EXISTENCE OF GLOBAL MINIMUM FOR CALCULUS OF VARIATION

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ABSTRACT. The existence of a global minimum for problems in the calculus of variation is briefly reviewed in this note. The main ingredients are: 1. use coercivity w.r.t. the derivative variable to get the boundedness of a minimizing sequence; 2. use weak compactness to get a weak convergent sub-sequence; 3. use convexity w.r.t. the derivative variable to prove the functional is lower weak semi-continuous.

CONTENTS

1. INTRODUCTION

We consider the calculus of variations problem given by the expression:

$$\inf_{u\in\mathcal{M}} I(u), \tag{1}$$

where $I(u)$ is the integral functional defined by

$$I(u) = \int_\Omega L(x, u(x), \nabla u(x))\,\mathrm{d}x,$$

with

- $\Omega \subset \mathbb{R}^d$ an open and bounded Lipschitz domain;
- $\mathcal{M} = \{u \in W^{1,q}(\Omega) : \operatorname{tr} u = g \text{ on } \partial\Omega\}$ the set of admissible functions;
- $L : \Omega \times \mathbb{R} \times \mathbb{R}^d$ the Lagrangian.

Date: June 6, 2025.

When writing the Lagrangian as $L(x,u,p)$, the variables (x,u,p) are independent. In $L(x,u(x),\nabla u(x))$, the relation $p(x)=\nabla u(x)$ is substituted, making the second and third variables dependent on x. After integration, $I(u)$ maps a function to a real value. Our goal is to find an admissible function that minimizes this functional.

Analogous to the calculus of variations problem, consider the classical problem:

$$\inf_{x\in M} f(x), \tag{2}$$

where $M\subset\mathbb{R}^d$. Two sufficient conditions ensure the existence of a solution to (2):

1. The function f is continuous on M.
2. The set M is compact, i.e., closed and bounded.

Thus, to guarantee a minimum for f over M, it suffices that f is continuous and M is compact.

Finding a minimizer for (1) is more subtle. The admissible set $\mathcal{M}$ lies in an infinite-dimensional function space, whereas M in (2) is finite-dimensional. This change forces a reassessment of key ideas such as *compactness and continuity*.

A key difference is the notion of compactness. In finite-dimensional spaces, a closed and bounded set is compact. In infinite-dimensional spaces, this is no longer sufficient. For example, the sequence $\{e_n\}\subset\ell^2$, where $e_n=(0,0,\dots,1,0,\dots)$ with 1 in the n-th position, satisfies $\|e_n-e_m\|=\sqrt{2}$ for $n\neq m$, so it has no convergent subsequence.

To recover compactness, we introduce weaker topologies. A topology is a collection of subsets, called open sets, satisfying: the empty set and the whole space are open; arbitrary unions of open sets are open; finite intersections of open sets are open. A set is compact if every open covering has a finite sub-covering. Weaker topologies, having fewer open sets, increase the chance of fulfilling the 'every open covering has a finite sub-covering' condition.

A related notion is sequential compactness: every bounded sequence has a convergent subsequence whose limit lies in the set. Compactness implies sequential compactness. In norm-induced topologies, they are equivalent. While we want sequential compactness, thinking in terms of open sets is often more useful.

Now consider a map $T: X\to Y$ between topological spaces. It is continuous if the pre-image of every open set in Y is open in X. With fewer open sets in X (i.e., under a weaker topology), continuity becomes harder to attain. In particular, $I(u)$ may not be continuous under the weak topology. We relax this to lower weak semi-continuity (l.w.s.c.), which can be characterized geometrically by the closeness of the epigraph.

The balance of compactness and continuity from the topological point of view is summarized as follows:

Weak vs Strong Topology

Weak topology:	fewer open sets, better compactness, weaker continuity.
Strong topology:	more open sets, weaker compactness, better continuity.

To ensure the existence of a minimizer, we need conditions on L that guarantee:

- coercivity to ensure the boundedness of a minimizing sequence $\{u_k\}$;
- convexity to ensure the lower weak semi-continuity of $I(\cdot)$.

By the scaling argument, the dominant contribution to the functional comes from the gradient ∇u. Therefore, we impose structural conditions: coercivity and convexity on the Lagrangian $L(x,u,p)$ only with respect to the variable p.

2. Dirichlet's Principle

To begin with, we need to assume $I(\cdot)$ is bounded below. Otherwise, $\inf I(u) = -\infty$.

2.1. Minimizing sequence.

A minimizing sequence $\{u_k\} \subset \mathcal{M}$ is defined by the property that

$$\lim_{k\to\infty} I(u_k) = \inf_{u\in\mathcal{M}} I(u).$$

One standard approach to constructing such a sequence is to use a family of nested finite-dimensional subspaces or subsets $\{\mathcal{M}_k\}$ that approximate the admissible set $\mathcal{M}$. That is, we assume

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots, \quad \text{with} \quad \bigcup_{k=1}^{\infty} \mathcal{M}_k \text{ dense in } \mathcal{M}.$$

We then restrict the minimization problem (1) to each $\mathcal{M}_k$. As $\mathcal{M}_k$ lies in finite-dimensional spaces, finding $u_k = \arg\inf_{u\in\mathcal{M}_k} I(u)$ is relatively easy.

Due to the nested structure, we typically have

$$I(u_1) \geq \ldots \geq I(u_k) \geq I(u_{k+1}) \geq \ldots \geq \inf_{u\in\mathcal{M}} I(u) > -\infty,$$

so that $\{I(u_k)\}$ forms a monotone decreasing sequence. As $I(\cdot)$ is bounded below, it follows that $\lim_{k\to\infty} I(u_k)$ exists. Since $\bigcup_k \mathcal{M}_k$ is dense in $\mathcal{M}$, we conclude that

$$\lim_{k\to\infty} I(u_k) = \inf_{u\in\mathcal{M}} I(u),$$

and thus $\{u_k\}$ is a minimizing sequence.

