Ch 9 Canonical Formalism

44 The integral invariant of Poincaré-Cartan

$\mathbb{R}^3$ $v(x)$ vector field

$r = \text{curl}\, v$ vector field $\qquad \dfrac{dx}{dt} = r(x(t))$

defines vortex lines

vortex lines passing through γ_1 form vortex tube

Conservation of circulation

$$\oint_{\gamma_1} v \cdot ds = \oint_{\gamma_2} v \cdot ds$$

Proof. $\partial\sigma = \gamma_1 - \gamma_2$. $\int_{\partial\sigma} v = \int_\sigma \text{curl}\, v \cdot n\, ds$

but $\dot{x} = \text{curl}\, v(x)$ i.e. curl v is tangent to vortex lines and thus $\text{curl}\, v \cdot n = 0$. #

$\mathbb{R}^N$. $\omega \in \Lambda^2(\mathbb{R}^N)$, $\omega = \sum_{i<j} a_{ij}\, dx^i \wedge dx^j$. $A = (a_{ij})_{N\times N}$

$\omega \leftrightarrow A$: skew-symmetric matrix.

For $\omega^1 \in \Lambda^1$, $\omega^1 = \sum u_i\, dx^i$, $\omega^1 \leftrightarrow (u_1, u_2, \cdots, u_N)^T$

$d\omega^1 \in \Lambda^2 \leftrightarrow du = (\partial_j u_i - \partial_i u_j)$.

Lemma. $\omega \in \Lambda^2(\mathbb{R}^{2n+1})$. Then $\exists\, \xi \neq 0$ s.t. $\omega(\xi, \eta) = 0 \quad \forall \eta \in \mathbb{R}^{2n+1}$.

Pf. A is skew-symmetric so $\det A = \det A^T = -\det A$, $\Rightarrow \det A = 0$

Then ξ is the corresponding eigenvector. #

Null space of $\omega \in \Lambda^2 = \ker(A_\omega)$. ω is non-singular if $\dim(A_\omega)$ is the minimal possible.

M^{2n+1}: $\omega = d\omega^1$ and assume it is non-singular. Then $\exists\, \xi$, s.t.

$$d\omega^1(\xi, \eta) = 0 \quad \forall \eta \in TM_x$$

ξ is called the "vortex direction" of ω^1 and

vortex lines passing through γ_1 form vortex tube

conservation of circulation becomes $\oint_{\gamma_1} \omega^1 = \oint_{\gamma_2} \omega^1$

if $\gamma_1 - \gamma_2 = \partial\sigma$, where σ is a piece of the vortex tube.

Hamilton's equations in M^{2n+1}

$(p_1, \cdots, p_n, q_1, \cdots, q_n, t) \qquad H = H(p, q, t)$

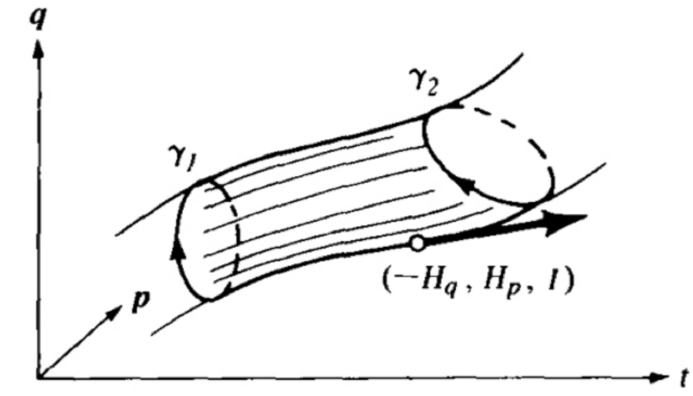
Let $\omega^1 = p\,dq - H\,dt$. The vortex line of ω^1 is the phase flow $(p(t), q(t), t)$ of the Hamiltonian system.

