Numer. Math. (2008) 109:167–191
DOI 10.1007/s00211-007-0118-6

Numerische
Mathematik

Stability and accuracy of adapted finite element methods for singularly perturbed problems

Long Chen · Jinchao Xu

Received: 18 November 2005 / Published online: 9 February 2008

Abstract The stability and accuracy of a standard finite element method (FEM) and a new streamline diffusion finite element method (SDFEM) are studied in this paper for a one dimensional singularly perturbed connvection-diffusion problem discretized on arbitrary grids. Both schemes are proven to produce stable and accurate approximations provided that the underlying grid is properly adapted to capture the singularity (often in the form of boundary layers) of the solution. Surprisingly the accuracy of the standard FEM is shown to depend crucially on the uniformity of the grid away from the singularity. In other words, the accuracy of the adapted approximation is very sensitive to the perturbation of grid points in the region where the solution is smooth but, in contrast, it is robust with respect to perturbation of properly adapted grid inside the boundary layer. Motivated by this discovery, a new SDFEM is developed based on a special choice of the stabilization bubble function. The new method is shown to have an optimal maximum norm stability and approximation property in the sense that $\|u - u_N\|_\infty \leq C \inf_{v_N \in V^N} \|u - v_N\|_\infty$, where u_N is the SDFEM approximation in linear finite element space V^N of the exact solution u. Finally several optimal convergence results for the standard FEM and the new SDFEM are obtained and an open question about the optimal choice of the monitor function for the moving grid method is answered.

Mathematics Subject Classification (2000) 65L10 · 65L20 · 65L60 · 76R99

This work was supported in part by NSF DMS-0209497 and NSF DMS-0215392 and the Changjiang Professorship through Peking University.

L. Chen (✉) · J. Xu
Department of Mathematics, University of California at Irvine, Irvine, CA, USA
e-mail: chenlong@math.uci.edu

J. Xu
The School of Mathematical Science, Peking University, Beijing, China
e-mail: xu@math.psu.edu

1 Introduction

It has been numerically observed and sometimes theoretically verified that the standard finite element method (FEM) [10,40,56,57] and the streamline diffusion finite element method (SDFEM) [39,54] often give good and stable approximations of singularly perturbed boundary-value problems if the underlying grid is properly adapted to capture the singularity of the solution such as sharp boundary layers. In this paper, we give a careful analysis of this phenomenon and further develop improved algorithms.

The model problem we will study is a linear convection-dominated stationary convection-diffusion problem:

$$-\varepsilon u'' - bu' = f \quad \text{in } (0,1), \tag{1}$$

$$u(0) = g_0, \quad u(1) = g_1, \tag{2}$$

where the diffusion constant ε satisfies $0 < \varepsilon \ll b$. For the simplicity of analysis, we assume the convection coefficient b is constant and positive. Most results in this paper remain true if b is a smooth enough function with a positive lower bound [13].

The solution to (1)–(2) typically has a boundary layer at $x = 0$ and thus the standard FEM approximation on a uniform grid will yield nonphysical oscillation unless the mesh size compares with ε (see, e.g. [41,42,47]). To obtain a reliable numerical approximation, one approach is to use highly non-uniform mesh which is adapted to capture the boundary layer. Examples of this approach are layer-adapted grids [1,38,46,52] or grids by the equidistribution of monitor functions [7,17,18,55]. Another approach is to invoke some form of upwinding to stabilize the scheme. The SDFEM, introduced first by Hughes and Brooks in [28], is one of such stabilized methods and it can also be derived from more general approaches based on, for example, residual-free bubble finite element method [5,22] and multiscale variational methods [27,29].

The above approaches and their combinations have been observed to work well in practice. Their error analysis, however, is not so easy. The classic convergence result for the standard FEM is not appropriate in the sense that the accuracy depends not only on the number of the grid nodes N but also on the parameter ε. When ε is small, the error bound becomes prohibitively large. This paper is devoted to the ε-uniform convergence. All the error bounds in this paper are ε-uniform unless it is explicitly expressed otherwise.

1.1 Smooth solutions

To isolate the stability issue, the first question we would like to ask is about a smooth solution to this equation. Namely, if the solution to (1)–(2) happens to be smooth (say, $\|u^{(3)}\|_\infty$ is uniformly bounded), does the standard FEM on a quasiuniform grid $\mathcal{T}_N$ give an accurate approximation to the solution?

Let u_N be the standard finite element approximation of the exact solution u based on the grid $\mathcal{T}_N$. Here are the answers to the above question.

1. The stability of the standard FEM depends on the parity of the number of unknowns for uniform grids. Namely $\|u-u_N\|_\infty \le CN^{-2}$ if the number of unknowns is even, while the method is not ε-uniformly stable if the number of unknowns is odd.
2. When the number of unknowns is even, the method can be stabilized if we only move any one grid point within $O(\varepsilon)$ to one of its neighbor.
3. For quasi-uniform grids, we show that $\|u-u_N\|_\infty = O(N^{-1})$ at best in general. Although for the nodal interpolation u_I the error is of second order, namely $\|u-u_I\|_\infty \le CN^{-2}$.

1.2 Solutions with boundary layers

In most cases of interest, such as the homogeneous boundary condition with a smooth source term, the solution to (1)–(2) has a boundary layer at $x = 0$. In order to capture the boundary layer highly nonuniform layer-adapted grids need to be adopted. Among them, Bakhvalov grid [1] and Shishkin grid [52] are two commonly used grids. For these two grids, the uniform convergence of the standard FEM is well known [33,38,41]: the following two error estimates

$$\|u-u_N\|_\infty \le CN^{-2} \quad \text{and} \quad \|u-u_N\|_\infty \le CN^{-2}\ln^2 N, \tag{3}$$

are valid for Bakhvalov grid and Shishkin grid respectively.

A careful analysis in this paper will provide some further insights to this type of results. Namely the optimality of the convergence rate in (3) depends crucially on the uniformity of the grid in the smooth part. If the grid is only quasiuniform outside of the boundary layer, we can only expect in general

$$\|u-u_N\|_\infty = O(N^{-1}).$$

But if the grid is indeed uniform away from the boundary layer, the estimate (3) remains valid even if the grid is locally perturbed (in a locally quasi-uniform manner) within the boundary layer. From both theoretical and practical points of view, we find this is a rather significant phenomenon for singularly perturbed problems.

1.3 Uniform stability of a new SDFEM

For singularly perturbed problems, special stablized methods such as the streamline diffusion finite element method (SDFEM) are more often used than the standard finite element method. Many convergence estimates of the SDFEM [3,4,30,31,44,58] have been done for quasiuniform meshes which show that the SDFEM is able to capture the main feature of the solution without using layer-adapted meshes to resolve the boundary layer. Nevertheless, very few ε-uniformly pointwise convergence results are obtained inside the boundary layer [39,53,54].

We will propose a new SDFEM for the problem (1)–(2) and analyze it on *arbitrary* grids $\mathcal{T}_N$. With a special choice of the stabilization bubble function, we will prove that the new SDFEM approximation $\tilde{u}_N$ is nearly optimal (or so-called quasi-optimal) in the maximum norm, namely

$$\|u - \tilde{u}_N\|_\infty \le C \inf_{v_N \in V^N} \|u - v_N\|_\infty, \tag{4}$$

where V^N is the linear finite element space based on $\mathcal{T}_N$ with appropriate boundary conditions. We would like to explicitly point out again that here C is a constant that is independent on both ε and N.

The estimate (4) is the most desirable estimate we may expect to obtain for the problem (1)–(2). Such types of estimates have been known for "diffusion dominated" problem in both one and two dimensions [50,51]. But we have not seen such an estimate for singularly perturbed problems. The added difficulty is of course the uniformity with respect to ε. Effort to obtain such type of result can also be found in [48,49].

1.4 Convergence of the new SDFEM and optimal monitor function

Thanks to (4) the convergence of the new SDFEM becomes an approximation problem which is well studied in the literature (see, e.g. [11,19]). If, for example, the function $|u''|^{1/2}$ is monotone, there exists a grid such that

$$\|u - u_I\|_\infty \le C\|u''\|_{1/2} N^{-2}, \tag{5}$$

and thus by (4)

$$\|u - \tilde{u}_N\|_\infty \le C\|u - u_I\|_\infty \le C\|u''\|_{1/2} N^{-2}, \tag{6}$$

where $\|u''\|_{1/2} := \left(\int_0^1 |u''|^{1/2} \mathrm{d}x\right)^2$. Note that $\|u''\|_{1/2}$ is ε-uniformly bounded in many cases, the convergence (6) is indeed ε-uniform.