An important question remains: can we conclude that the sequence $\{u_k\}$ or a subsequence converges to some limit u, and the limit u is a global minimizer?

2.2. Dirichlet's principle.

Historically, the existence of minimizers was justified using Dirichlet's principle, which asserts:

> **Dirichlet's principle**
>
> If $I(\cdot)$ is bounded below, then there exists a minimizer $u \in \mathcal{M}$ such that
> $$I(u) = \inf_{v\in\mathcal{M}} I(v).$$

A classic 'proof' of Dirichlet's principle, attributed to Riemann [3], proceeds as follows:

> Choose a minimizing sequence $\{u_k\} \subset \mathcal{M}$ such that $I(u_k) \to \inf_{u\in\mathcal{M}} I(u)$. Since the sequence $\{u_k\}$ is bounded, there exists a convergent subsequence $u_{k_j} \to u_0$. Then u_0 is the desired minimizer, i.e., $I(u_0) = \inf_{u\in\mathcal{M}} I(u)$.

However, this argument is flawed for several reasons:

1. Boundedness of $\{I(u_k)\}$ does not imply boundedness of $\{\|u_k\|_{H^1(\Omega)}\}$.
2. Even if $\{\|u_k\|_{H^1(\Omega)}\}$ is bounded, this does not guarantee the existence of a strongly convergent subsequence in $H^1(\Omega)$.

To address these issues, we introduce two crucial notions:

- **Coercivity** — to ensure that bounded functional implies boundedness in the function space norm, thus addressing issue 1.

- **Weak convergence** $u_k \rightharpoonup u_0$ to replace strong convergence in non-compact infinite-dimensional settings, thus addressing issue 2.

However, passing to the weak limit introduces a new obstacle:

3. Weak convergence $u_k \rightharpoonup u_0$ does not imply $I(u_k) \to I(u_0)$. That is, the functional $I(\cdot)$ may not be weakly continuous.

This happens because in the weak topology, the collection of open sets is coarser, and therefore, continuous functionals in the strong topology may fail to be continuous in the weak topology.

So the existence of minimizer from the minimizing sequence is not guaranteed without additional assumptions. We will first explore several examples to illustrate situations in which the minimum may fail to exist. Typically, we begin with examples from finite-dimensional calculus and then proceed to examples in the calculus of variations.

2.3. **Examples.** We first show the function value can be bounded below, but the minimizing sequence may not be bounded.

Example 2.1 (Minimizing sequence is unbounded)**.** Consider $f(x) = e^{-x}$ and $M = \mathbb{R}_+$. Function f is bounded below: $f \geq 0$ and $\inf f = 0$. We can take $M_k = [-k, k]$ and get $x_k = k$. But no global minimizer. The minimizing sequence $\{x_k\}$ is not bounded. Here f is strictly convex but not μ-strongly convex for any $\mu > 0$.

Example 2.2 (Minimizing sequence is unbounded)**.** Consider $I(u) = \int_{-1}^{1} x^2 (u')^2 \,\mathrm{d}x$ with $\mathcal{M} = \{u \in H^1(-1,1), u(-1) = -1, u(1) = 1\}$. The Lagrangian $L(x, u, p)$ is convex in p except at $x = 0$ and thus not μ-strongly convex.

Construct a piecewise linear function with slop $1/h$ from $(-h, h)$ to connect -1 and 1. Then $I(u_h) \to 0$. If u_0 is a minimum of I on M, then $u_0' = 0$ and u_0 is constant which cannot satisfy the boundary condition. Notice that the minimizing sequence $\{u_h\}$ is bounded in $L^2(0,1)$ but not in $H^1(0,1)$.

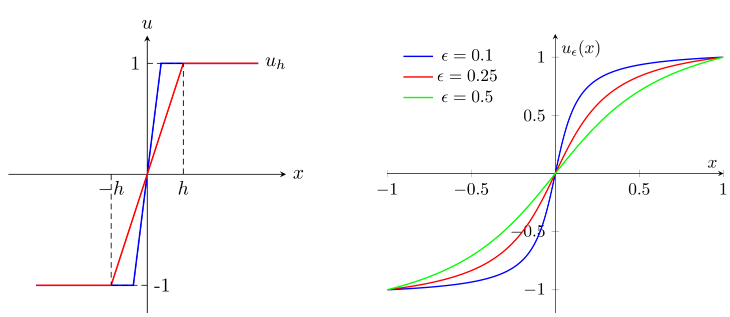


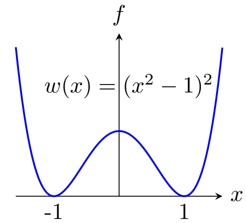
FIGURE 1. A minimizing sequence with unbounded H^1-norm.

In the original example constructed by Weierstrass, it is a smooth function

$$u_\epsilon = \frac{\arctan(x/\epsilon)}{\arctan(1/\epsilon)}.$$

Example 2.3 (Loss of continuity in a weak convergence)**.** Consider the functional

$$I(u) = \int_0^1 u^2 + \left[(u')^2 - 1\right]^2 \,\mathrm{d}x$$



with boundary conditions $u(0) = u(1) = 0$. The function $\left(x^2-1\right)^2$ is often referred to as a double-well potential. We can construct a zig-zag piecewise linear function u_h such that $|u_h(x)| \le h$ and $|u_h'(x)| = 1$.

Then $\{u_h\}$ forms a minimizing sequence with $I(u_h) \to 0$ as $h \to 0$. What is the limit of u_h? The sequence $\{u_h\}$ converges to zero in L^2, since $\|u_h\|_{L^2(0,1)} \le \|u_h\|_{L^\infty(0,1)} \le h$, but not in H^1, as $\|u_h'\|_{L^2(0,1)} = 1$ does not vanish. This shows that the minimizing sequence does not converge strongly in H^1 but does in a weaker L^2 topology.