Pf. $A_{d\omega^1} = \begin{pmatrix} 0 & -I & H_p \\ I & 0 & H_q \\ -H_p & -H_q & 0 \end{pmatrix}$, $\xi = \begin{pmatrix} -H_q \\ H_p \\ 1 \end{pmatrix}$. Phase flow: $\dot{y} = \xi(y)$ is the Hamiltonian system.

Theorem (Integral invariant of Poincaré-Cartan)

$$\oint_{\gamma_1} p\,dq - H\,dt = \oint_{\gamma_2} p\,dq - H\,dt, \quad \gamma_1 - \gamma_2 = \partial\sigma,\ \sigma \text{ piece of vortex tube.}$$

In particular γ_i lying in the planes $t = \text{const}$. Then $dt = 0$.



Hamiltonian field and vortex lines of the form $\mathbf{p}\, d\mathbf{q} - H\, dt$.

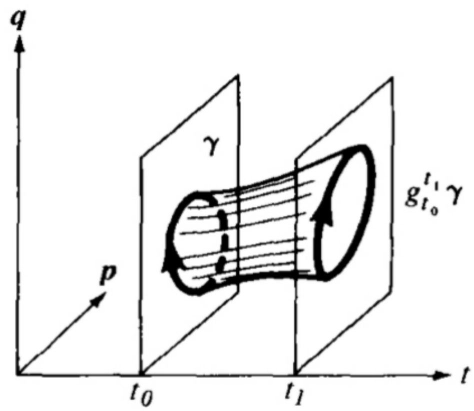


Figure 183 Poincaré's integral invariant

$$\oint_\gamma p\,dq = \oint_{g^t(\gamma)} p\,dq \qquad g^t : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \text{ phase flow} \qquad \gamma \text{ is closed.}$$

$\downarrow$ stokes' theorem

$$\int_\sigma dp \wedge dq = \int_{g^t(\sigma)} dp \wedge dq$$

- $p\,dq$: Poincaré's relative integral inv.
- $dp \wedge dq$: absolute integral invariant
- $p\,dq - H\,dt$: Poincaré-Cartan invariant

Canonical (Simplectic) transformations

$g : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is canonical if g preserves $\omega^2 = \sum dp_i \wedge dq_i$.

① $g^* \omega^2 = \omega^2$ ② $\iint_\sigma \omega^2 = \iint_{g(\sigma)} \omega^2$ ③ $\oint_\gamma p\,dq = \oint_{g(\gamma)} p\,dq$.

Theorem. The transformation induced by the phase flow is canonical.

Corollary. Canonical transformations preserve the volume element in phase space: $\text{vol}(g(D)) = \text{vol}(D) \quad \forall D$.

45 Applications of the integral invariant of Poincaré-Cartan

A Changes of variables

Oringal coordinate $(p, q, t) \longrightarrow (P, Q, T)$

Hamiltonian $H(p, q, t) \longrightarrow K(P, Q, T)$

$$\begin{cases} \dfrac{dp}{dt} = -\dfrac{\partial H}{\partial q} \\ \dfrac{dq}{dt} = \dfrac{\partial H}{\partial p} \end{cases} \quad (1) \qquad \begin{cases} \dfrac{dP}{dT} = -\dfrac{\partial K}{\partial Q} \\ \dfrac{dQ}{dT} = \dfrac{\partial K}{\partial P} \end{cases} \quad (2)$$

relation: $p\,dq - H\,dt = P\,dQ - K\,dT + dS$

Pf. view as vortex lines of 1-form. since $d(dS) = 0$, dS has no influence on the vortex line. #

In particular, $T = t$. $(p, q) \longrightarrow (P, Q)$ is canonical (simpletic) then $(1) \rightarrow (2)$ with $K(P, Q, t) = H(p, q, t)$.

