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FINITE ELEMENTS FOR DIV DIV CONFORMING SYMMETRIC TENSORS IN THREE DIMENSIONS

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Abstract. Finite element spaces on a tetrahedron are constructed for div div -conforming symmetric tensors in three dimensions. The key tools of the construction are the decomposition of polynomial tensor spaces and the characterization of the trace operators. First, the div div Hilbert complex and its corresponding polynomial complexes are presented. Several decompositions of polynomial vector and tensor spaces are derived from the polynomial complexes. Second, traces for the div div operator are characterized through a Green's identity. Besides the normal-normal component, another trace involving combination of first order derivatives of the tensor is continuous across the face. Due to the smoothness of polynomials, the symmetric tensor element is also continuous at vertices, and on the plane orthogonal to each edge. Besides, a finite element for sym curl-conforming trace-free tensors is constructed following the same approach. Putting all together, a finite element div div complex, as well as the bubble functions complex, in three dimensions is established.

1. Introduction

In this paper, we shall construct finite element subspaces for the space

$$\boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S}) := \{\boldsymbol{\tau}\in \boldsymbol{L}^2(\Omega;\mathbb{S}) : \operatorname{div}\operatorname{div}\boldsymbol{\tau}\in L^2(\Omega)\},\ \Omega\subset\mathbb{R}^3,$$

which consists of symmetric tensors such that $\operatorname{div}\operatorname{div}\boldsymbol{\tau}\in L^2(\Omega)$ with the inner div applied row-wisely to $\boldsymbol{\tau}$ resulting in a column vector for which the outer div operator is applied. $H(\operatorname{div}\operatorname{div})$-conforming finite elements can be applied to discretize the linearized Einstein-Bianchi system [21, Section 4.11] and the mixed formulation of the biharmonic equation [19].

Recently Christiansen and Hu [7] constructed a conforming discrete strain complex on Clough-Tocher split in two dimensions which is the rotation of a two-dimensional div div complex. Chen and Huang [6] constructed two-dimensional $H(\operatorname{div}\operatorname{div})$-conforming finite elements and a finite element div div complex in two dimensions. The construction in three dimensions is much harder. The essential difficulty arises from the three-dimensional div div Hilbert complex

$$\boldsymbol{RT} \xrightarrow{\subset} \boldsymbol{H}^1(\Omega;\mathbb{R}^3) \xrightarrow{\operatorname{dev}\operatorname{grad}} \boldsymbol{H}(\operatorname{sym}\operatorname{curl},\Omega;\mathbb{T}) \xrightarrow{\operatorname{sym}\operatorname{curl}} \boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S}) \xrightarrow{\operatorname{div}\operatorname{div}} L^2(\Omega)\to 0,$$

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where $\boldsymbol{RT} = \{a\boldsymbol{x} + \boldsymbol{b} : a \in \mathbb{R}, \boldsymbol{b} \in \mathbb{R}^3\}$, $\boldsymbol{H}^1(\Omega; \mathbb{R}^3)$ and $L^2(\Omega)$ are standard Sobolev spaces, and $\boldsymbol{H}(\operatorname{sym}\operatorname{curl}, \Omega; \mathbb{T})$ is the space of traceless tensor $\boldsymbol{\sigma} \in L^2(\Omega; \mathbb{T})$ such that $\operatorname{sym}\operatorname{curl}\boldsymbol{\sigma} \in L^2(\Omega; \mathbb{S})$ with the row-wise curl operator. In the three-dimensional div div complex, the Sobolev space before $\boldsymbol{H}(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S})$ consists of tensor functions, whereas it consists of vector functions in two dimensions. For the sake of comparison, the div div Hilbert complex in two dimensions is

$$\boldsymbol{RT} \xrightarrow{\subset} \boldsymbol{H}^1(\Omega; \mathbb{R}^2) \xrightarrow{\operatorname{sym}\operatorname{curl}} \boldsymbol{H}(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S}) \xrightarrow{\operatorname{div}\operatorname{div}} L^2(\Omega) \to 0.$$

Finite element spaces for $\boldsymbol{H}^1(\Omega; \mathbb{R}^2)$ are relatively mature. Then the design of a div div conforming finite element in two dimensions is relatively easy; see [6] and also Section 5.4.

We start our construction from the following polynomial complexes

$$\boldsymbol{RT} \underset{\boldsymbol{\pi}_{RT}}{\overset{\subset}{\rightleftarrows}} \mathbb{P}_{k+2}(\Omega; \mathbb{R}^3) \underset{\cdot \boldsymbol{x}}{\overset{\operatorname{dev}\operatorname{grad}}{\rightleftarrows}} \mathbb{P}_{k+1}(\Omega; \mathbb{T}) \underset{\times \boldsymbol{x}}{\overset{\operatorname{sym}\operatorname{curl}}{\rightleftarrows}} \mathbb{P}_k(\Omega; \mathbb{S}) \underset{\boldsymbol{x}\boldsymbol{x}^{\intercal}}{\overset{\operatorname{div}\operatorname{div}}{\rightleftarrows}} \mathbb{P}_{k-2}(\Omega) \underset{\supset}{\rightleftarrows} 0 \tag{1}$$

and reveal several decompositions of polynomial vector and tensor spaces from (1). We then present a Green's identity

$$(\operatorname{div}\operatorname{div}\boldsymbol{\tau}, v)_K = (\boldsymbol{\tau}, \nabla^2 v)_K - \sum_{F\in\mathcal{F}(K)} \sum_{e\in\mathcal{E}(F)} (\boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}, v)_e$$
$$- \sum_{F\in\mathcal{F}(K)} \left[(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}, \partial_n v)_F - (2\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}) + \partial_n(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}), v)_F\right],$$

and give a characterization of two traces for $\boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{div}\operatorname{div}, K; \mathbb{S})$

$$\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n} \in H_n^{-1/2}(\partial K), \quad \text{and } 2\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}) + \partial_n(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}) \in H_t^{-3/2}(\partial K),$$

see Section 4.3 for detailed definitions of these negative Sobolev space for traces.

Based on the decomposition of polynomial tensors and the characterization of traces, we are able to construct two types of $H(\operatorname{div}\operatorname{div})$-conforming finite element spaces on a tetrahedron. Here we present the BDM-type (full polynomial) space below. Let K be a tetrahedron and let $k \geq 3$ be an integer. The shape function space is $\mathbb{P}_k(K; \mathbb{S})$. The set of edges of K is denoted by $\mathcal{E}(K)$, the set of faces by $\mathcal{F}(K)$, and the set of vertices by $\mathcal{V}(K)$. For each edge, we choose two normal vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$. The degrees of freedom (DoFs) are given by

$$\boldsymbol{\tau}(\delta) \quad \forall\, \delta \in \mathcal{V}(K), \tag{2}$$
$$(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j, q)_e \quad \forall\, q \in \mathbb{P}_{k-2}(e), e \in \mathcal{E}(K),\ i, j = 1, 2, \tag{3}$$
$$(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}, q)_F \quad \forall\, q \in \mathbb{P}_{k-3}(F), F \in \mathcal{F}(K), \tag{4}$$
$$(2\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}) + \partial_n(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}), q)_F \quad \forall\, q \in \mathbb{P}_{k-1}(F), F \in \mathcal{F}(K), \tag{5}$$
$$(\boldsymbol{\tau}, \boldsymbol{\varsigma})_K \quad \forall\, \boldsymbol{\varsigma} \in \nabla^2\mathbb{P}_{k-2}(K), \tag{6}$$
$$(\boldsymbol{\tau}, \boldsymbol{\varsigma})_K \quad \forall\, \boldsymbol{\varsigma} \in \operatorname{sym}(\mathbb{P}_{k-2}(K; \mathbb{T}) \times \boldsymbol{x}), \tag{7}$$
$$(\boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{n} \times \boldsymbol{x}q)_{F_1} \quad \forall\, q \in \mathbb{P}_{k-2}(F_1), \tag{8}$$

where $F_1 \in \mathcal{F}(K)$ is an arbitrary but fixed face. The last degree of freedom (8) will be regarded as an interior degree of freedom to the tetrahedron K. Namely even a face F is chosen in different elements, the degree of freedom (8) is double-valued when defining the global finite element space. The RT-type (incomplete polynomial) space can be obtained by further reducing the index of degrees of freedom by 1 except the moment with $\nabla^2\mathbb{P}_{k-2}(K)$. To the best of our knowledge, these are the first $H(\operatorname{div}\operatorname{div})$-conforming finite elements for symmetric tensors in three dimensions. After our work, in [17], a new family of divdiv-conforming finite elements is introduced for triangular and tetrahedral grids in a more unified way. The constructed finite element spaces there are in $\boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})\cap\boldsymbol{H}(\operatorname{div},\Omega;\mathbb{S})$, while ours is in $\boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$ only which is more natural.

To help the understanding of our construction, we sketch a decomposition of a finite element space associated to a generic differential operator d in Fig. 1, where d^* is the formal adjoint of d. The boundary degrees of freedom (4)-(5) are

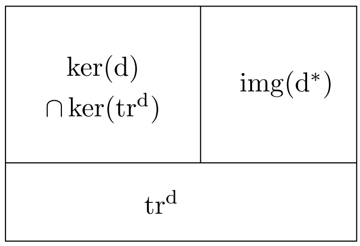

FIGURE 1. Decomposition of a generic finite element space

obviously motivated by the Green's formula and the characterization of the trace of $\boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$. The extra continuity (2)-(3) is to ensure the cancellation of the edge term when adding element-wise Green's identity over a mesh. All together (2)-(5) will determine the trace on the boundary of a tetrahedron, i.e., the bottom box in Fig. 1.