However, the functional I is not continuous in the L^2 topology. Although $u_h \to 0$ in L^2, we have

$$I(0) = \int_0^1 0 + (0-1)^2 \,\mathrm{d}x = 1 \neq 0 = \lim_{h\to 0} I(u_h).$$

This illustrates that a minimum may not be attained even when a convergent minimizing sequence exists. The convergence could be in a weaker topology and the lack of continuity in this weaker topology is the key obstruction in this example.

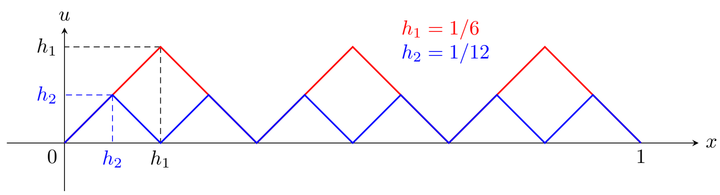


FIGURE 2. A minimizing sequence $\{u_h\}$ with $u_h \to 0$ in L^2 but $I(\cdot)$ is not continuous in L^2.

3. Coercivity and Convexity

In this section, we present an existence proof, which corrects the flawed proof based on Dirichlet's principle. The following presentation closely follows the works in [1, Chapter 9] and [4, Chapter 8].

3.1. **Coercivity.** Using scaling arguments, we observe that boundedness often arises from control over ∇u, which implies that most structural conditions are imposed on the p variable in the Lagrangian $L(x, u, p)$.

The Lagrangian $L(x, u, p)$ is said to be *coercive* with respect to the variable p if there exist constants $\alpha > 0$, $\beta \ge 0$, and $1 < q < \infty$ such that

$$L(x, u, p) \ge \alpha |p|^q - \beta, \quad \forall\, p \in \mathbb{R}^n,\ u \in \mathbb{R},\ x \in \Omega. \tag{3}$$

As a consequence, for any $w \in W^{1,q}(\Omega)$, the functional

$$I(w) := \int_\Omega L(x, w(x), \nabla w(x)) \,\mathrm{d}x$$

satisfies the lower bound

$$I(w) \geq \delta \|\nabla w\|_{L^q(\Omega)}^q - \gamma \geq -\gamma,$$

where $\delta := \alpha > 0$ and $\gamma := \beta|\Omega|$.

In particular, if $I(w) < \infty$, then $\|\nabla w\|_{L^q(\Omega)} < \infty$. Now fix a reference function $u_g \in \mathcal{M}$ such that $I(u_g) < \infty$. Then for any admissible $w \in \mathcal{M}$, the difference $w - u_g \in W_0^{1,q}(\Omega)$. By the Poincaré inequality,

$$\|w - u_g\|_{L^q(\Omega)} \leq C\|\nabla(w - u_g)\|_{L^q(\Omega)},$$

and hence

$$\|w\|_{L^q(\Omega)} \leq \|w - u_g\|_{L^q(\Omega)} + \|u_g\|_{L^q(\Omega)} < \infty.$$

In summary, the coercivity assumption (3) on p guarantees the boundedness of minimizing sequences in $W^{1,q}(\Omega)$ and also ensures that the functional $I(\cdot)$ is bounded below.

3.2. **Weak compactness.** Let X be a Banach space (i.e., a complete normed space) and X^* its dual space, consisting of all linear and continuous functionals on X. A sequence $\{u_k\} \subset X$ is said to converge weakly to $u \in X$, denoted

$$u_k \rightharpoonup u,$$

if

$$\langle f, u_k \rangle \to \langle f, u \rangle \quad \text{for all } f \in X^*,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X^* and X.

In contrast, strong convergence is defined by

$$u_k \to u \quad \text{if} \quad \lim_{k\to\infty} \|u_k - u\|_X = 0.$$

For weak convergence sequence $u_k \rightharpoonup u$, although we cannot derive $\|u_k - u\|_X \to 0$, we can conclude the norm is uniformly bounded

$$\sup_k \{\|u_k\|_X\} < \infty, \qquad \text{if } u_k \rightharpoonup u.$$

Exercise 3.1 (Exercise 1, Chapter 8 of [4])**.** *This problem illustrates that a weakly convergent sequence can be rather badly behaved.*

(1) *Show that $u_k(x) = \sin(kx)$ converges weakly to zero in $L^2(0,1)$ as $k \to \infty$.*
(2) *Fix constants $a, b \in \mathbb{R}$ and $0 < \lambda < 1$. Define a sequence of piecewise constant functions:*

$$u_k(x) := \begin{cases} a & \text{if } \frac{j}{k} \leq x < \frac{j+\lambda}{k}, \\ b & \text{if } \frac{j+\lambda}{k} \leq x < \frac{j+1}{k}, \end{cases} \quad \text{for } j = 0, \ldots, k-1.$$

Prove that $u_k \rightharpoonup \lambda a + (1-\lambda) b$ in $L^2(0,1)$.

The duality pairing

$$\langle f, u \rangle, \quad f \in X^*, \ u \in X,$$

can be used to define a natural embedding $X \hookrightarrow X^{**} := (X^*)^*$ by mapping each $u \in X$ to the element $\hat{u} \in X^{**}$ defined by

$$\hat{u}(f) := \langle f, u \rangle \quad \text{for all } f \in X^*.$$

If this mapping is surjective, that is, every element of X^{**} arises from some $u \in X$, then we say that X is *reflexive*.

Exercise 3.2. *The space $L^p(\Omega)$, $1 < p < +\infty$, is reflexive. But not including the endpoints $p = 1$ and $p = +\infty$. We do have*

$$(L^1(\Omega))^* = L^\infty(\Omega), \quad \textit{but} \quad (L^\infty(\Omega))^* \supset L^1(\Omega).$$

Find out $(L^\infty(\Omega))^*$.