Pf. The transform is simplectic, then $\oint_\gamma p\,dq - P\,dQ = 0$

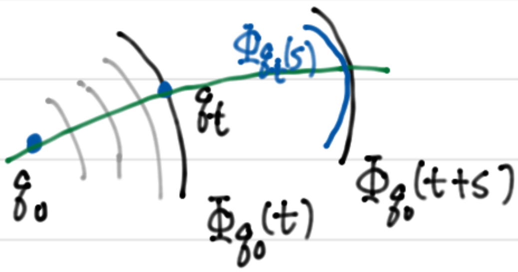
$S = \int_{(p_0, q_0)}^{(p_1, q_1)} p\,dq - P\,dQ$ is well defined and $dS = p\,dq - P\,dQ$

Consequently $p\,dq - H\,dt = P\,dQ - H\,dt + dS$.

46 Huygen's principle

Geometric optics. Fermat principle: light travels from a point q_0 to a point q_1 in the shortest possible time.

$\vec{v} = \vec{v}(\vec{q})$: inhomogeneous medium and/or anisotropic medium



Theorem. $\Phi_{q_0}(s+t)$ is the envelope of $\Phi_q(s)$, $q \in \Phi_{q_0}(t)$

Pf. Contradiction. 

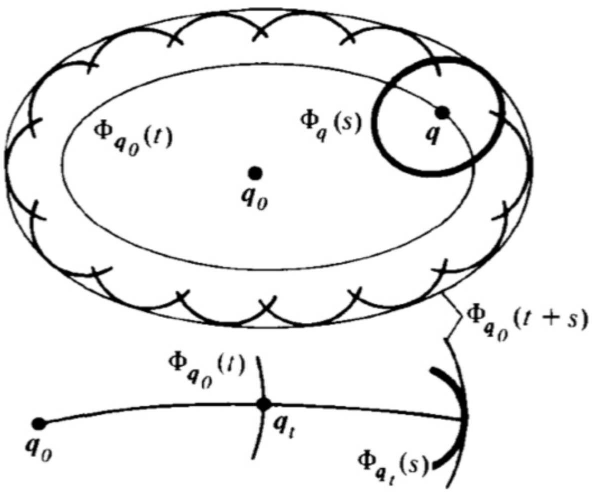


Figure 193 Envelope of wave fronts

In geometry, an envelope of a family of curves in the plane is a curve that is tangent to each member of the family at some point, and these points of tangency together form the whole envelope.

Two descriptions of the process of propagation.

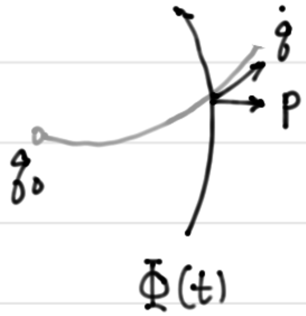
1. trace the rays: velocity vector $\dot{q}$.
2. trace the wave fronts: velocity of motion of the wave front p.

Fix q_0 and define $S_{q_0}(q)$ = the least time of the light from q_0 to q.

$\Phi_{q_0}(t) = \{ q : S_{q_0}(q) = t \}$. $\quad p = \frac{\partial S}{\partial q}$ normal slowness of the front

$$\boxed{\begin{array}{l} S_{q_0} : q \to t \\ \Phi_{q_0} : t \to q \end{array}}$$

p large means $q \to q + \Delta q$ needs more time.



direction of p and $\dot{q}$ are conjugate

Optics	Mechanics
Optical medium	Extended configuration space $\{(\mathbf{q}, t)\}$
Fermat's principle	Hamilton's principle $\delta \int L\, dt = 0$
Rays	Trajectories $\mathbf{q}(t)$
Indicatrices	Lagrangian L
Normal slowness vector $\mathbf{p}$ of the front	Momentum $\mathbf{p}$
Expression of $\mathbf{p}$ in terms of the velocity of the ray, $\dot{\mathbf{q}}$	Legendre transformation
1-form $\mathbf{p}\, d\mathbf{q}$	1-form $\mathbf{p}\, d\mathbf{q} - H\, dt$

Hamilton-Jacobi Equation

Action function $S(q,t) = S_{(q_0,t_0)}(q,t) = \int_\gamma L\,dt$

where γ: extremal curve connecting (q_0,t_0) and (q,t).

