NOETHER'S THEOREM

LONG CHEN

ABSTRACT.

1. NOETHER'S THEOREM

1.1. **Variation of the domain.** Consider a smooth diffeomorphism

$$\begin{aligned}
&x \to y(x,\varepsilon)\\
&y(x,0) = x, \quad \partial_\varepsilon y(x,0) = \vec{v}(x)\\
&y(x,\varepsilon) = x + \varepsilon \vec{v}(x) + 0(\varepsilon)
\end{aligned}$$

Through this mapping: $\Omega \to \Omega_\varepsilon = \{y = y(x,\varepsilon) : x \in \Omega\}$. The difference $y - x$ is called displacement and the parameter ε indicates we are considering small deformation of the domain.

Lemma 1.1.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_{\Omega_\varepsilon} f(y)\,\mathrm{d}y\bigg|_{\varepsilon=0} &= \int_\Omega \nabla\cdot(f\vec{v})\,\mathrm{d}x\\
&= \int_{\partial\Omega} f\vec{v}\cdot n\,\mathrm{d}s
\end{aligned}$$

Proof. We apply the integration by parts to get

$$\int_{\Omega_\varepsilon} f(y)\,\mathrm{d}y = \int_\Omega f(y(x,\varepsilon))|J(x,\varepsilon)|\,\mathrm{d}x$$

where $J = \left(\frac{\partial y}{\partial x}\right)$ is the Jacobian matrix and $|J| = |\det J|$. By definition

$$J(x,0) = I, |J(x,0)| = 1, \partial_\epsilon J(x,0) = \left(\frac{\partial \vec{v}}{\partial x}\right), |\partial_\epsilon J(x,0)| = \nabla\cdot\vec{v}.$$

Now the domain is independent of ε. By the product rule, we have

$$\nabla f \partial_\varepsilon y|J| + f\partial\varepsilon|J||_{\varepsilon=0} = \quad \nabla f(x)\vec{v} + f(x)\nabla\cdot\vec{v} = \nabla\cdot(f\vec{v}).$$

□

As a corollary, we have

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_{\Omega_\varepsilon} L(y,u(y),\nabla u(y))\,\mathrm{d}y\,|_{\varepsilon=0} = \int_\Omega \nabla\cdot\Big[L(x,u(x),\nabla u(x))\,\vec{v}(x)\Big]\,\mathrm{d}x. \tag{1}$$

Date: June 9, 2023.

1.2. **Variation of functions.** Consider the following non-linear perturbation of u

$$\begin{aligned}&u(x) \to w(x,\varepsilon),\\ &w(x,0) = u(x), \quad m(x) = \partial_\varepsilon w(x,0),\\ &w(x,\varepsilon) = u(x) + \varepsilon m(x) + o(\varepsilon).\end{aligned}$$

Lemma 1.2.

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_\Omega L\left(x, w(x,\varepsilon), \nabla_x w(x,\varepsilon)\right)\,\mathrm{d}x\Big|_{\varepsilon=0}\\ &= \int_\Omega L_u m + L_p \nabla m \,\mathrm{d}x\\ &= \int_\Omega (-\nabla\cdot L_p + L_u)m + \nabla\cdot(L_p\, m)\,\mathrm{d}x.\end{aligned}$$

Proof. We compute the derivative w.r.t. ε

$$L_u \partial_\varepsilon w(x,\varepsilon) + L_p \partial_\varepsilon \nabla_x w(x,\varepsilon)$$

and switch $\partial_\varepsilon \nabla_x$ to $\nabla_x \partial\varepsilon$ as w is smooth enough.

The procedure is the same as the linear perturbation $u + \varepsilon\phi$ except when apply integration by parts

$$\int_\Omega L_p \nabla m \,\mathrm{d}x = \int_\Omega -\nabla\cdot L_p m \,\mathrm{d}x + \int_{\partial\Omega} n\cdot L_p\, m \,\mathrm{d}S.$$

And the boundary term is $\int_{\partial\Omega} n\cdot L_p m\,\mathrm{d}S = \int_\Omega \nabla\cdot(L_p\, m)$. □

1.3. **Noether's theorem.**

Theorem 1.3. *If there exists domain variation $y(x,\varepsilon)$ and function variation $w(x,\varepsilon)$ such that*

$$\int_\Omega L\left(x, w(x,\varepsilon), \nabla_x w(x,\varepsilon)\right)\,\mathrm{d}x = \int_{\Omega_\varepsilon} L\left(y, u(y), \nabla u(y)\right)\,\mathrm{d}y$$

then

(1)

$$\int_\Omega \mathrm{EL}(u) m \,\mathrm{d}x + \int_\Omega \nabla\cdot\left(L_p m - L\vec{v}\right)\,\mathrm{d}x = 0,$$

where $\mathrm{EL}(u)$ stands for the Euler-Lagrange equation

$$\mathrm{EL}(u) := -\nabla\cdot L_p(x,u,\nabla u) + L_u(x,u,\nabla u).$$

(2) For u solves the Euler-Lagrange equation, i.e. $\mathrm{EL}(u) = 0$, then

$$\int_\Omega \nabla\cdot\left(L_p m - L\vec{v}\right)\,\mathrm{d}x = 0.$$

Furthermore if it holds for arbitrary domain, then we have the divergence identity