The interior moment of $\nabla^2\mathbb{P}_{k-2}(K)$ is to determine the image $\operatorname{div}\operatorname{div}(\mathbb{P}_k(K;\mathbb{S})\cap\ker(\mathrm{tr}))$, which is isomorphism to $\mathrm{img}(\nabla^2)$ – the upper right block in Fig. 1. Together with $\operatorname{sym}(\mathbb{P}_{k-2}(K;\mathbb{T})\times\boldsymbol{x})$, the volume moments can determine the polynomial of degree only up to $k-1$. We then use the vanished trace and the symmetry of the tensor to figure out the remaining degrees of freedom. The DoFs (7)-(8) will determine $\ker(\operatorname{div}\operatorname{div})\cap\ker(\mathrm{tr})$ – the upper left block in Fig. 1.

For the symmetric tensor space, it seems odd to have degrees of freedom not symmetric, as a face is singled out in (8). In view of Fig. 1 and the exactness of the polynomial div div complex (1), (7)-(8) can be replaced by

$$(\boldsymbol{\tau},\boldsymbol{\varsigma})_K \quad \forall\,\boldsymbol{\varsigma}\in\operatorname{sym}\operatorname{curl}\mathbb{B}_{k+1}(\operatorname{sym}\operatorname{curl},K;\mathbb{T}), \tag{9}$$

where $\mathbb{B}_{k+1}(\operatorname{sym}\operatorname{curl},K;\mathbb{T})=\mathbb{P}_{k+1}(K;\mathbb{T})\cap\boldsymbol{H}_0(\operatorname{sym}\operatorname{curl},K;\mathbb{T})$ is the so-called bubble function space and will be characterized precisely in Section 5.2. Although (9) is more symmetric, it is indeed not simpler than (7)-(8) in implementation as the formulation of $\operatorname{sym}\operatorname{curl}\mathbb{B}_{k+1}(\operatorname{sym}\operatorname{curl},K;\mathbb{T})$ is much more complicated than polynomials on a face.

With the help of the $H(\operatorname{div}\operatorname{div})$-conforming finite elements for symmetric tensors and two traces $\boldsymbol{n}\times\operatorname{sym}(\boldsymbol{\tau}\times\boldsymbol{n})\times\boldsymbol{n}$ and $\boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{n}$ of space $\boldsymbol{H}(\operatorname{sym}\operatorname{curl},K;\mathbb{T})$, we construct $H(\operatorname{sym}\operatorname{curl})$-conforming finite elements for trace-free tensors. The space of shape functions is $\mathbb{P}_{\ell+1}(K;\mathbb{T})$ with $\ell\geq\max\{k-1,3\}$. The degrees of freedom are

$$\begin{aligned}
&\boldsymbol{\tau}(\delta) && \forall\, \delta \in \mathcal{V}(K),\\
&(\operatorname{sym}\operatorname{curl}\boldsymbol{\tau})(\delta) && \forall\, \delta \in \mathcal{V}(K),\\
&(\boldsymbol{n}_i^{\intercal}(\operatorname{sym}\operatorname{curl}\boldsymbol{\tau})\boldsymbol{n}_j, q)_e && \forall\, q \in \mathbb{P}_{\ell-2}(e), e \in \mathcal{E}(K), i,j=1,2,\\
&(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{t}, q)_e && \forall\, q \in \mathbb{P}_{\ell-1}(e), e \in \mathcal{E}(K), i=1,2,\\
&(\boldsymbol{n}_2^{\intercal}(\operatorname{curl}\boldsymbol{\tau})\boldsymbol{n}_1 + \partial_t(\boldsymbol{t}^{\intercal}\boldsymbol{\tau}\boldsymbol{t}), q)_e && \forall\, q \in \mathbb{P}_{\ell}(e), e \in \mathcal{E}(K),\\
&(\boldsymbol{n}\times\operatorname{sym}(\boldsymbol{\tau}\times\boldsymbol{n})\times\boldsymbol{n}, \boldsymbol{\varsigma})_F && \forall\, \boldsymbol{\varsigma} \in (\nabla_F^{\perp})^2\,\mathbb{P}_{\ell-1}(F) \oplus \operatorname{sym}(\boldsymbol{x}\otimes\mathbb{P}_{\ell-1}(F;\mathbb{R}^2)),\\
&(\boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{n}, \boldsymbol{q})_F && \forall\, \boldsymbol{q} \in \nabla_F\mathbb{P}_{\ell-3}(F) \oplus \boldsymbol{x}^{\perp}\mathbb{P}_{\ell-1}(F), F \in \mathcal{F}(K),\\
&(\boldsymbol{\tau}, \boldsymbol{q})_K && \forall\, \boldsymbol{q} \in \mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K; \mathbb{T}).
\end{aligned}$$

Combining previous finite elements for tensors and the vectorial Hermite element in three dimensions, we arrive at a finite element div div complex in three dimensions

$$\boldsymbol{RT} \xrightarrow{\subset} \boldsymbol{V}_h \xrightarrow{\operatorname{dev}\operatorname{grad}} \boldsymbol{\Sigma}_h^{\mathbb{T}} \xrightarrow{\operatorname{sym}\operatorname{curl}} \boldsymbol{\Sigma}_h^{\mathbb{S}} \xrightarrow{\operatorname{div}\operatorname{div}} \mathcal{Q}_h \to 0$$

and the associated finite element bubble div div complex. Recently another finite element div div complex in three dimensions is devised in [16], where the $H(\operatorname{sym}\operatorname{curl})$-conforming finite elements for trace-free tensors and H^1-conforming finite elements for vectors employed in [16] are smoother than ours. Two-dimensional finite element div div complexes can be found in [4, 6, 17]. And the rotated version, discrete strain complexes, can be found in [7].

The rest of this paper is organized as follows. We present some operations for vectors and tensors in Section 2. Two polynomial complexes related to the div div complex and direct sum decompositions of polynomial spaces are shown in Section 3. We derive the Green's identity and characterize the trace of $\boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$ on polyhedrons in Section 4, and then construct the conforming finite elements for $\boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$ in three dimensions in Section 5. In Section 6 we construct conforming finite elements for $\boldsymbol{H}(\operatorname{sym}\operatorname{curl},\Omega;\mathbb{T})$. With previous devised finite elements for tensors, we form a finite element div div complex in three dimensions in Section 7.

2. Matrix and vector operations

In this section, we shall survey operations for vectors and tensors. In particular, we shall distinguish operators applied to columns and rows of a matrix.

2.1. Matrix-vector products. The matrix-vector product $\boldsymbol{A}\boldsymbol{b}$ can be interpreted as the inner product of $\boldsymbol{b}$ with the row vectors of $\boldsymbol{A}$. We thus define the dot operator $\boldsymbol{A}\cdot\boldsymbol{b} := \boldsymbol{A}\boldsymbol{b}$. Similarly we can define the row-wise cross product from the right $\boldsymbol{A}\times\boldsymbol{b}$. Here rigorously speaking when a column vector $\boldsymbol{b}$ is treated as a row vector, notation $\boldsymbol{b}^{\intercal}$ should be used. In most places, however, we will sacrifice this precision for the ease of notation. When the vector is on the left of the matrix, the operation is defined column-wise. For example, $\boldsymbol{b}\cdot\boldsymbol{A} := \boldsymbol{b}^{\intercal}\boldsymbol{A}$. For dot products, we will still mainly use the conventional notation, e.g. $\boldsymbol{b}\cdot\boldsymbol{A}\cdot\boldsymbol{c} = \boldsymbol{b}^{\intercal}\boldsymbol{A}\boldsymbol{c}$. But for the cross products, we emphasize again the cross product of a vector from the left is column-wise and from the right is row-wise. The transpose rule still works, i.e. $\boldsymbol{b}\times\boldsymbol{A} = -(\boldsymbol{A}^{\intercal}\times\boldsymbol{b})^{\intercal}$. Here again, we mix the usage of column vector $\boldsymbol{b}$ and row vector $\boldsymbol{b}^{\intercal}$.

The ordering of performing the row and column products does not matter which leads to the associative rule of the triple products

$$\boldsymbol{b} \times \boldsymbol{A} \times \boldsymbol{c} := (\boldsymbol{b} \times \boldsymbol{A}) \times \boldsymbol{c} = \boldsymbol{b} \times (\boldsymbol{A} \times \boldsymbol{c}).$$

Similar rules hold for $\boldsymbol{b} \cdot \boldsymbol{A} \cdot \boldsymbol{c}$ and $\boldsymbol{b} \cdot \boldsymbol{A} \times \boldsymbol{c}$ and thus parentheses can be safely skipped when no differentiation is involved.