A commonly used approach to constructing such a grid is the use of the monitor function $M(x)$ and the equidistribution principle. The grid $\mathcal{T}_N = \{0 = x_0 < \cdots < x_{N+1} = 1\}$ is chosen such that

$$\int_{x_i}^{x_{i+1}} M(x)\,\mathrm{d}x = \text{constant}, \quad i = 0, 1, 2, \ldots, N.$$

In the literature, the monitor function resulting a first order uniform convergence is the arc-length function $M = \sqrt{1 + |u'|^2}$ or its discrete analogue [14,15,35,45]. The optimal choice of the monitor function for a second order uniform convergent scheme remains open. Based on our convergence result (6) $M = |u''|^{1/2}$ is evidently a monitor function that leads to the optimal rate of convergence.

The layout of the rest of this paper is as follows: in the next section we will study the standard FEM for smooth solutions and solutions with boundary layers. In Sect. 3, we will develop a new SDFEM and analyze its stability and convergence. The last section contains some concluding remarks.

2 Stability analysis of the standard FEM

In this section we will study the stability of the standard FEM applied to the problem (1)–(2) on arbitrary grids. The approach used here mainly follows the work of Kopteva for central difference discretization in [33].

Let us first introduce some notation. For a positive integer N, let $\mathcal{T}_N := \{x_i \,|\, 0 = x_0 < x_1 < \cdots < x_{N+1} = 1\}$ be an arbitrary grid and let φ_i be the nodal basis function at point x_i. The linear finite element space $V^N := \{v_N | v_N = \sum_{i=0}^{N+1} v_i \varphi_i\}$. For a function $u \in C[0, 1]$, let $u_i := u(x_i)$ be the nodal values and let $u_I := \sum_{i=0}^{N+1} u_i \varphi_i$ be the nodal interpolant of u. The discrete maximum norm of u is denoted by $\|u\|_{\infty,\mathcal{T}_N} := \max_{0 \le i \le N+1} |u_i|$. For an index set $I \subset \{0, 1, \ldots, N + 1\}$, $\|u\|_{\infty,I} := \max_{i \in I} |u_i|$. On the other hand, given a discrete function $\{v_i, i = 0, 1, \ldots, N + 1\}$, the same letter without the subindex will denote the piecewise linear and global continuous function in V^N, i.e. $v := \sum_{i=0}^{N+1} v_i \varphi_i$. Thus the discrete maximum norm for the discrete function v_i will be written as $\|v\|_{\infty,\mathcal{T}_N}$.

2.1 Basic error equation

Let $H^1 = \{v \,|\, v \in L^2(0, 1), v' \in L^2(0, 1)\}$ and $H_0^1 = \{v \,|\, v \in H^1, v(0) = v(1) = 0\}$. The weak solution to the problem (1)–(2) is a function $u \in H^1$ satisfying $u(0) = g_0, u(1) = g_1$ and

$$a(u, v) = (f, v) \quad \forall\, v \in H_0^1, \tag{7}$$

where $(\cdot, \cdot)$ is the L^2 inner product and $a(u, v) = \varepsilon(u', v') + (bu, v')$. The existence and uniqueness of the weak solution are easy to establish.

The finite element discretization of (7) is to find a $u_N \in V^N$ such that $u_N(0) = g_0, u_N(1) = g_1$ and

$$a(u_N, v_N) = (f, v_N) \quad \forall\, v_N \in V^N \cap H_0^1. \tag{8}$$

Let $e = u_I - u_N = \sum_{i=0}^{N+1} e_i \varphi_i$. Since $a(u - u_N, v_N) = 0$, we obtain the error equation

$$a(e, \varphi_i) = a(u_I - u, \varphi_i), \quad i = 1, 2, \ldots, N, \tag{9}$$

$$e_0 = e_{N+1} = 0. \tag{10}$$

Let $a_{i,j} = a(\varphi_j, \varphi_i)$, $i, j = 1, \ldots, N$ and $h_i = x_i - x_{i-1}$, $i = 1, \ldots, N + 1$. It is easy to get

$$a(e, \varphi_i) = a_{i,i-1} e_{i-1} + a_{i,i} e_i + a_{i,i+1} e_{i+1}, \qquad \text{for } i = 1, 2, \ldots, N,$$

where

$$a_{i,i-1} = -\frac{\varepsilon}{h_i} + \frac{b}{2}, \quad a_{i,i} = \frac{\varepsilon}{h_i} + \frac{\varepsilon}{h_{i+1}}, \quad a_{i,i+1} = -\frac{\varepsilon}{h_{i+1}} - \frac{b}{2},$$

with standard modifications for $i = 1$ and $i = N$. It is well known that if $h = \max_i h_i \le 2\varepsilon/b$, the matrix $A = (a_{i,j})$ will be an M-matrix and thus the scheme satisfies a discrete maximum principle. We are more interested in the case $\varepsilon \ll h$ where in general the discrete maximum principle is not valid. We will solve the error equation directly. This procedure is essentially an LU factorization of a tridiagonal system.

Lemma 1 *The error equation* (9)–(10) *can be written as*

$$(D^N e)_i - (D^N e)_{i+1} = r_i - r_{i+1}, \quad i = 1, 2, \dots, N, \tag{11}$$

$$e_0 = e_{N+1} = 0, \tag{12}$$

where

$$(D^N e)_i = \left(\frac{\varepsilon}{bh_i} + \frac{1}{2}\right) e_i - \left(\frac{\varepsilon}{bh_i} - \frac{1}{2}\right) e_{i-1} \quad \textit{and} \quad r_i = \frac{1}{h_i}\int_{x_{i-1}}^{x_i} (u_I - u)(x)\,\mathrm{d}x.$$

Proof Since $a(\varphi_i, 1) = 0$, we get $a(\varphi_i, \varphi_{i-1} + \varphi_{i+1}) = -a(\varphi_i, \varphi_i)$, namely $a_{i,i} = -a_{i-1,i} - a_{i+1,i}$. Therefore

$$\begin{aligned} a(e, \varphi_i) &= a_{i,i-1}e_{i-1} + a_{i,i}e_i + a_{i,i+1}e_{i+1} \\ &= a_{i,i-1}e_{i-1} - a_{i-1,i}e_i - a_{i+1,i}e_i + a_{i,i+1}e_{i+1} \\ &= b\left[(D^N e)_i - (D^N e)_{i+1}\right]. \end{aligned}$$

On the other hand

$$\int_{x_{i-1}}^{x_i} (u_I - u)'\varphi_i'(x)\,\mathrm{d}x = (u_I - u)|_{x_{i-1}}^{x_i} - \int_{x_{i-1}}^{x_i} (u_I - u)\varphi_i''(x)\,\mathrm{d}x = 0,$$

since $(u_I - u)(x_k) = 0$ for $k = i-1, i$ and $\varphi_i'' = 0$ in $[x_{i-1}, x_i]$. The right hand side of (9) becomes

$$a(u_I - u, \varphi_i) = \int_{x_{i-1}}^{x_{i+1}} b(u_I - u)\varphi_i' = b(r_i - r_{i+1}).$$

The desired result then follows. □

It is easy to see that $(D^N e)_i = r_i - C$ with an appropriate constant C such that $e_0 = e_{N+1} = 0$. However it is difficult to determine C explicitly. Instead we use the following splitting of e_i.

Lemma 2

$$e_i = W_i - \frac{W_{N+1}}{V_{N+1}} V_i, \quad i = 1, 2, \ldots, N,$$

where V_i solves the difference equation

$$(D^N V)_i = 1, \quad i = 1, 2, \ldots, N+1, \ \ V_0 = 0,$$

and W_i solves the difference equation

$$(D^N W)_i = r_i, \quad i = 1, 2, \ldots, N+1, \ \ W_0 = 0.$$

Proof It is clear that $e_i = W_i - C V_i$. Since $e_{N+1} = 0$, we get $C = W_{N+1}/V_{N+1}$. □

Lemma 3 *Let*

$$\lambda_i = \left(\frac{\varepsilon}{bh_i} - \frac{1}{2}\right)\left(\frac{\varepsilon}{bh_i} + \frac{1}{2}\right)^{-1}, \quad S_j^i = \prod_{k=j}^{i} \lambda_k, \ \ i, j = 1, 2, \ldots, N+1,$$

(*with the convention that if $j > i$, $S_j^i = 1$*) *then for $i = 0, 1, \ldots, N+1$,*

$$V_i = 1 - S_1^i,$$

and

$$W_i = \sum_{j=1}^{i} \left[r_j (1 - \lambda_j) S_{j+1}^i\right] = r_i - r_1 S_1^i + \sum_{j=1}^{i-1} \left[(r_j - r_{j+1}) S_{j+1}^i\right].$$

Proof By the definition of W_i, we have

$$\left(\frac{\varepsilon}{bh_i} + \frac{1}{2}\right) W_i - \left(\frac{\varepsilon}{bh_i} - \frac{1}{2}\right) W_{i-1} = r_i, \quad i = 1, 2, \ldots, N+1,$$

and thus

$$W_i = \lambda_i W_{i-1} + (1 - \lambda_i) r_i, \quad i = 1, 2, \ldots, N+1.$$

Here we use the relation $1 - \lambda_i = \left(\frac{\varepsilon}{bh_i} + \frac{1}{2}\right)^{-1}$. Since $W_0 = 0$, we get

$$W_i = \sum_{j=1}^{i} \left[r_j (1 - \lambda_j) S_{j+1}^i\right] = \sum_{j=1}^{i} r_j \left(S_{j+1}^i - S_j^i\right).$$

By the discrete version of integration by parts (summation by parts), we get

$$W_i = r_i - r_1 S_1^i + \sum_{j=1}^{i-1} \Big[(r_j - r_{j+1}) S_{j+1}^i \Big].$$

The formula of V_i can be obtained by replacing $r_i = 1$ in the above identity. □

The following two lemmas concern the stability of the scheme.