In the dual space X^*, weak convergence is defined as follows:

$$f_k \rightharpoonup f \quad \text{if} \quad \langle f_k, u\rangle \to \langle f, u\rangle \quad \text{for all } u \in X^{**}.$$

If we restrict the class of test functions u to the subspace $X \subseteq X^{**}$ (via the natural embedding), we obtain the *weak-* convergence*:

$$f_k \rightharpoonup^* f \quad \text{if} \quad \langle f_k, u\rangle \to \langle f, u\rangle \quad \text{for all } u \in X \subseteq X^{**}. \tag{4}$$

If X is reflexive, then weak convergence is equivalent to weak-* convergence.

In finite-dimensional spaces, every bounded sequence has a convergent subsequence (by the Bolzano–Weierstrass theorem). However, in infinite-dimensional Banach spaces, this is no longer true in general for strong convergence. On the other hand, if the space X is reflexive, then every bounded sequence admits a *weakly* convergent subsequence.

Exercise 3.3. *Let ℓ^2 denote the space of square-summable sequences. Suppose $\{x_n\}_{n=1}^\infty \subset \ell^2$ is a bounded sequence, i.e.,*

$$\sup_n \|x_n\|_{\ell^2} < C.$$

Prove that there exists a subsequence $\{x_{n_k}\}$ and an element $x \in \ell^2$ such that for each coordinate $j \in \mathbb{N}$,

$$(x_{n_k})_j \to x_j \quad \textit{as } k \to \infty.$$

That is, prove that a bounded sequence in ℓ^2 admits a coordinate-wise convergent subsequence.

We now present a result under a stronger assumption: that the space X is separable. Recall that a normed space X is said to be *separable* if it contains a countable dense subset $Y \subset X$. When X is separable, it becomes possible to extract subsequences that converge pointwise on the dense subset and then apply a diagonal argument.

In functional analysis, the Banach–Alaoglu theorem (also known as Alaoglu's theorem) states that in a separable normed vector space, the closed unit ball of the dual space is compact in the weak-* topology.

Theorem 3.4 (Banach–Alaoglu)**.** *Let X^* be the dual space of a separable normed linear space X. Suppose $\{f_n\}_{n=1}^\infty \subset X^*$ is a norm-bounded sequence, i.e.,*

$$\sup_n \|f_n\| < \infty,$$

then there exists a subsequence that converges in the weak- topology of X^*.*

Proof. Since X is separable, there exists a countable dense subset $\{u_k\}_{k=1}^\infty \subset X$.

We proceed by constructing a diagonal subsequence. For the first point $u_1 \in X$, the sequence $\{\langle f_n, u_1\rangle\}_{n=1}^\infty$ is bounded in $\mathbb{R}$, hence there exists a subsequence $\{f_{n_j^1}\}$ such that

$$\langle f_{n_j^1}, u_1\rangle \to a_1 \quad \text{as } j \to \infty.$$

Next, consider the sequence $\{\langle f_{n_j^1}, u_2\rangle\}_{j=1}^\infty$. Again, this sequence is bounded, so it has a convergent subsequence $\{f_{n_j^2}\} \subset \{f_{n_j^1}\}$ such that

$$\langle f_{n_j^2}, u_2\rangle \to a_2 \quad \text{as } j \to \infty.$$

Continue this process inductively: for each k, extract a subsequence $\{f_{n_j^k}\} \subset \{f_{n_j^{k-1}}\}$ such that

$$\langle f_{n_j^k}, u_k\rangle \to a_k \quad \text{as } j \to \infty.$$

Now define a diagonal sequence by setting $f_{m_j} := f_{n_j^j}$. Then for each fixed k, the sequence $\langle f_{m_j}, u_k\rangle$ converges as $j \to \infty$.

For an arbitrary $u \in X$, since $\{u_k\}$ is dense in X and the functionals f_{m_j} are uniformly bounded, the convergence on the dense subset extends $\langle f_{m_j}, u\rangle$ to all elements u of X. Define a functional $f : X \to \mathbb{R}$ by

$$f(u) := \lim_{j\to\infty} \langle f_{m_j}, u\rangle.$$

This limit exists for all $u \in X$ because the f_{m_j} are uniformly bounded and converge on a dense subset. Moreover, f is linear and continuous:

$$|f(u)| \le \sup_j \|f_{m_j}\| \cdot \|u\| \le C\|u\|.$$

Therefore, $f \in X^*$ and

$$f_{m_j} \rightharpoonup^* f.$$

□

When X is reflexive, i.e., $X = X^{**}$, weak convergence and weak-* convergence coincide. The conditions "reflexive and separable" can be relaxed to "reflexive" only. Namely, in reflexive Banach spaces, every bounded sequence has a weakly convergent subsequence.

Theorem 3.5. *Let X be a reflexive Banach space, and let $\{u_k\} \subset X$ be a bounded sequence; that is,* $\sup_k \|u_k\| \le C$ *for some constant* $C > 0$. *Then there exists a subsequence* $\{u_{k_j}\} \subset \{u_k\}$ *and an element* $u \in X$ *such that*

$$u_{k_j} \rightharpoonup u.$$

We will skip the proof for Theorem 3.5 and refer to Theorem 3.4 for the case when X is reflexive and separable.

