If $g(p)$ is the Legendre transform of $f(x)$, then f is the Legendre transform of g.

Proof 1. $g(p) = p\,x(p) - f(x(p))$, relation $p = f'(x(p))$

$f(x) = p(x)\,x - g(p(x))$ if write $p = p(x)$.

Proof 2. Consider $G(x,p) = px - g(p)$. By definition, $G(x,p) = 0$ is the tangent line of f at $x(p)$. Due to the convexity,

$$f(x) \geqslant px - g(p) \quad \forall p, \forall x.$$

The "=" is acheived when $p = f'(x)$.

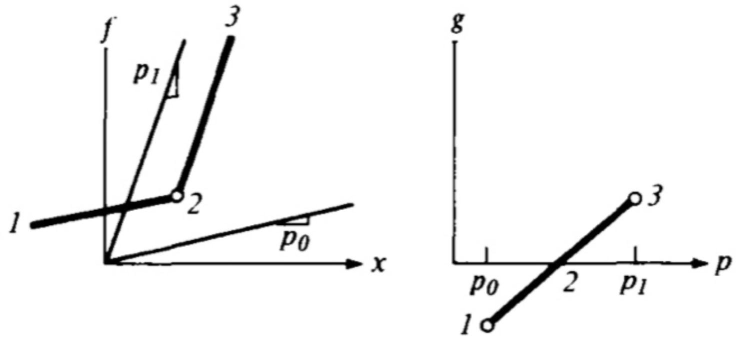


Figure 44 Legendre transformation taking an angle to a line segment

D Young's inequality

$f \underset{\text{Legendre}}{\overset{\text{Legendre}}{\rightleftarrows}} g$ dual

$$px \leqslant f(x) + g(p) \quad \forall x, p$$

= holds when $p = f'(x)$.

Example. $f(x) = \frac{1}{\alpha} x^\alpha$, $g(p) = \frac{1}{\beta} p^\beta$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Young's inequality $px \leqslant \frac{1}{\alpha} x^\alpha + \frac{1}{\beta} p^\beta$, $\forall x, p > 0$, $\alpha, \beta > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

15 Hamilton's Equations

A Equivalence of Lagrange's and Hamilton's Equations

Lagrangian $L(t, q, \dot{q}) \xleftrightarrow[\text{transform}]{\text{Legendre}}$ Hamiltonian $H(p, q, t) = p\dot{q}(p) - L(t, q, \dot{q}(p))$

$p = \frac{\partial L}{\partial \dot{q}}(t, q, \dot{q})$. $\frac{\partial^2 L}{\partial \dot{q}^2} > 0$ so solve $\dot{q} = \dot{q}(p)$

L is convex w.r.t $\dot{q}$.

E-L eqn

$$\dot{p} = \frac{\partial L}{\partial q} \quad \Leftrightarrow \quad \boxed{\begin{cases} \dot{p} = -\frac{\partial H}{\partial q} \\ \dot{q} = \frac{\partial H}{\partial p} \end{cases}}$$

Hamiltonian system

$$dH = \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial t} dt \quad \text{compare} \quad \cancel{p\,d\dot{q}} + \dot{q}\,dp - \frac{\partial L}{\partial t} dt - \frac{\partial L}{\partial q} dq \cancel{- \frac{\partial L}{\partial \dot{q}} d\dot{q}}$$

B Hamilton's function and energy

$L = T - U$, $T = \frac{1}{2}(A\dot{q}, \dot{q})$, $A = A(q,t)$ SPD, $U = U(q)$.

$H = T + U$. $p = \frac{\partial L}{\partial \dot{q}} = \frac{\partial T}{\partial \dot{q}} = A\dot{q}$, $\dot{q} = A^{-1}p$, $p\dot{q} = (A\dot{q}, \dot{q})$

$$H = p\dot{q} - L = (A\dot{q}, \dot{q}) - \frac{1}{2}(A\dot{q}, \dot{q}) + U = T + U.$$

$H = H(t, p, q)$ function of three variables

$H(t) = H(t, p(t), q(t))$ Hamiltonian along the trajectory.

$\frac{dH}{dt} = \frac{\partial H}{\partial t}$ by Hamilton system.

so if $H(t, p, q)$ is independent of t, then $H(t)$ is constant.

This is the generalization of conservation of energy $\rightarrow$ conservation of Hamiltonian.

C. Cyclic coordinates

q_1 is cyclic if $\frac{\partial H}{\partial q_1} = 0 \Leftrightarrow \frac{\partial L}{\partial q_1} = 0$

conservation of momentum

q_1 is cyclic, then from $\dot{p}_1 = \frac{\partial L}{\partial q_1} = -\frac{\partial H}{\partial q_1}$, we get $p_1 = c$.