Lemma 4 *If $\mathcal{T}_N$ satisfy the condition*

$$|V_{N+1}|^{-1} = \left| 1 - \prod_{i=1}^{N+1} \lambda_i \right|^{-1} \le \rho_{\text{stab}}, \tag{13}$$

then

$$\|e\|_{\infty,\mathcal{T}_N} \le 2(\rho_{\text{stab}} + 1)\|W\|_{\infty,\mathcal{T}_N}.$$

Proof It is easy to see $|\lambda_i| \le 1$ and thus $|V_i| \le 2$. If (13) holds, then from Lemma 2, we can easily get $|e_i| \le 2\rho_{\text{stab}}|W_{N+1}| + |W_i|$, which leads to the lemma. □

Lemma 5 *Let I be an index set and $l(I) := \sum_{i\in I} h_i$. If $\lambda_i \ge 0$, for $i \in I$, then $\mathcal{T}_N$ satisfies condition* (13) *with*

$$\rho_{\text{stab}} = (1 - \mathrm{e}^{-bl(I)/(2\varepsilon)})^{-1}.$$

Proof We note that, if $\lambda_i \ge 0$, then $bh_i \le 2\varepsilon$. Using the simple inequality that $\ln(1-x) \le -x$ for $x \in (0,1)$, we have

$$\begin{aligned}\sum_{i\in I} \ln \lambda_i &= \sum_{i\in I} \ln \left(1 - \frac{2bh_i}{2\varepsilon + bh_i} \right) \le -\sum_{i\in I} \frac{2bh_i}{2\varepsilon + bh_i} \\ &\le -\frac{b\sum_{i\in I} h_i}{2\varepsilon} = -\frac{bl(I)}{2\varepsilon}.\end{aligned}$$

Therefore

$$\left| 1 - \prod_{i=1}^{N+1} \lambda_i \right| \ge 1 - \left| \prod_{i\in I} \lambda_i \right| \ge 1 - \mathrm{e}^{-bl(I)/(2\varepsilon)},$$

as desired. □

When all $\lambda_i \ge 0$, the resulting matrix is an M-matrix. By Lemma 5, it is stable with $\rho_{\text{stab}} \approx 1$. To stabilize the scheme, according to Lemma 5, we only need $l(I) = O(\varepsilon)$. Note that a local grid refinement usually produces such grids. It justifies that the grid adaptation will enhance the stability of the scheme which is often observed in the numerical computation.

2.2 Smooth solutions

In this subsection we will consider the case when the solution to (1)–(2) is smooth and the grid is uniform.

Lemma 6 *If* $\|u^{(k)}\|_{\infty},\ k = 1, 2, 3$ *are uniformly bounded, for a uniform grid* $\mathcal{T}_N$*, we have*:

$$\|r\|_{\infty,\mathcal{T}_N} \leq \mathrm{CN}^{-2} \quad \textit{and} \quad \max_{1\leq i\leq N} |r_i - r_{i+1}| \leq \mathrm{CN}^{-3},$$

and thus

$$\|W\|_{\infty,\mathcal{T}_N} \leq \mathrm{CN}^{-2}.$$

Proof It is easy to see that

$$r_i = \frac{1}{2} h_i^2 u''(\xi_i), \text{ with some } \xi_i \in [x_{i-1}, x_i].$$

Therefore $|r_i| \leq \mathrm{CN}^{-2}$, and

$$|r_i - r_{i+1}| = |h_i^2 u''(\xi_i) - h_{i+1}^2 u''(\xi_{i+1})| \leq N^{-3}\|u^{(3)}\|_{\infty}.$$

For the estimate of $\|W\|_{\infty,\mathcal{T}_N}$, we use the fact that $|S_j^i| \leq 1$ to get

$$|W_i| \leq 2\|r\|_{\infty,\mathcal{T}_N} + \sum_{i=1}^{N} |r_i - r_{i+1}| \leq \mathrm{CN}^{-2}.$$

□

Lemma 7 *For a uniform grid* $\mathcal{T}_N$ *with* N *interior points, we have*

1. *if* N *is even, then*

$$\left|1 - \prod_{i=1}^{N+1} \lambda_i\right|^{-1} \leq C;$$

2. *if* N *is odd, for a fixed* N*, then* $\lim_{\varepsilon\to 0} \left(1 - \prod_{i=1}^{N+1} \lambda_i\right)^{-1} = \infty$.

Proof (1) If $\lambda_i \geq 0$, the stability result follows from Lemma 5. If $\lambda_i < 0$, since N is even,

$$1 - \prod_{i=1}^{N+1} \lambda_i = 1 - (-1)^{N+1} \prod_{i=1}^{N+1} |\lambda_i| = 1 + \prod_{i=1}^{N+1} |\lambda_i| > 1.$$

(2) When N is odd,

$$1 - \prod_{i=1}^{N+1} \lambda_i = 1 - (-1)^{N+1} \prod_{i=1}^{N+1} |\lambda_i| = 1 - \prod_{i=1}^{N+1} |\lambda_i|.$$

Note that $\lim_{\varepsilon \to 0} |\lambda_i| = 1$, we conclude that $\lim_{\varepsilon \to 0} \left(1 - \prod_{i=1}^{N+1} \lambda_i\right)^{-1} = \infty$. □

With Lemma 6 and 7, we immediately get the following result.

Theorem 1 *Let u be the solution of (7) and let u_N be the standard finite element approximation of u based on a uniform grid $\mathcal{T}_N$. Suppose $\|u^{(k)}\|_\infty$, $k = 1, 2, 3$ are uniformly bounded. Then $\|u - u_N\|_\infty \le \mathrm{CN}^{-2}$ if the number of unknowns is even, while the method is not ε-uniformly stable if the number of unknowns is odd.*

Lemma 7 says that for smooth solutions and uniform grids, when ε is small, the stability of the scheme depends on the parity of the grid. An intuitive way to understand this interesting phenomenon is to consider the limiting matrix as ε goes to zero which is for the reduced problem $-bu' = f$. Both the PDE operator and the corresponding matrix are skew symmetric. Thus if the dimension of the matrix is odd, it has a zero eigenvalue. The corresponding eigenvector spans the kernel of the discrete problem which makes the scheme unstable. For this simple example, the eigenvector is $(1, 0, 1, \ldots, 0, 1)$. Indeed, Lenferink [36] eliminates every other unknown to stabilize the scheme.

It is interesting to note that if we modify the grid such that one element has $O(\varepsilon)$ mesh size, the scheme will be stabilized.

Lemma 8 *Let $Pe_i := bh_i/(2\varepsilon)$ be the grid Peclét number. If there exists an element $[x_{k-1}, x_k]$ in the grid $\mathcal{T}_N$ such that*

$$\rho_0 \le Pe_k \le \rho_0^{-1}, \quad \textit{for some } \rho_0 \in (0, 1], \tag{14}$$

then $\mathcal{T}_N$ satisfies condition (13) with $\rho_{\text{stab}} = (1 + \rho_0^{-1})/2$.

Proof Note that the function $|x - 1|/(x + 1)$ is increasing for $x > 1$ and decreaseing for $x \le 1$. With the assumption (14), if $Pe_k > 1$ then

$$|\lambda_k| = \frac{Pe_k - 1}{Pe_k + 1} \le \frac{\rho_0^{-1} - 1}{\rho_0^{-1} + 1} = \frac{1 - \rho_0}{1 + \rho_0}.$$

Otherwise

$$|\lambda_k| = \frac{1 - Pe_k}{Pe_k + 1} \le \frac{1 - \rho_0}{1 + \rho_0}.$$

Therefore

$$\left|1 - \prod_{i=1}^{N+1} \lambda_i\right| > 1 - |\lambda_k| \ge \frac{2\rho_0}{1 + \rho_0},$$

as desired. □

Since the local mesh refinement will generate some grids with the same scaling of ε, in view of Lemma 8 it will stabilize the standard FEM.