3.3. **Lower semi-continuity.** A function $f : X \to \mathbb{R}$ is *lower semi-continuous* if

$$f(u) \le \liminf_{k\to\infty} f(u_k) \quad \text{whenever } u_k \to u.$$

A function $f : X \to \mathbb{R}$ is *lower weakly semi-continuous* if

$$f(u) \le \liminf_{k\to\infty} f(u_k) \quad \text{whenever } u_k \rightharpoonup u.$$

To include singular behavior, we extend f to take the value $+\infty$:

$$f : X \to \mathbb{R}_{+\infty} := \mathbb{R} \cup \{+\infty\}, \qquad \operatorname{dom}(f) := \{x \in X : f(x) < +\infty\}.$$

A function is *proper* if $\operatorname{dom}(f) \neq \varnothing$.

The *graph* of f is

$$G(f) = \{(x, f(x)) : x \in \operatorname{dom}(f)\} \subset X \times \mathbb{R}.$$

The *epigraph* (or supergraph) of a function $f : X \to [-\infty, \infty]$ valued in the extended real numbers $[-\infty, \infty] = \mathbb{R} \cup \{\pm\infty\}$ is defined as

$$\begin{aligned}\operatorname{epi}(f) &= \{(x,t) \in X \times \mathbb{R} : t \ge f(x)\} \\ &= \left[f^{-1}(-\infty) \times \mathbb{R}\right] \cup \bigcup_{x\in f^{-1}(\mathbb{R})} \left(\{x\} \times [f(x), \infty)\right),\end{aligned}$$

where all sets in the union are pairwise disjoint. In the union over $x \in f^{-1}(\mathbb{R})$, the set $\{x\} \times [f(x), \infty)$ can be viewed as a vertical ray starting at $(x, f(x))$ and extending upward in $X \times \mathbb{R}$.

The following equivalence follows directly from the definition.

Theorem 3.6. *For any $f : X \to \mathbb{R}_{+\infty}$, the following are equivalent:*

(1) f is lower semi-continuous.
(2) $\operatorname{epi}(f)$ is closed in $X \times \mathbb{R}_{+\infty}$.

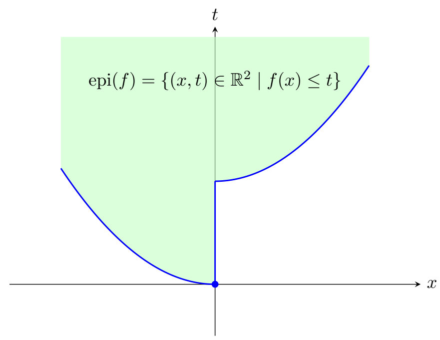


FIGURE 3. A lower semi-continuous function whose epigraph is closed.

The closedness of $\operatorname{epi}(f)$ implies a form of continuity along its boundary. Apparent vertical jumps in the graph of f may disappear after a suitable change of coordinates—in fact, f may become Lipschitz continuous in a new local chart. Extending f to $\mathbb{R}_{+\infty}$ allows us to handle functions such as $1/x$, whose behaviour can be understood through the closure of their epigraphs.

3.4. **Convexity.** A continuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ is *convex* if

$$f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y) \quad \forall\, x, y \in X,\ \forall\, \alpha \in [0,1]. \tag{5}$$

It is called *strictly convex* if the inequality is strict whenever $x \neq y$.

A convex function is called *μ-strongly convex* with parameter $\mu > 0$ if

$$f(\cdot) - \frac{\mu}{2}\|\cdot\|^2$$

is convex.

Exercise 3.7 (Strictly convex function)**.** *Show that if f is μ-strongly convex for some $\mu > 0$, then there exists a unique global minimizer of* $\min_{x \in X} f(x)$.

The convexity of a function can also be characterized through the geometry of its epigraph $\operatorname{epi}(f)$. The following equivalence follows directly from the definition.

Theorem 3.8. *Function f is convex and $\operatorname{dom}(f)$ is convex if and only if $\operatorname{epi}(f)$ is convex.*

3.5. **Equivalence between strong closedness and weak closedness of convex sets.** A helpful fact derived from convexity is the equivalence between strong closedness and weak closedness. More precisely, let X be a Banach space and $K \subset X$ be a convex set. Then, K is closed in the norm topology if and only if it is closed in the weak topology.

This equivalence follows from *Mazur's theorem*, which states that if $u_k \rightharpoonup u$ in X, then there exists a sequence of convex combinations of $\{u_k\}$, for example,

$$v_k = \sum_{n=k}^{N(k)} \alpha_n^{(k)} u_n,$$

such that $v_k \to u$ strongly in X. Mazur's theorem bridges weak and strong convergence by leveraging the convexity of the problem. Weak limits are "smoothed out" by convex combinations, which suppress oscillations or non-compactness responsible for weak-but-not-strong convergence.

Theorem 3.9 (Mazur). *Let X be a Banach space and $f : X \to \mathbb{R}$ be a convex function. Then f is lower semi-continuous (l.s.c.) if and only if f is lower weakly semi-continuous (l.w.s.c.).*

Proof. By the geometric characterization, f is l.s.c. if and only if its epigraph $\operatorname{epi}(f)$ is closed in the norm topology. Since $\operatorname{epi}(f)$ is convex, closedness in the norm topology is equivalent to closedness in the weak topology. Thus, f is l.s.c. if and only if f is l.w.s.c. □

3.6. **Lower semi-continuity of positive functional.** Suppose $u_k \to u$ strongly in L^1 and Lagrangian $L(x, u, p)$ are continuous function of variables (x, u, p). Can we exchange integral and limit to conclude

$$\lim_{k\to\infty} \int_\Omega L(u_k(x))\, \mathrm{d}x = \int_\Omega \lim_{k\to\infty} L(u_k(x))\, \mathrm{d}x = \int_\Omega L(u(x))\, \mathrm{d}x?$$

Here we skip (x, p) variables since the convergence is on u.

The answer is no as continuity of L alone does not imply a uniform bound or an integrable dominating function for

$$|L(u_k(x))| \leq g(x) \in L^1(\Omega)$$

Without such a bound, the Dominated Convergence Theorem (DCT) cannot be applied directly.