For two column vectors $\boldsymbol{u}, \boldsymbol{v}$, the tensor product $\boldsymbol{u} \otimes \boldsymbol{v} := \boldsymbol{u}\boldsymbol{v}^{\intercal}$ is a matrix which is also known as the dyadic product $\boldsymbol{u}\boldsymbol{v} := \boldsymbol{u}\boldsymbol{v}^{\intercal}$ with more clean notation (one $^{\intercal}$ is skipped). The row-wise product and column-wise product of $\boldsymbol{u}\boldsymbol{v}$ with another vector will be applied to the neighboring vector

$$\boldsymbol{x} \cdot (\boldsymbol{u}\boldsymbol{v}) = (\boldsymbol{x} \cdot \boldsymbol{u})\boldsymbol{v}^{\intercal}, \quad (\boldsymbol{u}\boldsymbol{v}) \cdot \boldsymbol{x} = \boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{x}), \tag{10}$$

$$\boldsymbol{x} \times (\boldsymbol{u}\boldsymbol{v}) = (\boldsymbol{x} \times \boldsymbol{u})\boldsymbol{v}, \quad (\boldsymbol{u}\boldsymbol{v}) \times \boldsymbol{x} = \boldsymbol{u}(\boldsymbol{v} \times \boldsymbol{x}). \tag{11}$$

2.2. **Differentiation.** We treat Hamilton operator $\nabla = (\partial_1, \partial_2, \partial_3)^{\intercal}$ as a column vector. For a vector function $\boldsymbol{u} = (u_1, u_2, u_3)^{\intercal}$, $\operatorname{curl} \boldsymbol{u} = \nabla \times \boldsymbol{u}$, and $\operatorname{div} \boldsymbol{u} = \nabla \cdot \boldsymbol{u}$ are standard differential operations. Define $\nabla \boldsymbol{u} := \nabla \boldsymbol{u}^{\intercal} = (\partial_i u_j)$, which can be understood as the dyadic product of Hamilton operator ∇ and column vector $\boldsymbol{u}$.

Applying matrix-vector operations to the Hamilton operator ∇, we get column-wise differentiation $\nabla \cdot \boldsymbol{A}, \nabla \times \boldsymbol{A}$, and row-wise differentiation $\boldsymbol{A} \cdot \nabla, \boldsymbol{A} \times \nabla$. Conventionally, the differentiation is applied to the function after the ∇ symbol. So a more conventional notation is

$$\boldsymbol{A} \cdot \nabla := (\nabla \cdot \boldsymbol{A}^{\intercal})^{\intercal}, \quad \boldsymbol{A} \times \nabla := -(\nabla \times \boldsymbol{A}^{\intercal})^{\intercal}.$$

By moving the differential operator to the right, the notation is simplified and the transpose rule for matrix-vector products can be formally used. Again the right most column vector ∇ is treated as a row vector $\nabla^{\intercal}$ to make the notation cleaner.

In the literature, differential operators are usually applied row-wisely to tensors. To distinguish with ∇ notation, we define operators in letters as

$$\begin{aligned}
\operatorname{grad} \boldsymbol{u} &:= \boldsymbol{u}\nabla^{\intercal} = (\partial_j u_i) = (\nabla \boldsymbol{u})^{\intercal}, \\
\operatorname{curl} \boldsymbol{A} &:= -\boldsymbol{A} \times \nabla = (\nabla \times \boldsymbol{A}^{\intercal})^{\intercal}, \\
\operatorname{div} \boldsymbol{A} &:= \boldsymbol{A} \cdot \nabla = (\nabla \cdot \boldsymbol{A}^{\intercal})^{\intercal}.
\end{aligned}$$

Note that for vector functions, the differentiation written in letters are equivalent to ∇ notation while for tensors they are slightly different. The double divergence operator can be written as

$$\operatorname{div} \operatorname{div} \boldsymbol{A} := \nabla \cdot \boldsymbol{A} \cdot \nabla.$$

As the column and row operations are independent, the ordering of operations is not important and parentheses can be skipped.

2.3. **Matrix decompositions.** Denote the space of all 3×3 matrices by $\mathbb{M}$, all symmetric 3×3 matrices by $\mathbb{S}$, all skew-symmetric 3×3 matrices by $\mathbb{K}$, and all trace-free 3×3 matrices by $\mathbb{T}$. For any matrix $\boldsymbol{B} \in \mathbb{M}$, we can decompose it into symmetric and skew-symmetric parts as

$$\boldsymbol{B} = \operatorname{sym}(\boldsymbol{B}) + \operatorname{skw}(\boldsymbol{B}) := \frac{1}{2}(\boldsymbol{B} + \boldsymbol{B}^{\intercal}) + \frac{1}{2}(\boldsymbol{B} - \boldsymbol{B}^{\intercal}).$$

We can also decompose it into a direct sum of a trace-free matrix and a scalar matrix as

$$\boldsymbol{B} = \operatorname{dev} \boldsymbol{B} + \frac{1}{3}\operatorname{tr}(\boldsymbol{B})\boldsymbol{I} := (\boldsymbol{B} - \frac{1}{3}\operatorname{tr}(\boldsymbol{B})\boldsymbol{I}) + \frac{1}{3}\operatorname{tr}(\boldsymbol{B})\boldsymbol{I}. \tag{12}$$

Define the sym curl operator for a matrix $\boldsymbol{A}$

$$\operatorname{sym}\operatorname{curl}\boldsymbol{A} := \frac{1}{2}(\nabla\times\boldsymbol{A}^{\intercal} + (\nabla\times\boldsymbol{A}^{\intercal})^{\intercal}) = \frac{1}{2}(\nabla\times\boldsymbol{A}^{\intercal} - \boldsymbol{A}\times\nabla).$$

We define an isomorphism between $\mathbb{R}^3$ and the space of skew-symmetric matrices $\mathbb{K}$ as follows: for a vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^{\intercal} \in \mathbb{R}^3$,

$$\operatorname{mskw}\boldsymbol{\omega} := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

Obviously mskw : $\mathbb{R}^3 \to \mathbb{K}$ is a bijection. We define vskw : $\mathbb{M} \to \mathbb{R}^3$ by vskw := $\operatorname{mskw}^{-1} \circ \operatorname{skw}$.

We will use the following identities for smooth enough vector or matrix functions

$$\operatorname{skw}(\operatorname{grad}\boldsymbol{u}) = \frac{1}{2}(\operatorname{mskw}\operatorname{curl}\boldsymbol{u}),$$

$$\operatorname{skw}(\operatorname{curl}\boldsymbol{A}) = \frac{1}{2}\operatorname{mskw}\left[\operatorname{div}(\boldsymbol{A}^{\intercal}) - \operatorname{grad}(\operatorname{tr}(\boldsymbol{A}))\right], \tag{13}$$

$$\operatorname{div}\operatorname{mskw}\boldsymbol{u} = -\operatorname{curl}\boldsymbol{u}, \tag{14}$$

$$\operatorname{curl}(u\boldsymbol{I}) = -\operatorname{mskw}\operatorname{grad}(u), \tag{15}$$

$$\operatorname{tr}(\boldsymbol{\tau}\times\boldsymbol{x}) = -2\boldsymbol{x}\cdot\operatorname{vskw}\boldsymbol{\tau}, \tag{16}$$

which can be verified by a direct calculation. More identities involving the matrix operation and differentiation are summarized in [1].

2.4. **Projections to a plane.** Given a plane F with normal vector $\boldsymbol{n}$, for a vector $\boldsymbol{v} \in \mathbb{R}^3$, we have the orthogonal decomposition

$$\boldsymbol{v} = \Pi_n\boldsymbol{v} + \Pi_F\boldsymbol{v} := (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n} + (\boldsymbol{n}\times\boldsymbol{v})\times\boldsymbol{n}.$$

The vector $\Pi_F^{\perp}\boldsymbol{v} := \boldsymbol{n}\times\boldsymbol{v}$ is also on the plane F and is a rotation of $\Pi_F\boldsymbol{v}$ by 90° counter-clockwise with respect to $\boldsymbol{n}$. We treat Hamilton operator $\nabla = (\partial_1, \partial_2, \partial_3)^{\intercal}$ as a column vector and define

$$\nabla_F^{\perp} := \boldsymbol{n}\times\nabla, \quad \nabla_F := \Pi_F\nabla = (\boldsymbol{n}\times\nabla)\times\boldsymbol{n}.$$

For a scalar function v,

$$\operatorname{grad}_F v := \nabla_F v = \Pi_F(\nabla v),$$
$$\operatorname{curl}_F v := \nabla_F^{\perp} v = \boldsymbol{n}\times\nabla v$$

are the surface gradient and surface curl, respectively. For a vector function $\boldsymbol{v}$, $\nabla_F\cdot\boldsymbol{v}$ is the surface divergence

$$\operatorname{div}_F\boldsymbol{v} := \nabla_F\cdot\boldsymbol{v} = \nabla_F\cdot(\Pi_F\boldsymbol{v}).$$

By the cyclic invariance of the mix product and the fact $\boldsymbol{n}$ is constant, the surface rot operator is

$$\operatorname{rot}_F\boldsymbol{v} := \nabla_F^{\perp}\cdot\boldsymbol{v} = (\boldsymbol{n}\times\nabla)\cdot\boldsymbol{v} = \boldsymbol{n}\cdot(\nabla\times\boldsymbol{v}),$$

which is the normal component of $\nabla \times \boldsymbol{v}$. The tangential trace of $\nabla \times \boldsymbol{v}$ is

$$\boldsymbol{n} \times (\nabla \times \boldsymbol{v}) = \nabla(\boldsymbol{n} \cdot \boldsymbol{v}) - \partial_n \boldsymbol{v}. \tag{17}$$

By definition,

$$\operatorname{rot}_F \boldsymbol{v} = -\operatorname{div}_F(\boldsymbol{n} \times \boldsymbol{v}), \quad \operatorname{div}_F \boldsymbol{v} = \operatorname{rot}_F(\boldsymbol{n} \times \boldsymbol{v}). \tag{18}$$

Note that the three-dimensional curl operator restricted to a two-dimensional plane F results in two operators: curl_F maps a scalar to a vector, which is a rotation of grad_F, and rot_F maps a vector to a scalar which can be thought of as a rotated version of div_F. The surface differentiations satisfy the property $\operatorname{div}_F \operatorname{curl}_F = 0$ and $\operatorname{rot}_F \operatorname{grad}_F = 0$ and when F is simply connected, $\ker(\operatorname{div}_F) = \operatorname{img}(\operatorname{curl}_F)$ and $\ker(\operatorname{rot}_F) = \operatorname{img}(\operatorname{grad}_F)$.