Lemma 9 *Let the grid* $\mathcal{T}_N$ *satisfy:* (1) *there exists an* h_k *satisfying* (14) *for some* $1 \le k \le N+1$ *and* (2) $h_i = \mathrm{CN}^{-1}$ *for* $i = 1, \ldots, N+1, i \ne k$. *Then*

$$\|W\|_{\infty,\mathcal{T}_N} \le \mathrm{CN}^{-2}.$$

Proof By the proof of Lemma 6, it is easy to see

$$\|r\|_{\infty,\mathcal{T}_N} \le \mathrm{CN}^{-2} \quad \text{and} \quad |r_i - r_{i+1}| \le \mathrm{CN}^{-3} \quad \text{for } i \ne k-1, k.$$

Therefore

$$\begin{aligned} |W_i| &\le 2\|r\|_{\infty,\mathcal{T}_N} + |r_{k-1} - r_k| + |r_k - r_{k+1}| + \sum_{i=0, i\ne k-1,k}^{N} |r_i - r_{i+1}| \\ &\le 6\|r\|_{\infty,\mathcal{T}_N} + \sum_{i=0, i\ne k-1,k}^{N} |r_i - r_{i+1}| \le \mathrm{CN}^{-2}. \end{aligned}$$

□

Combining Lemmas 8 and 9, we get the following result.

Theorem 2 *Let the grid* $\mathcal{T}_N$ *satisfy conditions in Lemma* 9 (*namely it is uniform except for one element that has size of* $\mathcal{O}(\varepsilon)$). *If* $\|u^{(k)}\|_\infty$, $k = 1, 2, 3$ *are uniformly bounded, then the standard FEM approximation* u_N *based on* $\mathcal{T}_N$ *satisfy*

$$\|u - u_N\|_\infty \le \mathrm{CN}^{-2}.$$

In the proof of Lemma 9, we use the uniformity of the grid to bound the summation of $|r_i - r_{i+1}|$ with few exceptions. We will show the uniformity of the grid is crucial for the second order convergence by the following example.

Example 1 There exist a sequence of quasiuniform grids $\{\mathcal{T}_N\}$ such that the standard finite element approximation u_N to the following equation:

$$-\varepsilon u''(x) - u'(x) = -2\varepsilon - 2x, \quad x \in (0, 1), \tag{15}$$

$$u(0) = 0, \quad u(1) = 1, \tag{16}$$

is only first order provided ε is small enough.

The real solution of (15)–(16) is $u = x^2$. Let N be an odd integer and $\mathcal{T}_N$ be the uniform grid with equal size h. We modify

$$x_{2i+1} = (i + 0.25)h, \quad i = 0, 1, \ldots, \frac{N+1}{2} - 1.$$

In this case

$$r_i - r_{i+1} = (-1)^i h^2, \quad i = 1, \ldots, N.$$

Let us choose small ε such that $\lambda_i < 0$. Note that $S_i^{N+1} = (-1)^i |S_i^{N+1}|$ also oscillates, we have

$$(r_i - r_{i+1}) S_i^{N+1} = h^2 \left| S_i^{N+1} \right| > h^2 q_\varepsilon^{N-i+1},$$

where $q_\varepsilon = (5 - 8\varepsilon N)(5 + 8\varepsilon N)^{-1}$. Therefore

$$|W_{N+1}| \geq h^2 \left(\sum_{i=1}^{N+1} |S_i^{N+1}| - 1 \right) \geq h^2 \left(\frac{1 - q_\varepsilon^{N+2}}{1 - q_\varepsilon} - 1 \right).$$

Since $\lim_{\varepsilon \to 0} (1 - q^N)(1 - q)^{-1} = N$, we may choose ε small enough such that $|W_{N+1}| \geq (N+1)h^2$. Note that $W_1 = (1 - \lambda_1) h_1^2 \geq 5/2h^2$, we have

$$\|u_I - u_N\|_\infty \geq |e_1| \geq |W_{N+1}| - |W_1| \geq (N - 4)h^2 \geq \mathrm{CN}^{-1}.$$

For smooth functions, we know that, on a quasi-uniform grid, the interpolation error is still of second order, namely $\|u - u_I\| \leq \mathrm{CN}^{-2}$. The optimal convergence rate we would like to expect for the numerical solution is also of second order. Example 1 tells us for singularly perturbed problems, when $\varepsilon \ll 1$, we may lose one order of accuracy for the standard FEM if we perturb the uniform grid to be a quasi-uniform one. In other words, the standard FEM is not stable with respect to the perturbation of the grid.

2.3 Solutions with boundary layers

In this subsection, we will consider the solutions with boundary layers and prove the convergence of the standard FEM on two special layer adapted grids: Bakhvalov grid and Shishkin grid. The convergence result is known; see for example [33]. However we will show the accuracy of the standard FEM depends crucially on the uniformality of the grid in the smooth part.

More speficially we will consider equation (1)–(2) with homogeneous boundary condition and smooth source term f. Namely

$$-\varepsilon u'' - bu' = f \quad \text{in } (0, 1), \tag{17}$$

$$u(0) = u(1) = 0. \tag{18}$$

The solution to (17)–(18) typically has a boundary layer near $x = 0$. The following a priori bound of the derivatives of the solution is well known in the literature, see for example [32,42,47] or [41].

Lemma 10 *Let u be the solution to equation* (17)–(18), *we have the following a priori bound*:

$$|u^{(k)}(x)| \le C(1+\varepsilon^{-k}\mathrm{e}^{-bx/\varepsilon}) \quad \forall x \in [0,1], \;\; k=0,1,2,3.$$

Layer-adapted grids are needed to capture the boundary layer. The first such grid is Bakhvalov grid [1] . Let N be an even integer and

$$\theta = \min\left\{\frac{1}{2}, \frac{2\varepsilon \ln \varepsilon^{-1}}{b}\right\}.$$

In $[0,\theta]$ we put $N/2$ elements such that

$$\int\limits_{x_{i-1}}^{x_i} \varepsilon^{-1}\mathrm{e}^{-bx/(2\varepsilon)}\,\mathrm{d}x = \frac{2}{N}\int\limits_{0}^{\theta} \varepsilon^{-1}\mathrm{e}^{-bx/(2\varepsilon)}\,\mathrm{d}x,$$

namely

$$x_i = -\frac{2\varepsilon}{b}\ln\left(1-2(1-\varepsilon)\frac{i}{N}\right), \quad i=0,1,\ldots,N/2.$$

In $[\theta,1]$, we put $N/2$ uniform grids and denote the mesh size $H = 2(1-\theta)/N \le \mathrm{CN}^{-1}$.

The interpolation error estimate on Bakhvalov grid is known [38] which can be also derived from Theorem 7 in Sect. 3.2.

Lemma 11 *Let u be the solution to* (17)–(18). *For Bakhvalov grid,*

$$\|u-u_I\|_{L^\infty} \le \mathrm{CN}^{-2}.$$

To prove the convergence, we need the following technical lemma.

Lemma 12 *For $j<i$, let $I=\{j,j+1,\ldots,i\}$ and $h_I = \max_{k\in I} h_k$. We have*

1.
$$|W_i| \le |W_j| + |r_i| + |r_j| + \sum_{k\in I\setminus\{i\}} |r_k - r_{k+1}|, \tag{19}$$

2. *If $\lambda_k \ge 0$, $k \in I$, then*
$$|W_i| \le |W_j| + 2\max_{k\in I\setminus\{j\}} |r_k|. \tag{20}$$

Proof By Lemma 3, we get

$$\begin{aligned} W_i &= S^i_{j+1}W_j + \sum_{k=j+1}^{i} r_k\left(S^i_{k+1} - S^i_k\right) \\ &= S^i_{j+1}W_j + r_i - r_j S^i_j + \sum_{k=j}^{i-1}\left[(r_k - r_{k+1})S^i_{k+1}\right]. \end{aligned}$$

Hence (19) follows from the fact $|S^i_j| \le 1$.

If $\lambda_k \geq 0,\ k \in I$, then S_k^i is monotone decreasing with respect to k, and thus

$$|W_i| \leq |W_j| + \max_{j+1\leq k\leq i} |r_k| \left| \sum_{k=j}^{i-1} \left(S_k^i - S_{k+1}^i \right) \right| \leq |W_j| + 2 \max_{k\in I\setminus\{j\}} |r_k|.$$

□

Theorem 3 *For Bakhvalov grid, the standard FEM approximation is uniformly optimal. Namely*

$$\|u - u_N\|_\infty \leq \mathrm{CN}^{-2}.$$

Proof We divide $\mathcal{T}_N$ into boundary layer $I_1 = \{0, 1, \ldots, N/2\}$ and smooth part $I_2 = \{N/2 + 1, \ldots, N + 1\}$ by the transition point θ. In I_1, it is easy to see $h_i/\varepsilon = \mathrm{CN}^{-1}$ by the mean value theorem. Thus $\lambda_i > 0, i \in I_1$ for sufficient large N (independent of ε). Since $l(I_1) = \theta$, the stability of the scheme follows from Lemma 5.