Theorem. $dS = p\,dq - H\,dt.$ (1)

Hamilton-Jacobi eqn

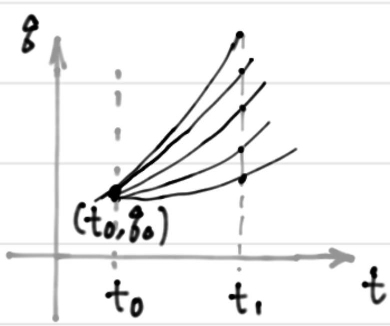
$$dS = \partial_t S\,dt + \partial_q S\,dq$$

$$p = \partial_q S, \quad -H = \partial_t S \qquad \text{compare (1)}$$

$$\boxed{\partial_t S + H(\partial_q S, q, t) = 0}$$

Proof of Theorem.

① $S(q,t)$ is well defined at least for $|t-t_0|$ is small enough.



E-L eqn $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = \frac{\partial L}{\partial q}$ 2nd order ODE

$q(t)$, $t \in (t_0, t_1)$ is determined by E-L eqn

and either initial condition $q(t_0) = q_0$, $\dot q(t_0) = \dot q_0$

or boundary condition $q(t_0) = q_0$, $q(t_1) = q$.

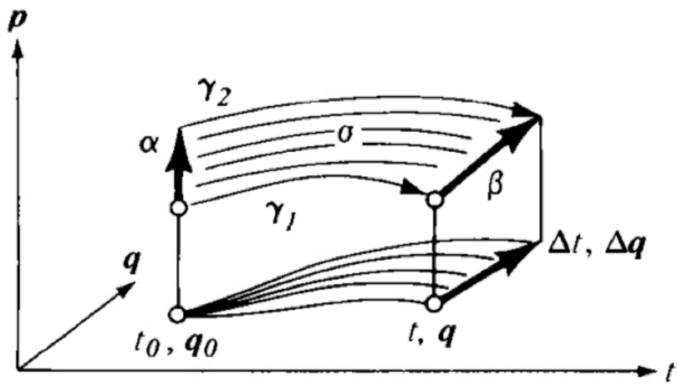
The existence and uniqueness holds at least locally.

But it may happen  more than one extremal. (conjugate pts)

② $\omega^1 = p\,dq - H\,dt$. $\int_\gamma \omega^1 = \int_\gamma L\,dt$ when γ is an extremal

Now the curve is $(p(t), q(t), t)$ in (p, q, t) space p and q are independent in general. If $p = \frac{\partial L}{\partial \dot q}$, which is the case when γ is an extremal curve, then $(p\dot q - H)\,dt = L\,dt$ as $H = p\dot q - L$.

③



σ consists of vortex lines of ω^1 and thus $\iint_\sigma d\omega^1 = 0$

By Stokes' theorem, $\int_\sigma d\omega^1 = \int_{\partial\sigma} \omega^1 = \int_{\gamma_1 - \gamma_2 + \beta - \alpha} \omega^1$

α: $dq = dt = 0$ as $\alpha \subset \{t = t_0, q = q_0\}$

$\int_{\gamma_i} \omega^1 = \int_{\gamma_i} L\,dt$. $\int_{\gamma_1} L\,dt = S(t, q)$, $\int_{\gamma_2} L\,dt = S(t+\Delta t, q+\Delta q)$

So $\int_\beta \omega^1 = \int_{\gamma_2} \omega^1 - \int_{\gamma_1} \omega^1 = S(t+\Delta t, q+\Delta q) - S(t, q)$

$$\left.\begin{aligned} &= \partial_t S\,\Delta t + \partial_q S\,\Delta q \\ \int_\beta p\,dq - H\,dt &= p\Delta q - H\Delta t + o(\Delta t, \Delta q) \end{aligned}\right\} dS = \omega^1.$$

Solution to the Hamilton-Jacobi equation

$S = S(q, t)$ time dependent H-J eqn is:

$$\begin{cases} \dfrac{\partial S}{\partial t} + H\left(\dfrac{\partial S}{\partial q}, q, t\right) = 0 \\ S(q, t_0) = S_0(q) \end{cases} \iff \begin{cases} \dot{p} = -\partial_q H \\ \dot{q} = \partial_p H \end{cases}$$

$$q(t_0) = q_0, \quad p(t_0) = \nabla_q S_0 |_{q_0}$$

action function with initial condition S_0, integrating along the characteristic leading to A.