Example 3.10. Let $\Omega = [0, 1]$, and define $L(u) = u^2$. Consider the sequence of functions

$$u_k(x) := k^{2/3}\, \chi_{[0,\,1/k]}(x),$$

where $\chi_{[0,\,1/k]}$ is the characteristic function of the interval $[0, 1/k]$. We observe that $\int_0^1 u_k \,\mathrm{d}x = k^{-1/3}$ and thus $u_k \to 0$ in $L^1(\Omega)$, but

$$\int_0^1 u_k^2(x)\, \mathrm{d}x = k^{1/3} \to \infty \neq 0.$$

We can prove the lower semi-continuity if we assume $L \geq 0$. The integral $\int_\Omega L(u_k)\, \mathrm{d}x$ may be unbounded and goes to $+\infty$. But that will not affect the lower semi-continuity.

Before we get into technical details, we present *Littlewood's Three Principles of Analysis*, which capture key heuristics in measure theory and real analysis.

1. **Every measurable set is nearly a finite union of intervals (or more generally, compact sets).**
   For any measurable set $E \subset \mathbb{R}$ and any $\varepsilon > 0$, there exists a finite union of closed intervals F such that the Lebesgue measure $m(E \setminus F) < \varepsilon$.
2. **Every measurable function is nearly continuous.** *(Formalized by Lusin's Theorem.)*
   Let $u : \Omega \to \mathbb{R}$ be a measurable function on a bounded domain $\Omega \subset \mathbb{R}^n$ with finite Lebesgue measure. Then for every $\varepsilon > 0$, there exists a closed set $K \subset \Omega$ such that

- $|\Omega \setminus K| < \varepsilon$,
- $u|_K$ is uniformly continuous.

3. **Every pointwise convergent sequence of measurable functions is nearly uniformly convergent.** *(Formalized by Egorov's Theorem.)*
   Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $\{u_n\}$ be a sequence of measurable functions $u_n : \Omega \to \mathbb{R}$ such that
   $$u_n(x) \;\longrightarrow\; u(x) \quad \text{for almost every } x \in \Omega.$$
   Then, for every $\varepsilon > 0$, there exists a *closed* set $K \subset \Omega$ such that
   - $|\Omega \setminus K| < \varepsilon$,
   - $u_n \to u$ uniformly on K.

As we assume Ω is bounded, the closed and bounded set K is also compact. On compact sets, a continuous function is bounded and DCT can apply. The non-negativity is used to have the lower bound.

Lemma 3.11 (Lower semicontinuity for non–negative integrands)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded domain, and let $u_k, u \in L^1(\Omega)$ satisfy*
$$u_k \;\longrightarrow\; u \quad \textit{strongly in } L^1(\Omega).$$
Assume L is continuous and $L \geq 0$. Then
$$\int_\Omega L\big(u(x)\big)\,\mathrm{d}x \;\leq\; \liminf_{k\to\infty} \int_\Omega L\big(u_k(x)\big)\,\mathrm{d}x.$$

Proof. [1] **Pick a "good" set.** Fix $\varepsilon > 0$. By Lusin's theorem there is a closed set $K_1 \subset \Omega$ such that
$$|\Omega \setminus K_1| < \varepsilon, \qquad u \text{ is continuous on } K_1.$$
Applying Egorov's theorem to $\{u_k\}$ on K_1, we obtain a compact subset $K \subset K_1$ with the same measure property
$$|\Omega \setminus K| < \varepsilon,$$
such that $u_k \to u$ uniformly on K.

[2] **Convergence on K.** Because L is continuous and $u_k \to u$ uniformly on the compact set K,
$$\sup_{x\in K}\Big|L\big(u_k(x)\big) - L\big(u(x)\big)\Big| \;\xrightarrow{k\to\infty} 0.$$
Hence
$$\int_K L\big(u_k(x)\big)\,\mathrm{d}x \;\longrightarrow\; \int_K L\big(u(x)\big)\,\mathrm{d}x.$$

[3] **Take the $\liminf$.** Split each integral:
$$\int_\Omega L\big(u_k(x)\big)\,\mathrm{d}x = \int_K L\big(u_k(x)\big)\,\mathrm{d}x \;+\; \int_{\Omega\setminus K} L\big(u_k(x)\big)\,\mathrm{d}x,$$
$$\int_\Omega L\big(u(x)\big)\,\mathrm{d}x = \int_K L\big(u(x)\big)\,\mathrm{d}x \;+\; \int_{\Omega\setminus K} L\big(u(x)\big)\,\mathrm{d}x.$$
Because $L \geq 0$, the second term in the first line is non–negative; thus
$$\liminf_{k\to\infty} \int_\Omega L\big(u_k(x)\big)\,\mathrm{d}x \;\geq\; \lim_{k\to\infty} \int_K L\big(u_k(x)\big)\,\mathrm{d}x \;=\; \int_K L\big(u(x)\big)\,\mathrm{d}x.$$

Therefore

$$\int_\Omega L\big(u(x)\big)\,\mathrm{d}x \;\le\; \liminf_{k\to\infty}\int_\Omega L\big(u_k(x)\big)\,\mathrm{d}x \;+\; \int_{\Omega\setminus K} L\big(u(x)\big)\,\mathrm{d}x.$$

$\boxed{4}$ **Let $\varepsilon \to 0$.** Because $L(u(\cdot)) \in L^1(\Omega)$, the Lebesgue integral is absolutely continuous with respect to measure and $|\Omega \setminus K| < \varepsilon$. Therefore, we can make

$$\int_{\Omega\setminus K} L\big(u(x)\big)\,\mathrm{d}x < \delta$$

for any given $\delta > 0$ by choosing $\varepsilon > 0$ small enough. Sending $\delta \to 0$ gives the desired inequality. □

Example 3.12. Let $L(u) = -u$. Consider $\Omega = [0,1]$ and define

$$u_k(x) = k\chi_{[0,1/k]}(x), \quad u(x) = 0.$$

Then $u_k \to u$ strongly in $L^1(\Omega)$, $\int_\Omega L(u_k)\,\mathrm{d}x = -1$ for all k, $\int_\Omega L(u)\,\mathrm{d}x = 0$, and $\liminf_{k\to\infty}\int_\Omega L(u_k)\,\mathrm{d}x = -1 < 0$. Thus the conclusion of Lemma 3.11 fails when L takes negative values.