With $W_0 = 0$ and (20) we have

$$\|W\|_{\infty,I_1} \leq \|r\|_{\infty,I_1} \leq C\|u - u_I\|_\infty \leq \mathrm{CN}^{-2}. \tag{21}$$

In the smooth part $[\theta, 1]$

$$\begin{aligned}\|W\|_{\infty,I_2} &\leq |W_{N/2+1}| + |r_{N/2+1}| + |r_{N+1}| + \sum_{i\in I_2\setminus\{N+1\}} |r_i - r_{i+1}| \\ &\leq |W_{N/2}| + C\|r\|_{\infty,\mathcal{T}_N} + \sum_{i\in I_2} C\|u^{(3)}\|_{\infty,[x_{i-1},x_{i+1}]} H^3.\end{aligned}$$

By (21) the first two terms are bounded by CN^{-2}. We now estimate the third term

$$\begin{aligned}\sum_{i\in I_2} \|u^{(3)}\|_{\infty,[x_{i-1},x_{i+1}]} H^3 &\leq H^2 \left(\sum_{i\in I_2} (1 + \varepsilon^{-3} \mathrm{e}^{-bx/\varepsilon}) H \right) \\ &\leq N^{-2} \int_\theta^1 (1 + \varepsilon^{-3} \mathrm{e}^{-bx/\varepsilon})\, \mathrm{d}x \\ &\leq \mathrm{CN}^{-2}.\end{aligned}$$

Combining those estimates together, we get

$$\|u_I - u_N\|_\infty \leq C\|W\|_{\infty,\mathcal{T}_N} \leq \mathrm{CN}^{-2}.$$

The result then follows from the triangle inequality

$$\|u - u_N\|_\infty \leq \|u - u_I\|_\infty + \|u_I - u_N\|_\infty,$$

and the interpolation error estimate for $\|u - u_I\|_\infty$. □

Another simple layer adapted grid is Shishkin grid [52]. Let N be an even integer and the transition point

$$\theta = \min\left\{\frac{1}{2}, \frac{2\varepsilon \ln N}{b}\right\}.$$

In practice, ε is so small that $\theta = 2b^{-1}\varepsilon \ln N$. Then $[0, \theta]$ and $[\theta, 1]$ are divided into $N/2$ equidistant subintervals. The following interpolation error estimate for Shishkin grid is well known [37,38,41,47].

Lemma 13 *Let u be the solution to* (17)–(18). *For Shishkin grid,*

$$\|u - u_I\|_{L^\infty(x_{i-1},x_i)} \leq \begin{cases} \mathrm{CN}^{-2}\ln^2 N, & i = 1, 2, \ldots, N/2, \\ \mathrm{CN}^{-2}, & i = N/2 + 1, \ldots, N + 1. \end{cases}$$

Similar to the proof of Theorem 3, we can get the convergence of the standard FEM approximation on Shishkin grid.

Theorem 4 *For Shishkin grid, the standard FEM approximation u_N is an almost seconder order approximation*

$$\|u - u_N\|_\infty \leq \mathrm{CN}^{-2}\ln^2 N,$$

From the proof of Theorem 3, we see inside the boundary layer, we use $|r_i| \leq \|u - u_I\|_{\infty,[x_i,x_{i+1}]}$ and only need to bound the interpolation error. Therefore the grid inside the boundary layer can be relaxed to be quasiuniform. While in the smooth part, we need the uniformity of the grid to ensure $|r_i - r_{i+1}| \leq \mathrm{CN}^{-3}$. Actually, the convergence rate highly depends on the uniformality of the grid in the smooth part.

We can construct an example similar to Example 1 to show that when the smooth part of layer adapted grids is only quasiuniform, the convergence rate will degrade to the first order for small ε.

Example 2 There exist a sequence of Bakhvalov type grids $\{\mathcal{T}_N\}$ such that the standard FEM approximation u_N to the following equation:

$$-\varepsilon u''(x) - u'(x) = -2\varepsilon - 2x, \quad x \in (0, 1), \tag{22}$$

$$u(0) = 1, \quad u(1) = 1, \tag{23}$$

is only of first order provided ε is small enough. Namely

$$\|u - u_N\|_\infty \geq \mathrm{CN}^{-1}, \quad \text{but } \|u - u_I\|_\infty \leq \mathrm{CN}^{-2}.$$

The real solution to (22)–(23) is

$$u = \frac{e^{-x/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}} + x^2,$$

which contains a boundary layer near $x = 0$. We modify the Bakhvalov grid in the smooth part by moving all odd grid points (except x_{N+1}) with a right offset $h/4$. In this case, $r_i - r_{i+1} = (-1)^i N^{-2}$, $i > N/2+1$ and for small ε, the error in the smooth part will accumulate to N^{-1}. The proof is the same as that in Example 1.

Note that for such a modified Bakhvalov grid, the interpolation errror is still of second order, namely $\|u - u_I\|_\infty \le \mathrm{CN}^{-2}$. Example 2 implies that the accuracy of the adapted approximation of standard FEM is very sensitive to the perturbation of grid points in the region where the solution is smooth.

3 A new SDFEM and error analysis

In this section, we will propose a new SDFEM based on a special choice of the stabilization bubble functions and prove that the new method produces a nearly optimal approximation.

3.1 The new SDFEM and its uniform stability

To introduce the SDFEM, we first modify our bilinear form to be

$$\tilde{a}(u, v) := a(u, v) - \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \delta_i(-\varepsilon u'' - bu')bv',$$

where δ_i is a stabilization function in $[x_{i-1}, x_i]$. We will discuss the choice of δ_i in a moment. For the exact solution u of (1)–(2), it satisfies

$$\tilde{a}(u, v) = \tilde{f}(v) \quad \forall v \in H_0^1,$$

where $\tilde{f}(v) = (f, v) - \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} \delta_k f bv'$. The SDFEM is to find $\tilde{u}_N \in V^N$ such that $\tilde{u}_N(0) = g_0$, $\tilde{u}_N(1) = g_1$ and

$$\tilde{a}(u_N, v_N) = \tilde{f}(v_N) \quad \forall v_N \in V^N \cap H_0^1. \tag{24}$$

In the traditional SDFEM, δ_i is chosen to be a proper constant, such as h_i, on each interval $[x_{i-1}, x_i]$. The key in the new SDFEM is that δ_i is chosen to be a bubble function on each $[x_{i-1}, x_i]$ defined as follows

$$\delta_i = \min\left\{\frac{h_i}{2\varepsilon}, \frac{1}{b}\right\} h_i(\varphi_i\varphi_{i-1})(x). \tag{25}$$

Recall that φ_i is the nodal basis function at point x_i, so δ_i is a quadratic bubble function with scale h_i. It is interesting to note that the bubble function is used in some general stabilized method such as residual-free bubble finite element method [5,22] and multiscale variational methods [27,29].

With such a special choice of δ_i, we have the most desirable stability estimate stated in the following theorem.

Theorem 5 *Let $\tilde{u}_N$ be the SDFEM approximation to equation (1)–(2) with stabilization function δ_i determined by (25) and let $V_D^N := \{v_N \in V^N, v_N(0) = g_0, v_N(1) = g_1\}$, we have*

$$\|u - \tilde{u}_N\|_\infty \le C \inf_{v_N \in V_D^N} \|u - v_N\|_\infty.$$

The rest of this section is devoted to the proof of Theorem 5. Similar to the standard FEM, we write the error equation for $e = u_I - \tilde{u}_N$ as:

$$\tilde{a}(e, \varphi_i) = \tilde{a}(u_I - u, \varphi_i), \quad i = 1, 2, \ldots, N \tag{26}$$

$$e_0 = e_{N+1} = 0. \tag{27}$$

Lemma 14 *The error equation (26) can be written as*

$$(\tilde{D}^N e)_i - (\tilde{D}^N e)_{i+1} = \tilde{r}_i - \tilde{r}_{i+1}, \quad i = 1, 2, \ldots, N$$

$$e_0 = e_{N+1} = 0,$$

where

$$(\tilde{D}^N e)_i = \left(\frac{\varepsilon + b^2\bar{\delta}_i}{bh_i} + \frac{1}{2}\right) e_i - \left(\frac{\varepsilon + b^2\bar{\delta}_i}{bh_i} - \frac{1}{2}\right) e_{i-1},$$

$$\bar{\delta}_i = \frac{1}{h_i}\int_{x_{i-1}}^{x_i} \delta_i(x)\,\mathrm{d}x,$$

and

$$\tilde{r}_i = \frac{1}{h_i}\left[\int_{x_{i-1}}^{x_i} (u_I - u)(x)\,\mathrm{d}x \right. \tag{28}$$

$$+ \int_{x_{i-1}}^{x_i} \delta_i \varepsilon u''\,\mathrm{d}x \tag{29}$$

$$\left. - \int_{x_{i-1}}^{x_i} b\delta_i (u_I - u)'\,\mathrm{d}x\right]. \tag{30}$$

Let $\tilde{\varepsilon}_i = \varepsilon + b^2\bar{\delta}_i$,

$$\tilde{\lambda}_i = \left(\frac{\tilde{\varepsilon}_i}{bh_i} - \frac{1}{2}\right)\left(\frac{\tilde{\varepsilon}_i}{bh_i} + \frac{1}{2}\right)^{-1},$$

and $\tilde{S}^i_j$, $\tilde{W}_i$, $\tilde{V}_i$ be defined similarly. We can follow the same lines in Sect. 2 to solve the error equation and get similar results in the last section.