$$\nabla\cdot\left(L_p m - L\vec{v}\right) = 0.$$

Proof. Combination of the two variation formulae. □

2. EXAMPLES

2.1. **Classic mechanics.** Consider ℓ particles with mass: $M_1, M_2, \cdots, M_l$ moving in space . The position function is denoted by $\boldsymbol{q} = (q_1, q_2, \cdots, q_\ell)$ and each $q_i(t) = \begin{pmatrix} x_i(t) \\ y_i(t) \\ z_i(t) \end{pmatrix}$ is a function $\mathbb{R} \to \mathbb{R}^3$. Define the kinetic energy: $T(\dot{\boldsymbol{q}}) = \frac{1}{2}\sum M_j |\dot{q}_j(t)|^2$ and the potential energy: $V(\boldsymbol{q}) = -C\sum_{i<j} \frac{M_i M_j}{|q_i - q_j|}$ where C is a universal constant and for simplicity will be skipped. The Lagrangian $L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = T(\dot{\boldsymbol{q}}) - V(\boldsymbol{q})$

$$I(\boldsymbol{q}) = \int_a^b L(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))\,\mathrm{d}t.$$

The trajectory of the ℓ particles will be determined by the Euler-Lagrange equation and for $\ell \geq 3$, we may not be able to find a closed form solution.

Using Noether's theorem, we can still find out conservation laws without solving the E-L equation.

Translation invariance in time. Consider the variation

$$t \to \tau = t + \varepsilon, \qquad v = \partial_\epsilon \tau(t, 0) = 1,$$
$$\boldsymbol{q}(t) \to w(t, \epsilon) = \boldsymbol{q}(t + \varepsilon), \qquad m(t) = \partial_\varepsilon w(t, 0) = \dot{\boldsymbol{q}}(t).$$

The domain change is $\Omega = (a, b) \to \Omega_\varepsilon = (a + \epsilon, b + \epsilon)$. Then we verify the invariance as

$$\int_a^b T(\dot{\boldsymbol{q}}(t+\varepsilon)) - V(\boldsymbol{q}(t+\varepsilon))\,\mathrm{d}t = \int_{a+\varepsilon}^{b+\varepsilon} T(\dot{\boldsymbol{q}}(\tau)) - V(\boldsymbol{q}(\tau))\,\mathrm{d}\tau.$$

So we will have the conservation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(L_p\dot{\boldsymbol{q}} - L\right) = \frac{\mathrm{d}}{\mathrm{d}t} H(\boldsymbol{p}(t), \boldsymbol{q}(t)) = 0.$$

For $\boldsymbol{q}$ solves the E-L equation, by definition, $L_p\dot{\boldsymbol{q}} - L = H(\boldsymbol{p}(t), \boldsymbol{q}(t))$. Indeed the Hamiltonian is conserved for all time independent Lagrangian.

For this example, the Hamiltonian

$$H = T + V$$

is the total energy. Conservation of energy is deduced from the translation invariance in time.

Translation invariance in space. Consider the variation

$$t \to \tau = t, \qquad v = \partial_\epsilon \tau(t, 0) = 0,$$
$$q_i(t) \to w_i(t, \epsilon) = q_i(t) + \varepsilon \boldsymbol{e}, \qquad m(t) = \partial_\varepsilon w_i(t, 0) = \boldsymbol{e},$$

where $\boldsymbol{e} \in \mathbb{R}^3$ is arbitrary but constant vector.

The invariance is obvious as $\dot{w}(t, \epsilon) = \dot{\boldsymbol{q}}(t)$ and $q_i - q_j = w_i - w_j$ as the location is shifted by a constant vector $\boldsymbol{e}$.

So the conservation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(L_p \boldsymbol{e}\right) = 0 \Rightarrow \sum M_i \dot{q}_i \cdot \boldsymbol{e} = \text{const}.$$

By choosing $\boldsymbol{e} = (1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$, we obtain the conservation of the total momentum as $M_i \dot{q}_i$ is the momentum of the i-th particle.

Translation invariance in rotation. Consider the variation

$$t \to \tau = t, \qquad v = \partial_\epsilon \tau(t,0) = 0,$$

$$\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} \to \begin{cases} \widetilde{x}_i = x_i \cos\varepsilon + y_i \sin\varepsilon, \\ \widetilde{y}_i = -x_i \sin\varepsilon + y_i \cos\varepsilon, \\ \widetilde{z}_i = z_i \end{cases} \qquad m(t) = \partial_\varepsilon w_i(t,0) = R_\varepsilon'(0),$$

where $\{R_\varepsilon\}$ is a family of space-time coordinate transformations depending on the parameter ε :

$$R_\varepsilon = \begin{pmatrix} \cos\varepsilon & \sin\varepsilon \\ -\sin\varepsilon & \cos\varepsilon \end{pmatrix}, \quad R_\varepsilon'(0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Therefore the conservation property becomes

$$\sum_{i=1}^{\ell} M_i \left(y_i \dot{x}_i - x_i \dot{y}_i \right) = \text{ const.}$$

Likewise, for rotations in the yz-plane and the xz-plane, we also have similar identities. This is the conservation of angular momentum:

$$\sum_{i=1}^{\ell} M_i \boldsymbol{q}_i \wedge \dot{\boldsymbol{q}}_i = \text{ const.}$$