Differentiation for two-dimensional tensors can be defined similarly.

3. Divdiv complex and polynomial complexes

In this section, we shall consider the div div complex and establish two related polynomial complexes. We assume $\Omega \subset \mathbb{R}^3$ is a bounded and Lipschitz domain, which is topologically trivial in the sense that it is homeomorphic to a ball. Without loss of generality, we also assume $\mathbf{0} = (0, 0, 0) \in \Omega$.

Recall that a Hilbert complex is a sequence of Hilbert spaces connected by a sequence of linear operators satisfying the property: the composition of two consecutive operators vanishes. As all complexes considered in this paper are Hilbert complexes, we will abbreviate a Hilbert complex as a complex. If the range of each map is the kernel of the succeeding map, then a complex is called exact. As Ω is topologically trivial, the following de Rham Complex of Ω is exact

$$0 \to H^1(\Omega) \xrightarrow{\operatorname{grad}} \boldsymbol{H}(\operatorname{curl}; \Omega) \xrightarrow{\operatorname{curl}} \boldsymbol{H}(\operatorname{div}; \Omega) \xrightarrow{\operatorname{div}} L^2(\Omega) \to 0, \tag{19}$$

where $\boldsymbol{H}(\operatorname{curl}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega; \mathbb{R}^3) : \operatorname{curl} \boldsymbol{v} \in \boldsymbol{L}^2(\Omega; \mathbb{R}^3)\}, \boldsymbol{H}(\operatorname{div}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega; \mathbb{R}^3) : \operatorname{div} \boldsymbol{v} \in L^2(\Omega)\}$.

3.1. The div div complex.

The div div complex in three dimensions reads as [1,19]

$$\boldsymbol{RT} \xrightarrow{\subset} \boldsymbol{H}^1(\Omega; \mathbb{R}^3) \xrightarrow{\operatorname{dev} \operatorname{grad}} \boldsymbol{H}(\operatorname{sym} \operatorname{curl}, \Omega; \mathbb{T}) \xrightarrow{\operatorname{sym} \operatorname{curl}} \boldsymbol{H}(\operatorname{div} \operatorname{div}, \Omega; \mathbb{S}) \xrightarrow{\operatorname{div} \operatorname{div}} L^2(\Omega) \to 0, \tag{20}$$

where $\boldsymbol{RT} := \{a\boldsymbol{x} + \boldsymbol{b} : a \in \mathbb{R}, \boldsymbol{b} \in \mathbb{R}^3\}$ is the space of shape functions of the lowest order Raviart-Thomas element [22]. For completeness, we prove the exactness of the complex (20) following [19].

Theorem 3.1. *Assume Ω is a bounded and topologically trivial Lipschitz domain in $\mathbb{R}^3$. Then (20) is an exact complex.*

Proof. We verify that the composition of consecutive operators vanishes from left to right. Take a function $\boldsymbol{v} = a\boldsymbol{x} + \boldsymbol{b} \in \boldsymbol{RT}$, then $\operatorname{grad} \boldsymbol{v} = a\boldsymbol{I}$ and $\operatorname{dev} \boldsymbol{I} = \mathbf{0}$. For any $\boldsymbol{v} \in \mathcal{C}^2(\Omega; \mathbb{R}^3)$, it holds from (15) that

$$\begin{aligned}
\operatorname{sym} \operatorname{curl} \operatorname{dev} \operatorname{grad} \boldsymbol{v} &= \operatorname{sym} \operatorname{curl} \left(\operatorname{grad} \boldsymbol{v} - \frac{1}{3} (\operatorname{div} \boldsymbol{v}) \boldsymbol{I} \right) = -\frac{1}{3} \operatorname{sym} \operatorname{curl}((\operatorname{div} \boldsymbol{v}) \boldsymbol{I}) \\
&= \frac{1}{3} \operatorname{sym} \operatorname{mskw}(\operatorname{grad}(\operatorname{div} \boldsymbol{v})) = \mathbf{0}.
\end{aligned}$$

By the density argument, we get $\operatorname{sym}\operatorname{curl}\operatorname{dev}\operatorname{grad}\boldsymbol{H}^1(\Omega;\mathbb{R}^3)=\mathbf{0}$. For any $\boldsymbol{\tau}\in\mathcal{C}^3(\Omega;\mathbb{T})$,

$$\operatorname{div}\operatorname{div}\operatorname{sym}\operatorname{curl}\boldsymbol{\tau}=\frac{1}{2}\nabla\cdot(\nabla\times\boldsymbol{\tau}^{\intercal}-\boldsymbol{\tau}\times\nabla)\cdot\nabla=0.$$

Again by the density argument, $\operatorname{div}\operatorname{div}\operatorname{sym}\operatorname{curl}\boldsymbol{H}(\operatorname{sym}\operatorname{curl},\Omega;\mathbb{T})=0$. Thus (20) is a complex.

We then verify the exactness of (20) from the right to the left.

(1) $\operatorname{div}\operatorname{div}\boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})=L^2(\Omega)$.

Recursively applying the exactness of de Rham complex (19), we can prove $\operatorname{div}\operatorname{div}\boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{M})=L^2(\Omega)$ without the symmetry requirement, where the space $\boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{M})=\{\boldsymbol{\tau}\in L^2(\Omega;\mathbb{M}):\operatorname{div}\operatorname{div}\boldsymbol{\tau}\in L^2(\Omega)\}$.

Any skew-symmetric $\boldsymbol{\tau}$ can be written as $\boldsymbol{\tau}=\operatorname{mskw}\boldsymbol{v}$ for $\boldsymbol{v}=\operatorname{vskw}(\boldsymbol{\tau})$. Assume $\boldsymbol{v}\in\mathcal{C}^2(\Omega;\mathbb{R}^3)$; it follows from (14) that

$$\operatorname{div}\operatorname{div}\boldsymbol{\tau}=\operatorname{div}\operatorname{div}\operatorname{mskw}\boldsymbol{v}=-\operatorname{div}(\operatorname{curl}\boldsymbol{v})=0. \tag{21}$$

Since $\operatorname{div}\operatorname{div}\boldsymbol{\tau}=0$ for any smooth skew-symmetric tensor field $\boldsymbol{\tau}$, we obtain

$$\operatorname{div}\operatorname{div}\boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})=\operatorname{div}\operatorname{div}\boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{M})=L^2(\Omega).$$

(2) $\boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})\cap\ker(\operatorname{div}\operatorname{div})=\operatorname{sym}\operatorname{curl}\boldsymbol{H}(\operatorname{sym}\operatorname{curl},\Omega;\mathbb{T})$, *i.e. if* $\operatorname{div}\operatorname{div}\boldsymbol{\sigma}=0$ *and* $\boldsymbol{\sigma}\in\boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$, *then there exists a* $\boldsymbol{\tau}\in\boldsymbol{H}(\operatorname{sym}\operatorname{curl},\Omega;\mathbb{T})$, *s.t.* $\boldsymbol{\sigma}=\operatorname{sym}\operatorname{curl}\boldsymbol{\tau}$.

Since $\operatorname{div}(\operatorname{div}\boldsymbol{\sigma})=0$, by the exactness of the de Rham complex and identity (14), there exists $\boldsymbol{v}\in\boldsymbol{L}^2(\Omega;\mathbb{R}^3)$ such that

$$\operatorname{div}\boldsymbol{\sigma}=\operatorname{curl}\boldsymbol{v}=-\operatorname{div}(\operatorname{mskw}\boldsymbol{v}).$$

Namely $\operatorname{div}(\boldsymbol{\sigma}+\operatorname{mskw}\boldsymbol{v})=\mathbf{0}$. By the existence of regular potentials (cf. [10]), there exists $\widetilde{\boldsymbol{\tau}}\in\boldsymbol{H}^1(\Omega;\mathbb{M})$ such that

$$\operatorname{curl}\widetilde{\boldsymbol{\tau}}=\boldsymbol{\sigma}+\operatorname{mskw}\boldsymbol{v}.$$

By the symmetry of $\boldsymbol{\sigma}$, we have

$$\boldsymbol{\sigma}=\operatorname{sym}\operatorname{curl}\widetilde{\boldsymbol{\tau}}=\operatorname{sym}\operatorname{curl}(\operatorname{dev}\widetilde{\boldsymbol{\tau}})+\frac{1}{3}\operatorname{sym}\operatorname{curl}\left((\operatorname{tr}\widetilde{\boldsymbol{\tau}})\boldsymbol{I}\right).$$

From (15) we get

$$\operatorname{sym}\operatorname{curl}\left((\operatorname{tr}\widetilde{\boldsymbol{\tau}})\boldsymbol{I}\right)=-\operatorname{sym}(\operatorname{mskw}\operatorname{grad}(\operatorname{tr}\widetilde{\boldsymbol{\tau}}))=\mathbf{0},$$

which indicates $\boldsymbol{\sigma}=\operatorname{sym}\operatorname{curl}\boldsymbol{\tau}$ with $\boldsymbol{\tau}=\operatorname{dev}\widetilde{\boldsymbol{\tau}}\in\boldsymbol{H}^1(\Omega;\mathbb{T})$.