The following lemma is crucial to obtain our main theorem.

Lemma 15 *For δ_i given by (25), we have*

$$|\tilde{r}_i| \le \frac{C}{h_i}\int_{x_{i-1}}^{x_i} |u - u_I|\,\mathrm{d}x, \quad \textit{and thus } \|\tilde{r}\|_{\infty,\mathcal{T}_N} \le \|u - u_I\|_\infty. \tag{31}$$

Proof We will prove (31) by estimating the three terms (28)–(30) respectively. The proof for (28) is trivial. For (29), we have

$$\begin{aligned}
\left|\frac{1}{h_i}\int_{x_{i-1}}^{x_i} \varepsilon\delta_i u''\,\mathrm{d}x\right| &\le \left|\int_{x_{i-1}}^{x_i} \min\left\{\frac{1}{2}, \frac{\varepsilon}{bh_i}\right\} h_i(\varphi_i\varphi_{i-1})u''\,\mathrm{d}x\right| \\
&\le \left|\frac{1}{2h_i}\int_{x_{i-1}}^{x_i} (x - x_{i-1})(x - x_i)u''(x)\,\mathrm{d}x\right| \\
&\le \left|\frac{1}{2h_i}\int_{x_{i-1}}^{x_i} (x - x_{i-1})(x - x_i)(u - u_I)''(x)\,\mathrm{d}x\right| \\
&= \left|\frac{1}{h_i}\int_{x_{i-1}}^{x_i} (u_I - u)(x)\,\mathrm{d}x\right|.
\end{aligned}$$

The last step follows from integration by parts twice.

For (30), we have

$$\begin{aligned}
\left|\frac{1}{h_i}\int_{x_{i-1}}^{x_i} b\delta_i(u_I - u)'\,\mathrm{d}x\right| &= \left|\frac{1}{h_i}\int_{x_{i-1}}^{x_i} b\delta_i'(u_I - u)\,\mathrm{d}x\right| \\
&\le \left|\int_{x_{i-1}}^{x_i} (u_I - u)(\varphi_{i-1}\varphi_i)'\,\mathrm{d}x\right| \\
&\le \frac{1}{h_i}\int_{x_{i-1}}^{x_i} |u_I - u|\,\mathrm{d}x.
\end{aligned}$$

□

Theorem 6 *For the SDFEM with δ_i determined by (25), we have*

$$\|u_I - \tilde{u}_N\|_\infty \leq C\|u - u_I\|_\infty,$$

and thus

$$\|u - \tilde{u}_N\|_\infty \leq C\|u - u_I\|_\infty.$$

Proof If $\dfrac{h_i}{2\varepsilon} < \dfrac{1}{b}$, then

$$\bar{\delta}_i = \frac{h_i^2}{4\varepsilon} \quad \text{and} \quad \frac{\varepsilon + b^2\bar{\delta}_i}{h_i} = \frac{\varepsilon}{h_i} + \frac{b^2 h_i}{4\varepsilon} \geq \frac{b}{2}.$$

Otherwise

$$\bar{\delta}_i = \frac{h_i}{2b} \quad \text{and} \quad \frac{\varepsilon + b^2\bar{\delta}_i}{h_i} > \frac{b^2\bar{\delta}_i}{h_i} \geq \frac{b}{2}.$$

Thus $\tilde{\lambda}_i \geq 0$, for all $i = 1, \ldots, N+1$. By Lemma 4, we have $\|u_I - \tilde{u}_N\|_\infty \leq C\|\tilde{W}\|_{\infty,\mathcal{T}_N}$ and by Lemma 12 we have $\|\tilde{W}\|_{\infty,\mathcal{T}_N} \leq C\|\tilde{r}\|_{\infty,\mathcal{T}_N}$.

Now using Lemma 15, we have:

$$\|u_I - \tilde{u}_N\|_\infty \leq C\|\tilde{r_i}\|_{\infty,\mathcal{T}_N} \leq C\|u - u_I\|_\infty.$$

The second inequality in the theorem is obtained by the triangle inequality. □

We are now in a position to prove the main theorem in this section.

Proof of Theorem 5 Let us first consider the case $g_0 = g_1 = 0$. We denote the corresponding finite element space by V_0^N. We define the projection operator

$$P_N : H_0^1 \to V_0^N \text{ by } P_N u = \tilde{u}_N.$$

By Theorem 6,

$$\|u - \tilde{u}_N\|_\infty \leq C\|u - u_I\|_\infty \leq C(\|u\|_\infty + \|u_I\|_\infty) \leq C\|u\|_\infty.$$

Thus

$$\|P_N u\|_\infty = \|\tilde{u}_N\|_\infty \leq \|u\|_\infty + \|u - \tilde{u}_N\|_\infty \leq C\|u\|_\infty.$$

With the property $P_N^2 = P_N$, for any $v_N \in V_0^N$, we have

$$\|u - \tilde{u}_N\|_\infty = \|(I - P_N)(u - v_N)\|_\infty \leq C\|u - v_N\|_\infty.$$

Since it is true for any $v_N \in V_0^N$, the optimality result for homogeneous boundary condition is then obtained.

For general boundary conditions, we define $u_N^* = (g_1 - g_0)x + g_0$ which belongs to V_D^N. Note that $u - u_N^* \in H_0^1$ solving the following equation

$$-\varepsilon v'' - bv' = f + bu_N^* \ \text{ in } (0,1), \quad v(0) = 0, v(1) = 0.$$

Thus $P_N(u - u_N^*)$ is well defined. On the other hand $\tilde{u}_N - u_N^*$ is also a SDFEM approximation of the above equation. By the uniqueness, we have $\tilde{u}_N - u_N^* = P_N(u - u_N^*)$. Therefore

$$\begin{aligned} \|u - \tilde{u}_N\|_\infty &= \|(u - u_N^*) - P_N(u - u_N^*)\|_\infty \\ &\le C \inf_{v_N \in V_0^N} \|u - u_N^* - v_N\|_\infty \\ &= C \inf_{v_N \in V_D^N} \|u - v_N\|_\infty. \end{aligned}$$

□

3.2 The convergence of the new SDFEM

We first discuss how to adapt the grid to get optimal interpolation error estimates. Given a function $u \in C^2[0,1]$, a positive function $H(x)$ is called a majorant of the second order derivative of u, if $|u''(x)| \le H(x),\ x \in (0,1)$. For an element $\tau_i = [x_{i-1}, x_i]$, its length in the metric H are denoted by $|\tau_i|_H$, namely

$$|\tau_i|_H = \int_{x_{i-1}}^{x_i} H^{1/2}(x)\,\mathrm{d}x.$$

We need two basic assumptions to get a nearly optimal interpolation error estimate.

- **(A1)** H is monotone in each element τ_i, $\quad i = 1, 2, \ldots, N+1$.
- **(A2)** $|\tau_i|_H$ is nearly equidistributed in the sense that

$$\max_{1 \le i \le N+1} |\tau_i|_H \le \frac{C}{N} \sum_{i=1}^{N+1} |\tau_i|_H.$$

Theorem 7 [17,19] *Let* $u \in C^2[0,1]$ *and the mesh* $\mathcal{T}_N$ *satisfy assumptions* (A1) *and* (A2), *the following error estimate holds*:

$$\|u - u_I\|_\infty \le C\|H\|_{1/2} N^{-2}, \tag{32}$$

where

$$\|H\|_{L^{1/2}} := \left(\int_0^1 H^{1/2}\,\mathrm{d}x \right)^2.$$

Remark 1 This error estimate is optimal in the sense that for a strictly convex (or concave) function, the above inequality holds asymptotically in a reversed direction with $H = |u''|$.