3.7. **Convexity and lower weak semi-continuity.** If $\nabla u_k \to \nabla u$ and $L(x,u,p)$ is also continuous in p, then we can apply the lower semi-continuity as before for $u_k \to u$. However, the boundedness of $\|\nabla u_k\|_{L^q}$ can only lead to the weak convergence $\nabla u_k \rightharpoonup \nabla u$ (up to a sub-sequence). We need extra condition to recover the lower weak semi-continuity of $I(u)$.

In view of Mazur's Theorem 3.9, one sufficient condition is $L(x,u,p)$ is convex in p.

Lemma 3.13 (Lower weak semicontinuity of convex integral functionals)**.** *Let $\Omega \subset \mathbb{R}^n$ be measurable with finite measure. Suppose $L : \mathbb{R}^m \to \mathbb{R}$ is convex and continuously differentiable. Let $\{p_k\} \subset L^q(\Omega;\mathbb{R}^m)$ for some $1 < q < \infty$, and suppose:*

- $p_k \rightharpoonup p$ *weakly in* $L^q(\Omega;\mathbb{R}^m)$,
- $L(p_k) \in L^1(\Omega)$ *and* $\nabla L(p) \in L^{q'}(\Omega;\mathbb{R}^m)$.

Then

$$\int_\Omega L(p(x))\,\mathrm{d}x \;\le\; \liminf_{k\to\infty}\int_\Omega L(p_k(x))\,\mathrm{d}x.$$

Proof. By convexity and differentiability of $L : \mathbb{R}^m \to \mathbb{R}$, we have the inequality:

$$L(p_k(x)) \ge L(p(x)) + \nabla L(p(x))\cdot(p_k(x) - p(x)) \quad \text{for all } x \in \Omega.$$

Integrating both sides over Ω, we obtain:

$$\int_\Omega L(p_k(x))\,\mathrm{d}x \;\ge\; \int_\Omega L(p(x))\,\mathrm{d}x + \int_\Omega \nabla L(p(x))\cdot(p_k(x)-p(x))\,\mathrm{d}x.$$

Since $p_k \rightharpoonup p$ in $L^q(\Omega;\mathbb{R}^m)$ and $\nabla L(p) \in L^{q'}(\Omega;\mathbb{R}^m)$, the last term tends to zero:

$$\int_\Omega \nabla L(p(x))\cdot(p_k(x)-p(x))\,\mathrm{d}x \to 0.$$

Taking $\liminf$ on both sides yields:

$$\liminf_{k\to\infty}\int_\Omega L(p_k(x))\,\mathrm{d}x \;\ge\; \int_\Omega L(p(x))\,\mathrm{d}x.$$

□

One can also apply a variant of Mazur's theorem to find convex combination of p_k with strong convergence so that the assumption $\nabla L(p) \in L^{q'}(\Omega;\mathbb{R}^m)$ is not needed.

3.8. **Application to Calculus of Variations.** We aim to establish the lower weak semi-continuity (l.w.s.c.) for the functional $I(u)$. Since p will be substituted by ∇u, which has a dominant scaling, we only need the convexity of the Lagrangian as a function of p.

In 1-D, we can calculate the scaling of u and u' to demonstrate the difference in scaling. In multi-dimensions, the dominance of the scaling can be argued by Rellich-Kondrachov compactness embedding theorem. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary and $1 \leq q < \infty$. The embedding operator $W^{1,q}(\Omega) \hookrightarrow L^q(\Omega)$ is compact in the sense that every bounded sets is mapped to a sequential compact set. That is every bounded sequence in $W^{1,q}$ will imply a strong convergent subsequence in L^q.

As the weak convergence will imply the uniform boundedness of the norm, we conclude that if a sequence $\{u_k\} \subset W^{1,q}(\Omega)$ converges weakly to $u \in W^{1,q}(\Omega)$, i.e.,

$$u_k \rightharpoonup u \quad \text{in} \quad W^{1,q}(\Omega),$$

then there exists a subsequence $\{u_{k_j}\}$ that converges strongly in $L^q(\Omega)$:

$$u_{k_j} \to u \quad \text{in} \quad L^q(\Omega).$$

Note that the domain regularity (Ω bounded Lipschitz) and condition $q < \infty$ are essential.

The proof of Tonelli-Morrey is technical but the idea has been presented in Lemma 3.11 and 3.13. For $u_k \to u$, we switch to a good compact subset. For $\nabla u_k \rightharpoonup u$, we use the convexity. The non-negativity $L \geq 0$ is used to get the lower bound.

Theorem 3.14 (Tonelli-Morrey)**.** *Suppose* $L : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^1$ *satisfies*

(1) $L \in C^1\left(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n\right)$,
(2) $L \geq 0$,
(3) $\forall (x,u) \in \Omega \times \mathbb{R}, p \mapsto L(x,u,p)$ *is convex,*

then $I(u) = \int_\Omega L(x,u(x),\nabla u(x))\,\mathrm{d}x$ *is weakly sequentially lower semicontinuous in* $W^{1,q}(\Omega)\,(1 \leq q < \infty)$.