Solution curve $q(t)$ is an extrem curve $\delta \int L\, dt = 0$.

$$S(A) = S_0(q_0) + \int_{(q_0, t_0)}^{A} L(q, \dot{q}, t)\, dt \Leftarrow \text{characteristic lines}$$

47 The Hamilton-Jacobi method for integrating Hamilton's System

Consider (P, Q, t) and Hamiltonian $K(Q,t)$ depends on Q only.

$$\begin{cases} \dot{Q} = 0 \\ \dot{P} = \frac{\partial K}{\partial Q} \end{cases} \Rightarrow \begin{cases} Q(t) = Q(0) \\ P(t) = P(0) + \int_0^t \frac{\partial K}{\partial Q}\Big|_{Q(0)} dt \end{cases}$$

Now for (p, q, t) and $H(p, q, t)$, we are looking for a transformation given by a generating function $S(Q, q)$

$$\begin{cases} p = \frac{\partial S}{\partial q}(Q, q) & (1) \\ P = -\frac{\partial S}{\partial Q}(Q, q) & (2) \end{cases}$$

Want: $(p, q) \to (P, Q)$ is canonical (simplectic)

Claim: (1), (2) will determine such a transformation

① $(p, q) \to (P, Q)$. (p, q, P, Q) four variables. (1)-(2) two eqns

implicit function theorem: $\det\left(\frac{\partial^2 S}{\partial Q \partial q}\right)\Big|_{(Q_0, q_0)} \neq 0.$

② this transform is canonical.

Check $p\,dq - P\,dQ = \partial_q S\,dq + \partial_Q S\,dQ = dS(q, Q) = dS(q, Q(p, q))$.

So we look for $S(Q, q)$ satisfies static Hamilton-Jacobi equation

$$\boxed{H\left(\frac{\partial S(Q, q)}{\partial q}, q, t\right) = K(Q, t)}$$

Hamiltonian system → Hamilton-Jacobi eqn
ODE PDE

This is the most powerful method known for exact integration, and many problems which were solved by Jacobi cannot be solved by other methods.

> The main difficulty in integrating a given differential equation lies in introducing convenient variables, which there is no rule for finding. Therefore, we must travel the reverse path and after finding some noticeable substitution, look for problems to which it can be successful applied.
>
> (Jacobi, “Lectures on dynamics”).

48 Generating functions

$S_1(q,Q)$
$$\begin{cases} p = \frac{\partial S_1}{\partial q}(q,Q) \\ P = -\frac{\partial S_1}{\partial Q}(q,Q) \end{cases} \to Q = Q(p,q) \text{ if } \left(\frac{\partial^2 S_1}{\partial Q \partial q}\right) \text{ non-singular}$$

Canonical if $p\,dq - P\,dQ = dS(p,q) = dS_1(q,Q)$

$$S_1(q,Q) = S(p(q,Q),q)$$

In this setting, (q,Q) are independent. So can't deal with the identity map $P=p$, $Q=q$.

$S_2(P,q)$
$$\begin{cases} p = \frac{\partial S_2}{\partial q}(P,q) \\ Q = \frac{\partial S_2}{\partial P}(P,q) \end{cases} \to P = P(p,q) \text{ if } \left(\frac{\partial^2 S_2}{\partial P \partial q}\right) \text{ non-singular}$$

Canonical $p\,dq + Q\,dP = d(PQ + S(p,q)) = dS_2(P,q)$

$$S_2(P,q) = P\,Q(P,q) + S(p(P,q),q).$$

Now $S_2 = Pq$ gives the identity map.