(3) $\boldsymbol{H}(\operatorname{sym}\operatorname{curl},\Omega;\mathbb{T})\cap\ker(\operatorname{sym}\operatorname{curl})=\operatorname{dev}\operatorname{grad}\boldsymbol{H}^1(\Omega;\mathbb{R}^3)$, *i.e. if* $\operatorname{sym}\operatorname{curl}\boldsymbol{\tau}=\mathbf{0}$ *and* $\boldsymbol{\tau}\in\boldsymbol{H}(\operatorname{sym}\operatorname{curl},\Omega;\mathbb{T})$, *then there exists a* $\boldsymbol{v}\in\boldsymbol{H}^1(\Omega;\mathbb{R}^3)$, *s.t.* $\boldsymbol{\tau}=\operatorname{dev}\operatorname{grad}\boldsymbol{v}$.

Since $\operatorname{sym}(\operatorname{curl}\boldsymbol{\tau})=\mathbf{0}$ and $\operatorname{tr}\boldsymbol{\tau}=0$, we have from (13) that

$$\operatorname{curl}\boldsymbol{\tau}=\operatorname{skw}(\operatorname{curl}\boldsymbol{\tau})=\frac{1}{2}\operatorname{mskw}\left[\operatorname{div}(\boldsymbol{\tau}^{\intercal})-\operatorname{grad}(\operatorname{tr}(\boldsymbol{\tau}))\right]=\frac{1}{2}\operatorname{mskw}(\operatorname{div}(\boldsymbol{\tau}^{\intercal})).$$

Then by (14),

$$\operatorname{curl}(\operatorname{div}(\boldsymbol{\tau}^{\intercal}))=-\operatorname{div}(\operatorname{mskw}\operatorname{div}(\boldsymbol{\tau}^{\intercal}))=-2\operatorname{div}(\operatorname{curl}\boldsymbol{\tau})=\mathbf{0}.$$

Thus there exists $w\in L^2(\Omega)$ satisfying $\operatorname{div}(\boldsymbol{\tau}^{\intercal})=2\operatorname{grad}w$, which together with (15) implies

$$\operatorname{curl}\boldsymbol{\tau}=\operatorname{mskw}\operatorname{grad}w=-\operatorname{curl}(w\boldsymbol{I}).$$

Namely $\operatorname{curl}(\boldsymbol{\tau}+w\boldsymbol{I})=\mathbf{0}$. Hence there exists $\boldsymbol{v}\in\boldsymbol{H}^1(\Omega;\mathbb{R}^3)$ such that $\boldsymbol{\tau}=-w\boldsymbol{I}+\operatorname{grad}\boldsymbol{v}$. Noting that $\boldsymbol{\tau}$ is trace-free, we achieve

$$\boldsymbol{\tau}=\operatorname{dev}\boldsymbol{\tau}=\operatorname{dev}\operatorname{grad}\boldsymbol{v}.$$

(4) $\boldsymbol{H}^1(\Omega;\mathbb{R}^3)\cap\ker(\operatorname{dev}\operatorname{grad})=\boldsymbol{RT}$, *i.e. if* $\operatorname{dev}\operatorname{grad}\boldsymbol{v}=\mathbf{0}$ *and* $\boldsymbol{v}\in\boldsymbol{H}^1(\Omega;\mathbb{R}^3)$, *then* $\boldsymbol{v}\in\boldsymbol{RT}$.

Notice that

$$\operatorname{grad}\boldsymbol{v}=\frac{1}{3}(\operatorname{div}\boldsymbol{v})\boldsymbol{I}. \tag{22}$$

Apply curl on both sides of (22) and use (15) to get

$$-\operatorname{mskw}\operatorname{grad}(\operatorname{div}\boldsymbol{v})=\operatorname{curl}((\operatorname{div}\boldsymbol{v})\boldsymbol{I})=3\operatorname{curl}(\operatorname{grad}\boldsymbol{v})=\mathbf{0}.$$

Hence $\operatorname{div}\boldsymbol{v}$ is a constant, which combined with (22) implies that $\boldsymbol{v}$ is a linear function. Assume $\boldsymbol{v}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{b}$ with $\boldsymbol{A}\in\mathbb{M}$ and $\boldsymbol{b}\in\mathbb{R}^3$; then (22) becomes $\boldsymbol{A}=\frac{1}{3}\operatorname{tr}(\boldsymbol{A})\boldsymbol{I}$, and consequently $\boldsymbol{v}\in\boldsymbol{RT}$.

Thus the complex (20) is exact. □

The div div complex (20) is the so-called domain complex. By [1, Theorem 2], there exist bounded regular potentials. For example, for $\boldsymbol{\tau}\in\boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$ and $\operatorname{div}\operatorname{div}\boldsymbol{\tau}=0$, there exists a regular potential $\boldsymbol{\sigma}\in\boldsymbol{H}^1(\Omega;\mathbb{T})$ s.t. $\operatorname{sym}\operatorname{curl}\boldsymbol{\sigma}=\boldsymbol{\tau}$.

3.2. **A polynomial** div div **complex.** Given a bounded domain $G\subset\mathbb{R}^3$ and a non-negative integer m, let $\mathbb{P}_m(G)$ stand for the set of all polynomials in G with the total degree no more than m, and $\mathbb{P}_m(G;\mathbb{X})$ with $\mathbb{X}$ being $\mathbb{M}$, $\mathbb{S}$, $\mathbb{K}$, $\mathbb{T}$ or $\mathbb{R}^3$ denotes the tensor or vector version. Recall that $\dim\mathbb{P}_k(G)=\binom{k+3}{3}$, $\dim\mathbb{M}=9,\dim\mathbb{S}=6,\dim\mathbb{K}=3$, and $\dim\mathbb{T}=8$. For a linear operator T defined on a finite dimensional linear space V, we have the relation

$$\dim V=\dim\ker(T)+\dim\operatorname{img}(T), \tag{23}$$

which can be used to count $\dim\operatorname{img}(T)$ provided the space $\ker(T)$ is identified and vice versa.

The polynomial de Rham complex is

$$\mathbb{R}\xrightarrow{\subset}\mathbb{P}_{k+1}(\Omega)\xrightarrow{\operatorname{grad}}\mathbb{P}_k(\Omega;\mathbb{R}^3)\xrightarrow{\operatorname{curl}}\mathbb{P}_{k-1}(\Omega;\mathbb{R}^3)\xrightarrow{\operatorname{div}}\mathbb{P}_{k-2}(\Omega)\to 0. \tag{24}$$

As Ω is topologically trivial, complex (24) is also exact, i.e., the range of each map is the kernel of the succeeding map.

Lemma 3.2. *The polynomial* div div *complex*

$$\boldsymbol{RT}\xrightarrow{\subset}\mathbb{P}_{k+2}(\Omega;\mathbb{R}^3)\xrightarrow{\operatorname{dev}\operatorname{grad}}\mathbb{P}_{k+1}(\Omega;\mathbb{T})\xrightarrow{\operatorname{sym}\operatorname{curl}}\mathbb{P}_k(\Omega;\mathbb{S})\xrightarrow{\operatorname{div}\operatorname{div}}\mathbb{P}_{k-2}(\Omega)\to 0 \tag{25}$$

is exact.

Proof. Clearly (25) is a complex due to Theorem 3.1. We then verify the exactness.
(1) $\mathbb{P}_{k+2}(\Omega;\mathbb{R}^3)\cap\ker(\operatorname{dev}\operatorname{grad})=\boldsymbol{RT}$. By the exactness of the complex (20),

$$\boldsymbol{RT}\subseteq\mathbb{P}_{k+2}(\Omega;\mathbb{R}^3)\cap\ker(\operatorname{dev}\operatorname{grad})\subseteq\boldsymbol{H}^1(\Omega;\mathbb{R}^3)\cap\ker(\operatorname{dev}\operatorname{grad})=\boldsymbol{RT}.$$

(2) $\mathbb{P}_{k+1}(\Omega;\mathbb{T})\cap\ker(\operatorname{sym}\operatorname{curl})=\operatorname{dev}\operatorname{grad}\mathbb{P}_{k+2}(\Omega;\mathbb{R}^3)$, *i.e. if* $\operatorname{sym}\operatorname{curl}\boldsymbol{\tau}=\mathbf{0}$ *and* $\boldsymbol{\tau}\in\mathbb{P}_{k+1}(\Omega;\mathbb{T})$, *then there exists a* $\boldsymbol{v}\in\mathbb{P}_{k+2}(\Omega;\mathbb{R}^3)$, *s.t.* $\boldsymbol{\tau}=\operatorname{dev}\operatorname{grad}\boldsymbol{v}$.