Then $p_1 = c$ can be treat as a parameter. We can solve a reduced system for $p' = (p_2, \cdots, p_n)$ and $q' = (q_2, \cdots, q_n)$.

$$\begin{cases} \dot{q}' = \frac{\partial H}{\partial p'} \\ \dot{p}' = -\frac{\partial H}{\partial q'} \end{cases} \qquad \begin{aligned} \dot{q}_1 &= \frac{\partial H}{\partial p_1} = \frac{\partial H}{\partial p_1}(p_1, p'(t), q'(t), t) \\ \dot{p}_1 &= 0 \end{aligned}$$

no q_1 as $\frac{\partial H}{\partial q_1} = 0$

16 Liouville's Theorem

A The phase flow

$g^t : (p(0), q(0)) \to (p(t), q(t))$ where $\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} -\frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}$

forms one-parameter group of transformations

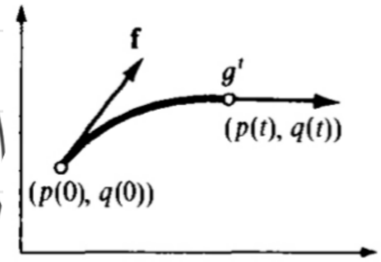


Figure 47 Phase flow

B Liouville's theorem

Theorem. The phase flow preserves volume: for any region D we have $\mathrm{vol}(g^t D) = \mathrm{vol}(D)$.

A general proposition: $g^t(x) = x + f(x)t + O(t^2)$ is the group of transformation corresponding to ODE $\dot{x} = f(x)$. Let $D(0)$ be a region and $v(0) = \mathrm{vol}(D(0))$, $D(t) = g^t D(0)$, $v(t) = \mathrm{vol}(D(t))$.

C. Proof

Lemma. $\det(I + At) = 1 + t\,\mathrm{tr}(A) + O(t^2)$ as $t \to 0$

Lemma. $\left.\frac{dv}{dt}\right|_{t=0} = \int_{D(0)} \mathrm{div} f \, dx$.

Pf. $v(t) = \int_{D(t)} 1 \, dy = \int_{D(0)} \det\left(\frac{\partial g^t(x)}{\partial x}\right) dx$, here $g^t : x \to y$

$$\left.\frac{dv}{dt}\right|_{t=0} = \int_{D(0)} \det\left(I + \frac{\partial f}{\partial x}\right) dx = \int_{D(0)} \mathrm{div} f \, dx.$$

Theorem. If $\operatorname{div} f \equiv 0$, then g^t preserves volume: $v(t) = v(0)$.

For Hamilton's equation $f = -\operatorname{curl} H$, $\operatorname{div} f = -\operatorname{div}\operatorname{curl} H = 0$.

D Poincaré's recurrence theorem

Theorem. $g: D \to D$. D is bounded; g is one-to-one and volume preserving. Then $\forall$ open set $U \subset D$, $\exists x \in U$, $n \in \mathbb{Z}$, s.t. $g^n x \in U$.

Proof. $U, g(U), g^2(U), \cdots g^n(U), \cdots$ have the same volume so $\exists\, k, l \in \mathbb{Z}$, $k > l$, s.t. $g^k(U) \cap g^l(U) \neq \emptyset$. Then $g^{k-l}(U) \cap U \neq \emptyset$.

E Application of Poincaré's theorem.

Example 1. Rotation of a circle: angle α. If $\alpha = 2\pi \frac{m}{n}$, then $g^n = id$.

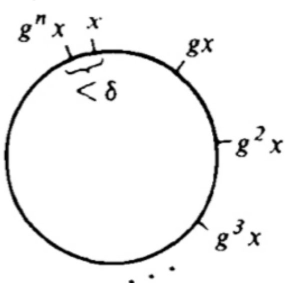


Figure 52 Dense set on the circle

If α is not commensurable with 2π, then $\forall \delta > 0, \exists n: |g^n x - x| < \delta$.

So if $\alpha \neq 2\pi \frac{m}{n}$, $g^k x$ is dense on the circle.

Example 2. Winding line on the torus.

$$\begin{cases} \dot{\varphi}_1 = \alpha_1, \\ \dot{\varphi}_2 = \alpha_2. \end{cases} \quad g^t: (\varphi_1, \varphi_2) \to (\varphi_1 + \alpha_1 t, \varphi_2 + \alpha_2 t)$$

is dense in the torus.

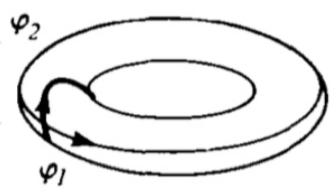


Figure 53 Torus