In our recent work [11], we have developed a general interpolation error estimate in any spatial dimension and for general L^p norms. In high dimensions, the new metric is given by a scaling of the majorant of the Hessian matrix H. The monotonicity in the assumption (A1) is replaced by no oscillation of H in each element. For details and applications, we refer to our recent work [8–12]. For other related works, we refer to [2,16,20,23,25,26,43].

The assumption (A2) can be used to direct our construction of the nearly optimal mesh. In the context of the so-called moving mesh method [6,24,26], it can be done by the equidistribution of a monitor function. A monitor function $M = M(u, u', u'', \ldots)$ is a function involving u and its derivatives. We say that the grid $\mathcal{T}_N$ nearly equidistributes the monitor function M if

$$\int_{x_i}^{x_{i+1}} M \, \mathrm{d}x \le \frac{C}{N} \int_0^1 M \, \mathrm{d}x, \quad i = 0, 1, 2, \ldots, N.$$

Based on the interpolation error estimates, an optimal monitor function for linear interpolant is $M = H^{1/2}$.

Theorem 8 *Let $\tilde{u}_N$ be the SDFEM approximation to the solution to* (1)–(2) *on a grid obtained by nearly equidistributing a monotone majorant H of the second derivative of u and δ_i is determined by* (25), *then*

$$\|u - \tilde{u}_N\|_\infty \le C \|H\|_{1/2} N^{-2}.$$

The convergence of the new SDFEM on different interesting cases can be obtained as corollaries of Theorem 8 and the interpolation error estimates. The convergence rate is not sensitive to the perturbation of the grid since it is controlled by the interpolation error. For equation (17)–(18), the convergence of the new SDFEM on different layer adapted grids is straightforward.

Corollary 1 *The SDFEM approximation $\tilde{u}_N$ with δ_i determined by* (25) *to* (17)–(18) *on the grid obtained by the nearly equidistribution of monitor function $M = \sqrt{1 + \varepsilon^{-2} \mathrm{e}^{-bx/\varepsilon}}$ satisfies*

$$\|u - \tilde{u}_N\|_\infty \le \mathrm{CN}^{-2}.$$

Corollary 2 *The SDFEM approximation $\tilde{u}_N$ with δ_i determined by* (25) *to* (17)–(18) *on Bakhvalov grid satisfies*

$$\|u - \tilde{u}_N\|_\infty \le \mathrm{CN}^{-2}.$$

Corollary 3 *The SDFEM approximation $\tilde{u}_N$ with δ_i determined by* (25) *to* (17)–(18) *on Shishkin grid satisfies*

$$\|u - \tilde{u}_N\|_\infty \le \text{CN}^{-2} \ln^2 N.$$

We would like to emphasis again that in the proof of the uniform optimality of the new SDFEM, we do not make use of the a priori information about $|u''|$ and the structure of the grid. The ε-uniform stability result Theorem 8 can be applied for non-smooth data f also.

To show the convergence, all we need to do is to adapt the grid to get a good interpolant. It could be done by a posterior error estimate [34] or a priori estimate of u''. For example, let us consider the following equation as studied in [48].

$$-\varepsilon u'' - bu' = f + \delta(\cdot - d) \;\text{ on } \Omega^- \cup \Omega^+, \tag{33}$$

$$u(0) = u(1) = 0, \tag{34}$$

where $\Omega = (0, 1), d \in \Omega, \Omega^- = (0, d), \Omega^+ = (d, 1)$ and $\delta(\cdot - d)$ denotes the Dirac-delta function at point d. Function f is sufficiently smooth on $\bar{\Omega}$. The equation (33)–(34) should be understood in the distribution sense and it is well known that it has a unique solution $u \in H_0^1(\Omega)$ which has an exponential interior layer at $x = d$ and boundary layer at $x = 0$. Furthermore, the following a priori estimate of the second derivative can be found at [48]:

$$|u^{(k)}(x)| \le C(1 + \varepsilon^{-k} \mathrm{e}^{-bx/\varepsilon}), \quad x \in \Omega^-, \; k = 0, 1, 2, 3, \text{ and}$$
$$|u^{(k)}(x)| \le C(1 + \varepsilon^{-k} \mathrm{e}^{-b(x-d)/\varepsilon}), \quad x \in \Omega^+, \; k = 0, 1, 2, 3.$$

With this information we can construct the corresponding layer-adapted grid to get an optimal interpolant u_I and thus obtain the optimal convergence of the new SDFEM. This example illustrates the usefulness of the near optimality of the SDFEM (Theorem 6) considering the fact that results for singularly perturbed problems with discontinuous right-hand side are relatively rare [21,48].

4 Concluding remarks

In this paper, we have shown the stabilization effect of the adaptive grid for the standard finite element method and developed an optimal streamline diffusion finite element method. The main results are listed below.

1. We found that the uniformity of the grid in the smooth part of the solution plays a crucial role for the optimality of the approximation.
2. In contrast, the new streamline diffusion finite element method that we developed inherits a quasi-optimal approximation property which is uniform with respect to ε.
3. With the optimal interpolation error estimate, we have answered an open question about the optimal choice of the monitor function for the singularly perturbed problem.

The above results raise many interesting questions for singularly perturbed problems in multiple dimensions. It is natural to expect that similar results should be still valid, but a rigorous theoretical analysis are still lacking and further research is still required. But at least our one dimensional results should provide some guidance to adaptive finite elements in high dimensions. For example, it is easy to construct a grid which is quasiuniform in the smooth part such that the convergence rate is deteriorated. Since for general domains in high dimensions, it is not easy to get uniform grids, the stabilization of the standard FEM is needed.

Acknowledgments The authors wish to express their sincere thanks to Professors Douglas N. Arnold, Franco Brezzi and Endre Süli for helpful discussions.

References

1. Bakhalov, N.S.: Towards optimization of methods for solving boundary value problems in the presence of boundary layers (in Russian). Zh. Vychisl. Mater. Mater. Fiz. **9**, 841–859 (1969)
2. Borouchaki, H., Castro-Diaz, M.J., George, P.L., Hecht, F., Mohammadi, B.: Anisotropic adaptive mesh generation in two dimensions for CFD. In: 5th International Conference On Numerical Grid Generation in Computational Field Simulations, vol. 3, pp. 197–206. Mississppi State University (1996)
3. Brezzi, F., Hughes, T.J.R., Marini, L.D., Russo, A., Süli, E.: A priori error analysis of residual-free bubbles for advection-diffusion problems. SIAM J. Numer. Anal. **36**(4), 1933–1948 (1999)
4. Brezzi, F., Marini, D., Süli, E.: Residual-free bubbles for advection-diffusion problems: the general error analysis. Numer. Math. **85**, 31–47 (2000)
5. Brezzi, F., Russo, A.: Choosing bubbles for advection-diffusion problems. Math. Models Methods Appl. Sci. **4**, 571–587 (1994)
6. Cao, W., Huang, W., Russell, R.D.: A study of monitor functions for two dimensional adaptive mesh generation. SIAM J. Sci. Comput. **20**, 1978–1994 (1999)
7. Carey, G.F., Dinh, H.T.: Grading functions and mesh redistribution. SIAM J. Numer. Anal. **22**(5), 1028–1040 (1985)
8. Chen, L.: Mesh smoothing schemes based on optimal Delaunay triangulations. In: 13th International Meshing Roundtable, pp. 109–120. Sandia National Laboratories, Williamsburg (2004)
9. Chen, L.: New analysis of the sphere covering problems and optimal polytope approximation of convex bodies. J. Approx. Theory **133**(1), 134–145 (2005)
10. Chen, L., Sun, P., Xu, J.: Multilevel homotopic adaptive finite element methods for convection dominated problems. In: The Proceedings for 15th Conferences for Domain Decomposition Methods. Lecture Notes in Computational Science and Engineering 40, pp. 459–468. Springer, Heidelberg (2004)
11. Chen, L., Sun, P., Xu, J.: Optimal anisotropic simplicial meshes for minimizing interpolation errors in L^p-norm. Math. Comput. **76**, 179–204 (2007)
12. Chen, L., Xu, J.: Optimal Delaunay triangulations. J. Comput. Math. **22**(2), 299–308 (2004)
13. Chen, L., Xu, J.: An optimal streamline diffusion finite element method for a singularly perturbed problem. In: AMS Contemporary Mathematics Series: Recent Advances in Adaptive Computation, vol. 383, pp. 236–246, Hangzhou (2005)
14. Chen, Y.: Uniform pointwise convergence for a singularly perturbed problem using arc-length equidistribution. In: Proceedings of the 6th japan-china joint seminar on numerical mathematics (tsukuba, 2002). J. Comput. Appl. Math. **159**(1), 25–34 (2003)
15. Chen, Y.: Uniform convergence analysis of finite difference approximations for singular perturbation problems on an adapted grid. Adv. Comput. Math. **24**, 197–212 (2006)
16. D'Azevedo, E.F., Simpson, R.B.: On optimal interpolation triangle incidences. SIAM J. Sci. Statist. Comput. **6**, 1063–1075 (1989)
17. de Boor, C.: Good approximation by splines with variable knots. Int. Seines Numer. Math, Birkhauser Verlag, Basel **21**, 57–72 (1973)