As $\operatorname{sym}\operatorname{curl}\boldsymbol{\tau}=\mathbf{0}$, there exists $\boldsymbol{v}\in\boldsymbol{H}^1(\Omega;\mathbb{R}^3)$ satisfying $\boldsymbol{\tau}=\operatorname{dev}\operatorname{grad}\boldsymbol{v}$, i.e. $\boldsymbol{\tau}=\operatorname{grad}\boldsymbol{v}-\frac{1}{3}(\operatorname{div}\boldsymbol{v})\boldsymbol{I}$. Then we get from (15) that

$$\operatorname{mskw}(\operatorname{grad}\operatorname{div}\boldsymbol{v})=-\operatorname{curl}((\operatorname{div}\boldsymbol{v})\boldsymbol{I})=3\operatorname{curl}(\boldsymbol{\tau}-\operatorname{grad}\boldsymbol{v})=3\operatorname{curl}\boldsymbol{\tau},$$

which implies $\operatorname{grad}\operatorname{div}\boldsymbol{v} = 3\operatorname{vskw}(\operatorname{curl}\boldsymbol{\tau}) \in \mathbb{P}_k(\Omega;\mathbb{R}^3)$. Hence $\operatorname{div}\boldsymbol{v} \in \mathbb{P}_{k+1}(\Omega)$. And thus $\operatorname{grad}\boldsymbol{v} = \boldsymbol{\tau} + \frac{1}{3}(\operatorname{div}\boldsymbol{v})\boldsymbol{I} \in \mathbb{P}_{k+1}(\Omega;\mathbb{M})$. As a result $\boldsymbol{v} \in \mathbb{P}_{k+2}(\Omega;\mathbb{R}^3)$.
(3) $\operatorname{div}\operatorname{div}\mathbb{P}_k(\Omega;\mathbb{S}) = \mathbb{P}_{k-2}(\Omega)$. Recursively applying the exactness of de Rham complex (24), we can prove $\operatorname{div}\operatorname{div}\mathbb{P}_k(\Omega;\mathbb{M}) = \mathbb{P}_{k-2}(\Omega)$. Then from (21) we have that

$$\operatorname{div}\operatorname{div}\mathbb{P}_k(\Omega;\mathbb{S}) = \operatorname{div}\operatorname{div}\mathbb{P}_k(\Omega;\mathbb{M}) = \mathbb{P}_{k-2}(\Omega).$$

(4) $\mathbb{P}_k(\Omega;\mathbb{S}) \cap \ker(\operatorname{div}\operatorname{div}) = \operatorname{sym}\operatorname{curl}\mathbb{P}_{k+1}(\Omega;\mathbb{T})$.

Obviously $\operatorname{sym}\operatorname{curl}\mathbb{P}_{k+1}(\Omega;\mathbb{T}) \subseteq (\mathbb{P}_k(\Omega;\mathbb{S}) \cap \ker(\operatorname{div}\operatorname{div}))$. As $\operatorname{div}\operatorname{div} : \mathbb{P}_k(\Omega;\mathbb{S}) \to \mathbb{P}_{k-2}(\Omega)$ is surjective by step (3), using (23), we have

$$\begin{aligned}\dim \mathbb{P}_k(\Omega;\mathbb{S}) \cap \ker(\operatorname{div}\operatorname{div}) &= \dim \mathbb{P}_k(\Omega;\mathbb{S}) - \dim \mathbb{P}_{k-2}(\Omega)\\ &= 6\binom{k+3}{3} - \binom{k+1}{3}\\ &= \frac{1}{6}(5k^3 + 36k^2 + 67k + 36).\end{aligned} \tag{26}$$

Thanks to results in steps (1) and (2), we can count the dimension of $\operatorname{sym}\operatorname{curl}$ $\mathbb{P}_{k+1}(\Omega;\mathbb{T})$

$$\begin{aligned}\dim \operatorname{sym}\operatorname{curl}\,\mathbb{P}_{k+1}(\Omega;\mathbb{T}) &= \dim \mathbb{P}_{k+1}(\Omega;\mathbb{T}) - \dim \operatorname{dev}\operatorname{grad}\mathbb{P}_{k+2}(\Omega;\mathbb{R}^3)\\ &= \dim \mathbb{P}_{k+1}(\Omega;\mathbb{T}) - (\dim \mathbb{P}_{k+2}(\Omega;\mathbb{R}^3) - \dim \boldsymbol{RT})\\ &= 8\binom{k+4}{3} - 3\binom{k+5}{3} + 4\\ &= \frac{1}{6}(5k^3 + 36k^2 + 67k + 36).\end{aligned} \tag{27}$$

We conclude that $\mathbb{P}_k(\Omega;\mathbb{S}) \cap \ker(\operatorname{div}\operatorname{div}) = \operatorname{sym}\operatorname{curl}\,\mathbb{P}_{k+1}(\Omega;\mathbb{T})$ as the dimensions match, cf. (26) and (27).

Therefore the complex (25) is exact. ☐

3.3. **A Koszul complex.** The Koszul complex corresponding to the de Rham complex (24) is

$$0 \to \mathbb{P}_{k-2}(\Omega) \xrightarrow{\boldsymbol{x}} \mathbb{P}_{k-1}(\Omega;\mathbb{R}^3) \xrightarrow{\times\boldsymbol{x}} \mathbb{P}_k(\Omega;\mathbb{R}^3) \xrightarrow{\cdot\boldsymbol{x}} \mathbb{P}_{k+1}(\Omega) \to 0, \tag{28}$$

where the operators are appended to the right of the polynomial, i.e. $v\boldsymbol{x}$, $\boldsymbol{v}\times\boldsymbol{x}$, or $\boldsymbol{v}\cdot\boldsymbol{x}$. The following complex is a generalization of the Koszul complex (28) to the $\operatorname{div}\operatorname{div}$ complex (25), where operator $\boldsymbol{\pi}_{RT} : \mathcal{C}^1(\Omega;\mathbb{R}^3) \to \boldsymbol{RT}$ is defined as

$$\boldsymbol{\pi}_{RT}\boldsymbol{v} := \boldsymbol{v}(0,0,0) + \frac{1}{3}(\operatorname{div}\boldsymbol{v})(0,0,0)\boldsymbol{x},$$

and other operators are appended to the right of the polynomial, i.e., $p\boldsymbol{x}\boldsymbol{x}^{\intercal}$, $\boldsymbol{\tau}\times\boldsymbol{x}$, or $\boldsymbol{\tau}\cdot\boldsymbol{x}$. The Koszul operator $\boldsymbol{x}\boldsymbol{x}^{\intercal}$ can also be obtained using the Poincaré operator constructed in [8], but others are simpler than those in [8].

Lemma 3.3. *The following polynomial sequence*

$$0 \xrightarrow{\subset} \mathbb{P}_{k-2}(\Omega) \xrightarrow{\boldsymbol{x}\boldsymbol{x}^{\intercal}} \mathbb{P}_k(\Omega;\mathbb{S}) \xrightarrow{\times\boldsymbol{x}} \mathbb{P}_{k+1}(\Omega;\mathbb{T}) \xrightarrow{\cdot\boldsymbol{x}} \mathbb{P}_{k+2}(\Omega;\mathbb{R}^3) \xrightarrow{\boldsymbol{\pi}_{RT}} \boldsymbol{RT} \to \boldsymbol{0} \tag{29}$$

is an exact complex.

Proof. In the sequence (29) only the mapping $\mathbb{P}_k(\Omega;\mathbb{S}) \xrightarrow{\times \boldsymbol{x}} \mathbb{P}_{k+1}(\Omega;\mathbb{T})$ is less obvious, which can be justified by the identity (16).

To verify (29) is a complex, we use the product rule (10)-(11):

$$p\boldsymbol{x}\boldsymbol{x}^{\intercal} \times \boldsymbol{x} = p\boldsymbol{x}(\boldsymbol{x}\times\boldsymbol{x})^{\intercal} = \mathbf{0}, \quad (\boldsymbol{\tau}\times\boldsymbol{x})\cdot\boldsymbol{x} = \mathbf{0}.$$

To verify $\boldsymbol{\pi}_{RT}(\boldsymbol{\tau}\cdot\boldsymbol{x}) = \mathbf{0}$ for $\boldsymbol{\tau}\in\mathbb{P}_{k+1}(\Omega;\mathbb{T})$, we use the formula

$$\operatorname{div}(\boldsymbol{\tau}\cdot\boldsymbol{x}) = \operatorname{div}(\boldsymbol{\tau}^{\intercal})\cdot\boldsymbol{x} + \operatorname{tr}\boldsymbol{\tau} = \boldsymbol{x}^{\intercal}\operatorname{div}(\boldsymbol{\tau}^{\intercal}), \tag{30}$$

and therefore evaluating at $\mathbf{0}$ is zero.

We then verify the exactness of (29).

(1) $\boldsymbol{\pi}_{RT}\mathbb{P}_{k+2}(\Omega;\mathbb{R}^3) = \boldsymbol{RT}$.

It is straightforward to verify

$$\boldsymbol{\pi}_{RT}\boldsymbol{v} = \boldsymbol{v} \quad \forall\, \boldsymbol{v}\in\boldsymbol{RT}. \tag{31}$$

Namely $\boldsymbol{\pi}_{RT}$ is a projector. Consequently, the operator $\boldsymbol{\pi}_{RT}: \mathbb{P}_{k+2}(\Omega;\mathbb{R}^3)\to\boldsymbol{RT}$ is surjective as $\boldsymbol{RT}\subset\mathbb{P}_1(\Omega;\mathbb{R}^3)$.