Proof. 1. By the Rellich-Kondrachov theorem, we can find a subsequence, still denoted by $\{u_j\}$ such that

$$u_j \to u, \quad L^q\left(\Omega, \mathbb{R}^N\right).$$

Then there exists a further subsequence, which is again denoted by $\{u_j\}$ such that

$$u_j(x) \to u(x) \text{ a.e. } x \in \Omega.$$

2. For $\varepsilon > 0$, there exists a compact subset $K \subset \Omega$ such that $|\Omega \backslash K| < \varepsilon$ and

1) $u_j \to u$ uniformly on K (Egorov's theorem),
2) u and ∇u are continuous on K (Luzin's theorem),
3) As $L \geq 0$, by the absolute continuity of Lebesgue integrals,

$$\int_K L(x,u(x),\nabla u(x))\,\mathrm{d}x \geq \int_\Omega L(x,u(x),\nabla u(x))\,\mathrm{d}x - \varepsilon.$$

The convexity implies that

$$L(x,u_k,\nabla u_k) \geq L(x,u_k,\nabla u) + L_p(x,u_k,\nabla u)\cdot(\nabla u_k - \nabla u).$$

Integrate over K to get

$$I(u_k) \geq \int_K L(x,u_k,\nabla u) + \int_K L_p(x,u,\nabla u)\cdot(\nabla u_k - \nabla u)$$
$$+ \int_K (L_p(x,u_k,\nabla u) - L_p(x,u,\nabla u))\cdot(\nabla u_k - \nabla u).$$

3. On $K, u_j \to u$ uniformly, and L is continuous, hence the constant C dominates the integrand:

$$\big|L\big(x,u_j(x),\nabla u(x)\big)\big| \;\leq\; C,$$

uniformly in j. We can use the Dominated Convergence Theorem to conclude the first term

$$\int_K L\left(x,u_j(x),\nabla u(x)\right)\,\mathrm{d}x \to \int_K L(x,u(x),\nabla u(x))\,\mathrm{d}x$$

4. Because $u, \nabla u$ are continuous on the compact set $K \subset \Omega$ and L is C^1, we have

$$L_p\big(x,u(x),\nabla u(x)\big) \;\in\; L^\infty(K;\mathbb{R}^{nN}).$$

Let χ_K be the characteristic function of K. Then

$$\phi(x) \;:=\; \chi_K(x)\,L_p\big(x,u(x),\nabla u(x)\big) \;\in\; L^\infty(\Omega;\mathbb{R}^{nN}) \;\subset\; L^{q'}(\Omega;\mathbb{R}^{nN}),$$

where $q' = q/(q-1)$ is the Hölder conjugate of q, i.e. $1/q' + 1/q = 1$.

Hence the duality pairing satisfies

$$\int_\Omega \phi(x)\left(\nabla u_j(x) - \nabla u(x)\right)\mathrm{d}x = \int_K L_p\!\big(x,u(x),\nabla u(x)\big)\left(\nabla u_j(x) - \nabla u(x)\right)\mathrm{d}x \longrightarrow 0,$$

because $\phi \in L^{q'}$ is fixed and $\nabla u_j - \nabla u \rightharpoonup 0$ in L^q. Therefore the second term vanishes:

$$\lim_{j\to\infty}\int_K L_p\!\big(x,u(x),\nabla u(x)\big)\left(\nabla u_j(x) - \nabla u(x)\right)\mathrm{d}x \;=\; 0.$$

5. Lastly, since a weakly convergent sequence is bounded,

$$\|\nabla u_j - \nabla u\|_1 \leq C_1\left(\|\nabla u_j - \nabla u\|_q\right) \leq C_1\left(\|\nabla u_j\|_q + \|\nabla u\|_q\right) \leq C_2.$$

Furthermore, since $L_p\left(x,u_j(x),\nabla u(x)\right) \to L_p(x,u(x),\nabla u(x))$ uniformly on K, it follows that the third term

$$\int_K \left(L_p\left(x,u_j(x),\nabla u(x)\right) - L_p(x,u(x),\nabla u(x))\right)\left(\nabla u_j - \nabla u\right)\,\mathrm{d}x$$
$$\leq \|L_p\left(\cdot,u_j(\cdot),\nabla u(\cdot)\right) - L_p(\cdot,u(\cdot),\nabla u(\cdot))\|_{\infty,K}\,\|\nabla u_j - \nabla u\|_{1,K} \to 0.$$

In summary,

$$\liminf_{j\to\infty} I\left(u_j\right) \geq \int_K L(x,u(x),\nabla u(x))\,\mathrm{d}x \geq I(u) - \varepsilon$$

Since $\varepsilon > 0$ is arbitrary,

$$\liminf_{j\to\infty} I\left(u_j\right) \geq I(u).$$

□

Put all components together, we summarize the existence theorem below.

Theorem 3.15 (Existence of minimizer). *Assume that L satisfies the coercivity inequality and is convex in the variable p. Suppose also the admissible set $\mathcal{M}$ is nonempty. Then there exists at least one function $u \in \mathcal{M}$ solving*

$$I(u) = \min_{w \in \mathcal{M}} I(v).$$

Proof. The proof can be treat a fix of the Dirichlet principle.

Choose a minimizing sequence $\{u_k\} \subset \mathcal{M}$ such that $I(u_k) \to \inf_{u \in \mathcal{M}} I(u)$. By the coercivity, we conclude the sequence $\{u_k\}$ is bounded in $W^{1,q}$. Then there exists a weakly convergent subsequence $\nabla u_{k_j} \rightharpoonup \nabla u_0$ in L^q and $u_{k_j} \to u_0$ strongly in L^q. By the convexity, we know $I(\cdot)$ is l.w.s.c. Then u_0 is the desired minimizer:

$$\inf_{u \in \mathcal{M}} I(u) \leq I(u_0) \leq \liminf_k I(u_k) = \inf_{u \in \mathcal{M}} I(u).$$

□

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