18. de Boor, C.: Good approximation by splines with variables knots II. In: Watson, G.A. (ed.) Proceedings of the Eleventh International Conference on Numerical Methods in Fluid Dynamics, vol. 363, pp. 12–20. Springer, Dundee (1974)
19. Devore, R.A., Lorentz, G.G.: Constructive Approximation. Springer, New York (1993)
20. Dolejšì, V., Felcman, J.: Anisotropic mesh adaptation for numerical solution of boundary value problems. Numer. Methods Partial Differ. Equ. **20**, 576–608 (2004)
21. Farrell, P.A., Hegarty, A.F., Miller, J.J.H., O'Riordan, E., Shishkin, G.I.: Singularly perturbed convection-diffusion problems with boundary and weak interior layers. J. Comput. Appl. Math. **166**, 131–151 (2004)
22. Franca, L.P., Russo, A.: Deriving upwinding, mass lumping and selective reduced integration by residual-free bubbles. Appl. Math. Lett. **9**, 83–88 (1996)
23. Habashi, W.G., Fortin, M., Dompierre, J., Vallet, M.G., Ait-Ali-Yahia, D., Bourgault, Y., Robichaud, M.P., Tam, A., Boivin, S.: Anisotropic mesh optimization for structured and unstructured meshes. In: 28th Computational Fluid Dynamics Lecture Series. von Karman Institute (1997)
24. Huang, W.: Practical aspects of formulation and solution of moving mesh partial differential equations. J. Comput. Phys. **171**, 753–775 (2001)
25. Huang, W.: Variational mesh adaptation: isotropy and equidistribution. J. Comput. Phys. **174**, 903–924 (2001)
26. Huang, W., Sun, W.: Variational mesh adaptation. II: Error estimates and monitor functions. J. Comput. Phys. **184**, 619–648 (2003)
27. Hughes, T.J.R.: Multiscale phenomena: Green's functions, the dirichlet-to-neumann formulation, subgrid scale models, bubbles and the origins of stabilized methods. Comput. Methods Appl. Mech. Eng, pp. 127, no. 1–4, 387–401 (1995)
28. Hughes, T.J.R., Brooks, A.: A multidimensional upwind scheme with no crosswind diffusion. In: Hughes, T.J.R. (ed.) Finite Element Methods for Convection Dominated Flows, AMD, vol. 34, pp. 19–35. ASME, New York (1979)
29. Hughes, T.J.R., Feijoo, G., Mazzei, L., Quincy, J.B.: The variational multiscale method—a paradigm for computational mechanics. Comput. Methods Appl. Mech. Eng. **166**, 3–24 (1998)
30. Johnson, C., Nvert, U.: An analysis of some finite element methods for advection-diffusion problems. In: Axelsson, O., Frank, L.S., Van der Sluis, A. (eds.) Analytical and Numerical Approaches to Asymptotic Problems in Analysis, pp. 99–116. NorthHolland, Amsterdam (1981)
31. Johnson, C., Schatz, A.H., Wahlbin, L.B.: Crosswind smear and pointwise errors in streamline diffusion finite element methods. Math. Comput. **49**, 25–38 (1987)
32. Kellogg, R.B., Tsan, A.: Analysis of some difference approximations for a singular perturbation problem without turning points. Math. Comput. **32**, 1025–1039 (1978)
33. Kopteva, N.V.: Uniform convergence with respect to a small parameter of a scheme with central difference on refining grids. Comput. Math. Math. Phys. **39**(10), 1594–1610 (1999)
34. Kopteva, N.V.: Maximum norm a posteriori error estimates for a one-dimensional convection-diffusion problem. SIAM J. Numer. Anal. **39**(2), 423–441 (2001)
35. Kopteva, N.V., Stynes, M.: A robust adaptive method for quasi-linear one-dimensional convection-diffusion problem. SIAM J. Numer. Anal. **39**, 1446–1467 (2001)
36. Lenferink, W.: Pointwise convergence of approximations to a convection-diffusion equation on a Shishkin mesh. Appl. Numer. Math. **32**(1), 69–86 (2000)
37. Linß, T.: Sufficient conditions for uniform convergence on layer-adapted grids. Appl. Numer. Math. **37**, 241–255 (2001)
38. Linß, T.: Layer-adapted meshes for convectioni-diffusion problems. Comput. Methods Appl. Mech. Eng. **192**, 1061–1105 (2003)
39. Linß, T., Stynes, M.: The SDFEM on Shishkin meshes for linear convection-diffusion problems. Numer. Math. **87**, 457–484 (2001)
40. Miller, J.J.H., O'Riordan, E., Shishkin, G.I.: On piecewise-uniform meshes for upwind- and central-difference operators for solving singularly perturbed problems. IMA J. Numer. Anal. **15**(1), 89–99 (1995)
41. Miller, J.J.H., O'Riordan, E., Shishkin, G.I.: Fitted Numerical Methods For Singular Perturbation Problems. World Scientific, Singapore (1996)
42. Morton, K.W.: Numerical Solution of Convection-Diffusion Problems, volume 12 of Applied Mathematics and Mathematical Computation. Chapman & Hall, London (1996)
43. Nadler, E.: Piecewise linear best L_2 approximation on triangulations. In: Chui, C.K., Schumaker, L.L., Ward, J.D. (eds.) Approximation Theory, vol. V, pp. 499–502. Academic, New York (1986)

44. Niijima, K.: Pointwise error estimates for a streamline diffusion finite element scheme. Numer. Math. **56**, 707–719 (1990)
45. Qiu, Y., Sloan, D.M., Tang, T.: Convergence analysis of an adaptive finite difference method for a singular perturbation problem. J. Comput. Appl. Math. **116**, 121–143 (2000)
46. Roos, H.G.: Layer-adapted grids for singular perturbation problems. ZAMM, Z. Angew. Math. Mech. **78**(5), 291–309 (1998)
47. Roos, H.G., Stynes, M., Tobiska, L.: Numerical Methods for Singularly Perturbed Differential Equations, volume 24 of Springer series in Computational Mathematics. Springer, Heidelberg (1996)
48. Roos, H.G., Zarin, H.: The streamline-diffusion method for a convection-diffusion problem with a point source. J. Comput. Appl. Math. **150**, 109–128 (2003)
49. Sangalli, G.: Quasi optimality of the supg method for the one-dimensional adavection-diffusion problem. SIAM J. Numer. Anal. **41**(4), 1528–1542 (2003)
50. Schatz, A.H., Wahlbin, L.B.: On the quasi-optimality in L_∞ of the $\overset{\circ}{H}{}^1$-projection into finite element spaces. Math. Comput. **38**(157), 1–22 (1982)
51. Schatz, A.H., Wahlbin, L.B.: On the finite element method for singularly perturbed reaction-diffusion problems in two and one dimensions. Math. Comput. **40**(161), 47–89 (1983)
52. Shishkin, G.I.: Grid approximation of singularly perturbed elliptic and parabolic equations (in Russian). PhD thesis, Second doctorial thesis, Keldysh Institute, Moscow (1990)
53. Stynes, M., Tobiska, L.: A finite difference analysis of a streamline diffusion method on a Shishkin mesh. Numer. Algorithms **18**, 337–360 (1998)
54. Stynes, M., Tobiska, L.: The SDFEM for a convection-diffusion problem with a boundary layer: optimal error analysis, enhancement of accuracy. SIAM J. Numer. Anal. **41**(5), 1620–1642 (2003)
55. White, A.B.: On selection of equidistributing meshes for two-point boundary-value problems. SIAM J. Numer. Anal. **16**, 472–502 (1979)
56. Zhang, Z.M.: Finite element superconvergence approximation for one-dimensional singularly perturbed problems. Numer. Meth. PDEs **18**, 374–395 (2002)
57. Zhang, Z.M.: Finite element superconvergence on Shishkin mesh for 2-D convection-diffusion problems. Math. Comput. **72**(243), 1147–1177 (2003)
58. Zhou, G., Rannacher, R.: Pointwise superconvergence of the streamline diffusion finite element method. Numer. Meth. PDEs 12, CMP **96**(05), 123–145 (1996)