(2) $\mathbb{P}_{k+2}(\Omega;\mathbb{R}^3)\cap\ker(\boldsymbol{\pi}_{RT}) = \mathbb{P}_{k+1}(\Omega;\mathbb{T})\cdot\boldsymbol{x}$, *i.e. if* $\boldsymbol{\pi}_{RT}\boldsymbol{v} = \mathbf{0}$ *and* $\boldsymbol{v}\in\mathbb{P}_{k+2}(\Omega;\mathbb{R}^3)$, *then there exists a* $\boldsymbol{\tau}\in\mathbb{P}_{k+1}(\Omega;\mathbb{T})$, *s.t.* $\boldsymbol{v} = \boldsymbol{\tau}\cdot\boldsymbol{x}$.

Since $\boldsymbol{v}(0,0,0) = \mathbf{0}$, by the fundamental theorem of calculus,

$$\boldsymbol{v} = \left(\int_0^1 \operatorname{grad}\boldsymbol{v}(t\boldsymbol{x})\,\mathrm{d}t\right)\boldsymbol{x}.$$

Using the decomposition (12), we conclude that there exist $\boldsymbol{\tau}_1\in\mathbb{P}_{k+1}(\Omega;\mathbb{T})$ and $q\in\mathbb{P}_{k+1}(\Omega)$ such that $\boldsymbol{v} = \boldsymbol{\tau}_1\boldsymbol{x} + q\boldsymbol{x}$. Again by (30), we have

$$\boldsymbol{\pi}_{RT}(q\boldsymbol{x}) = \boldsymbol{\pi}_{RT}\boldsymbol{v} - \boldsymbol{\pi}_{RT}(\boldsymbol{\tau}_1\boldsymbol{x}) = \mathbf{0},$$

which indicates $(\operatorname{div}(q\boldsymbol{x}))(0,0,0) = 0$. As $\operatorname{div}(q\boldsymbol{x}) = (\boldsymbol{x}\cdot\nabla)q + 3q$, we conclude $q(0,0,0) = 0$. Again using the fundamental theorem of calculus to conclude that there exists $\boldsymbol{q}_1\in\mathbb{P}_k(\Omega;\mathbb{R}^3)$ such that $q = \boldsymbol{q}_1^{\intercal}\boldsymbol{x}$. Taking $\boldsymbol{\tau} = \boldsymbol{\tau}_1 + \frac{3}{2}\boldsymbol{x}\boldsymbol{q}_1^{\intercal} - \frac{1}{2}\boldsymbol{q}_1^{\intercal}\boldsymbol{x}\boldsymbol{I}\in\mathbb{P}_{k+1}(\Omega;\mathbb{T})$, we get

$$\boldsymbol{\tau}\boldsymbol{x} = \boldsymbol{\tau}_1\boldsymbol{x} + \boldsymbol{x}\boldsymbol{q}_1^{\intercal}\boldsymbol{x} = \boldsymbol{\tau}_1\boldsymbol{x} + q\boldsymbol{x} = \boldsymbol{v}.$$

(3) $\mathbb{P}_k(\Omega;\mathbb{S})\cap\ker((\cdot)\times\boldsymbol{x}) = \mathbb{P}_{k-2}(\Omega)\boldsymbol{x}\boldsymbol{x}^{\intercal}$, *i.e. if* $\boldsymbol{\tau}\times\boldsymbol{x} = \mathbf{0}$ *and* $\boldsymbol{\tau}\in\mathbb{P}_k(\Omega;\mathbb{S})$, *then there exists a* $q\in\mathbb{P}_{k-2}(\Omega)$, *s.t.* $\boldsymbol{\tau} = q\boldsymbol{x}\boldsymbol{x}^{\intercal}$.

Thanks to $\boldsymbol{\tau}\times\boldsymbol{x} = \mathbf{0}$, there exists $\boldsymbol{v}\in\mathbb{P}_{k-1}(\Omega;\mathbb{R}^3)$ such that $\boldsymbol{\tau} = \boldsymbol{v}\boldsymbol{x}^{\intercal}$. By the symmetry of $\boldsymbol{\tau}$, it follows

$$(\boldsymbol{x}\boldsymbol{v}^{\intercal})\times\boldsymbol{x} = (\boldsymbol{v}\boldsymbol{x}^{\intercal})^{\intercal}\times\boldsymbol{x} = \boldsymbol{\tau}\times\boldsymbol{x} = \mathbf{0},$$

which indicates $\boldsymbol{v}\times\boldsymbol{x} = \mathbf{0}$. Then there exists $q\in\mathbb{P}_{k-2}(\Omega)$ satisfying $\boldsymbol{v} = q\boldsymbol{x}$. Hence $\boldsymbol{\tau} = q\boldsymbol{x}\boldsymbol{x}^{\intercal}$.

(4) $\mathbb{P}_{k+1}(\Omega;\mathbb{T})\cap\ker((\cdot)\cdot\boldsymbol{x}) = \mathbb{P}_k(\Omega;\mathbb{S})\times\boldsymbol{x}$.

It follows from steps (1) and (2) that

$$\begin{aligned}\dim(\mathbb{P}_{k+1}(\Omega;\mathbb{T})\cap\ker((\cdot)\cdot\boldsymbol{x})) &= \dim\mathbb{P}_{k+1}(\Omega;\mathbb{T}) - \dim(\mathbb{P}_{k+1}(\Omega;\mathbb{T})\boldsymbol{x})\\ &= \dim\mathbb{P}_{k+1}(\Omega;\mathbb{T}) - \dim(\mathbb{P}_{k+2}(\Omega;\mathbb{R}^3)\cap\ker(\boldsymbol{\pi}_{RT}))\\ &= \dim\mathbb{P}_{k+1}(\Omega;\mathbb{T}) - \dim\mathbb{P}_{k+2}(\Omega;\mathbb{R}^3) + 4\\ &= \frac{1}{6}(5k^3 + 36k^2 + 67k + 36).\end{aligned} \tag{32}$$

And by step (3),

$$\begin{aligned}\dim(\mathbb{P}_k(\Omega;\mathbb{S})\times\boldsymbol{x}) &= \dim\mathbb{P}_k(\Omega;\mathbb{S}) - \dim(\mathbb{P}_k(\Omega;\mathbb{S})\cap\ker((\cdot)\times\boldsymbol{x}))\\ &= \dim\mathbb{P}_k(\Omega;\mathbb{S}) - \dim(\mathbb{P}_{k-2}(\Omega)\boldsymbol{x}\boldsymbol{x}^{\intercal})\\ &= \frac{1}{6}(5k^3+36k^2+67k+36),\end{aligned}$$

which together with (32) implies $\mathbb{P}_{k+1}(\Omega;\mathbb{T})\cap\ker((\cdot)\cdot\boldsymbol{x}) = \mathbb{P}_k(\Omega;\mathbb{S})\times\boldsymbol{x}$.

Therefore the complex (29) is exact. □

3.4. **Decomposition of polynomial tensors.** Those two complexes (25) and (29) can be combined into one double-direction complex

$$\boldsymbol{RT} \underset{\boldsymbol{\pi}_{RT}}{\overset{\subset}{\rightleftarrows}} \mathbb{P}_{k+2}(\Omega;\mathbb{R}^3) \underset{\cdot\boldsymbol{x}}{\overset{\operatorname{dev}\operatorname{grad}}{\rightleftarrows}} \mathbb{P}_{k+1}(\Omega;\mathbb{T}) \underset{\times\boldsymbol{x}}{\overset{\operatorname{sym}\operatorname{curl}}{\rightleftarrows}} \mathbb{P}_k(\Omega;\mathbb{S}) \underset{\boldsymbol{x}\boldsymbol{x}^{\intercal}}{\overset{\operatorname{div}\operatorname{div}}{\rightleftarrows}} \mathbb{P}_{k-2}(\Omega) \underset{\supset}{\rightleftarrows} 0\,.$$

Unlike the Koszul complex for vector functions, we do not have the identity property applied to homogenous polynomials. Fortunately decomposition of polynomial spaces using Koszul and differential operators still holds.

Let $\mathbb{H}_k(\Omega) := \mathbb{P}_k(\Omega)/\mathbb{P}_{k-1}(\Omega)$ be the space of homogeneous polynomials of degree k. Then by Euler's formula

$$\boldsymbol{x}\cdot\nabla q = kq \quad \forall\, q\in\mathbb{H}_k(\Omega). \tag{33}$$

Due to (33), we have

$$\mathbb{P}_k(\Omega)\cap\ker(\boldsymbol{x}\cdot\nabla) = \mathbb{P}_0(\Omega), \tag{34}$$

$$\mathbb{P}_k(\Omega)\cap\ker(\boldsymbol{x}\cdot\nabla+\ell) = \{0\} \tag{35}$$

for any positive number ℓ.

It follows from (31) and the complex (29) that

$$\mathbb{P}_{k+2}(\Omega;\mathbb{R}^3) = \mathbb{P}_{k+1}(\Omega;\mathbb{T})\boldsymbol{x}\oplus\boldsymbol{RT}.$$

We then move to the space $\mathbb{P}_{k+1}(\Omega;\mathbb{T})$.

Lemma 3.4. *We have the decomposition*

$$\mathbb{P}_{k+1}(\Omega;\mathbb{T}) = (\mathbb{P}_k(\Omega;\mathbb{S})\times\boldsymbol{x})\oplus\operatorname{dev}\operatorname{grad}\mathbb{P}_{k+2}(\Omega;\mathbb{R}^3). \tag{36}$$

Proof. Let us count the dimension.

$$\dim\mathbb{P}_{k+1}(\Omega;\mathbb{T}) = 8\binom{k+4}{3},$$

while by the exactness of the Koszul complex (29)

$$\begin{aligned}\dim\mathbb{P}_k(\Omega;\mathbb{S})\times\boldsymbol{x} &= \dim\mathbb{P}_k(\Omega;\mathbb{S}) - \boldsymbol{x}\boldsymbol{x}^{\intercal}\mathbb{P}_{k-2}(\Omega)\\ &= 6\binom{k+3}{3} - \binom{k+1}{3},\\ \dim\operatorname{dev}\operatorname{grad}\mathbb{P}_{k+2}(\Omega;\mathbb{R}^3) &= \dim\mathbb{P}_{k+2}(\Omega;\mathbb{R}^3) - \ker(\operatorname{dev}\operatorname{grad})\\ &= 3\binom{k+5}{3} - 4.\end{aligned}$$

By a direct computation, the dimension of space on the left hand side is the summation of the dimension of the two spaces on the right hand side in (36). So we only need to prove that the sum in (36) is a direct sum.

Take $\boldsymbol{\tau} = \operatorname{dev}\operatorname{grad}\boldsymbol{q}$ for some $\boldsymbol{q} \in \mathbb{P}_{k+2}(\Omega;\mathbb{R}^3)$, and also assume $\boldsymbol{\tau} \in \mathbb{P}_k(\Omega;\mathbb{S})\times\boldsymbol{x}$. We have $\boldsymbol{\tau}\cdot\boldsymbol{x} = (\operatorname{dev}\operatorname{grad}\boldsymbol{q})\cdot\boldsymbol{x} = \boldsymbol{0}$, that is

$$(\operatorname{grad}\boldsymbol{q})\cdot\boldsymbol{x} = \frac{1}{3}(\operatorname{div}\boldsymbol{q})\boldsymbol{x}. \tag{37}$$

Since $\operatorname{div}((\operatorname{grad}\boldsymbol{q})\cdot\boldsymbol{x}) = (1+\boldsymbol{x}\cdot\operatorname{grad})\operatorname{div}\boldsymbol{q}$, applying the divergence operator div on both sides of (37) gives

$$(1+\boldsymbol{x}\cdot\operatorname{grad})\operatorname{div}\boldsymbol{q} = \frac{1}{3}(3+\boldsymbol{x}\cdot\operatorname{grad})\operatorname{div}\boldsymbol{q}.$$

Hence $(\boldsymbol{x}\cdot\operatorname{grad})\operatorname{div}\boldsymbol{q} = 0$, which together with (34) indicates $\operatorname{div}\boldsymbol{q}\in\mathbb{P}_0(\Omega)$. Due to (37), $(\operatorname{grad}\boldsymbol{q})\cdot\boldsymbol{x}$ is a linear function. It follows from (33) that $\boldsymbol{q}\in\mathbb{P}_1(\Omega;\mathbb{R}^3)$ and $\boldsymbol{\tau} = \operatorname{dev}\operatorname{grad}\boldsymbol{q}\in\mathbb{P}_0(\Omega;\mathbb{T})$, which together with $\boldsymbol{\tau}\cdot\boldsymbol{x}=\boldsymbol{0}$ implies $\boldsymbol{\tau}=\boldsymbol{0}$. ☐

Finally we present a decomposition of space $\mathbb{P}_k(\Omega;\mathbb{S})$. Let

$$\mathbb{C}_k(\Omega;\mathbb{S}) := \operatorname{sym}\operatorname{curl}\mathbb{P}_{k+1}(\Omega;\mathbb{T}),\quad \mathbb{C}_k^{\oplus}(\Omega;\mathbb{S}) := \boldsymbol{x}\boldsymbol{x}^{\intercal}\mathbb{P}_{k-2}(\Omega).$$

Their dimensions are

$$\dim\mathbb{C}_k(\Omega;\mathbb{S}) = \frac{1}{6}(5k^3+36k^2+67k+36),\quad \dim\mathbb{C}_k^{\oplus}(\Omega;\mathbb{S}) = \frac{1}{6}(k^3-k). \tag{38}$$

The calculation of $\dim\mathbb{C}_k^{\oplus}(\Omega;\mathbb{S})$ is easy and $\dim\mathbb{C}_k(\Omega;\mathbb{S})$ is detailed in (27).

Lemma 3.5. *We have*

(i) $\operatorname{div}\operatorname{div}(\boldsymbol{x}\boldsymbol{x}^{\intercal}q) = (k+4)(k+3)q$ *for any* $q\in\mathbb{H}_k(\Omega)$.
(ii) $\operatorname{div}\operatorname{div}:\mathbb{C}_k^{\oplus}(\Omega;\mathbb{S})\to\mathbb{P}_{k-2}(\Omega)$ *is a bijection.*
(iii) $\mathbb{P}_k(\Omega;\mathbb{S}) = \mathbb{C}_k(\Omega;\mathbb{S})\oplus\mathbb{C}_k^{\oplus}(\Omega;\mathbb{S})$.

Proof. Since $\operatorname{div}(\boldsymbol{x}\boldsymbol{x}^{\intercal}q) = (\operatorname{div}(\boldsymbol{x}q)+q)\boldsymbol{x}$ and $\operatorname{div}(\boldsymbol{x}q) = (\boldsymbol{x}\cdot\nabla)q+3q$, we get

$$\operatorname{div}\operatorname{div}(\boldsymbol{x}\boldsymbol{x}^{\intercal}q) = \operatorname{div}(((\boldsymbol{x}\cdot\nabla+4)q)\boldsymbol{x}) = (\boldsymbol{x}\cdot\nabla+3)(\boldsymbol{x}\cdot\nabla+4)q. \tag{39}$$

Hence property (i) follows from (33). Property (ii) is obtained by writing $\mathbb{P}_{k-2}(\Omega) = \bigoplus_{i=0}^{k-2}\mathbb{H}_i(\Omega)$. Now we prove property (iii). First the dimension of space on the left hand side is the summation of the dimension of the two spaces on the right hand side in (iii). Assume $q\in\mathbb{P}_{k-2}(\Omega)$ satisfies $\boldsymbol{x}\boldsymbol{x}^{\intercal}q\in\mathbb{C}_k(\Omega;\mathbb{S})$, which means

$$\operatorname{div}\operatorname{div}(\boldsymbol{x}\boldsymbol{x}^{\intercal}q) = 0.$$

Thus $q=0$ from (39) and (35) and consequently property (iii) holds. ☐

For the simplification of the degrees of freedom, we need another decomposition of the symmetric tensor polynomial space, which can be derived from the polynomial Hessian complex

$$\mathbb{P}_1(\Omega)\underset{\pi_1 v}{\overset{\subset}{\rightleftarrows}}\mathbb{P}_{k+2}(\Omega)\underset{\boldsymbol{x}^{\intercal}\boldsymbol{\tau}\boldsymbol{x}}{\overset{\text{hess}}{\rightleftarrows}}\mathbb{P}_k(\Omega;\mathbb{S})\underset{\operatorname{sym}(\boldsymbol{\tau}\times\boldsymbol{x})}{\overset{\text{curl}}{\rightleftarrows}}\mathbb{P}_{k-1}(\Omega;\mathbb{T})\underset{\operatorname{dev}(\boldsymbol{v}\boldsymbol{x}^{\intercal})}{\overset{\text{div}}{\rightleftarrows}}\mathbb{P}_{k-2}(\Omega;\mathbb{R}^3)\underset{\supset}{\rightleftarrows}0\,, \tag{40}$$

where $\pi_1 v := v(0,0,0)+\boldsymbol{x}^{\intercal}(\nabla v)(0,0,0)$. A proof of the exactness of (40) is similar to that of Lemma 3.3 and can be found in [5]. Based on (40), we have the following decomposition of symmetric polynomial tensors.

Lemma 3.6. *It holds*

$$\mathbb{P}_k(\Omega;\mathbb{S}) = \nabla^2\mathbb{P}_{k+2}(\Omega)\oplus\operatorname{sym}(\mathbb{P}_{k-1}(\Omega;\mathbb{T})\times\boldsymbol{x}). \tag{41}$$

Proof. Obviously the space on the right is contained in the space on the left. We then count the dimensions of spaces on both sides:

$$\dim \mathbb{P}_k(\Omega;\mathbb{S}) = 6\binom{k+3}{3} = (k+3)(k+2)(k+1),$$

$$\dim \nabla^2\mathbb{P}_{k+2}(\Omega) = \dim \mathbb{P}_{k+2}(\Omega) - \dim \mathbb{P}_1(\Omega) = \binom{k+5}{3} - 4,$$

$$\begin{aligned}\dim \operatorname{sym}(\mathbb{P}_{k-1}(\Omega;\mathbb{T})\times \boldsymbol{x}) &= \dim \mathbb{P}_{k-1}(\Omega;\mathbb{T}) - \dim \mathbb{P}_{k-2}(\Omega;\mathbb{R}^3)\\ &= 8\binom{k+2}{3} - 3\binom{k+1}{3} = \frac{1}{6}(k+1)k(5k+19).\end{aligned} \tag{42}$$

Then by a direct calculation,

$$\dim \nabla^2\mathbb{P}_{k+2}(\Omega) + \dim \operatorname{sym}(\mathbb{P}_{k-1}(\Omega;\mathbb{T})\times \boldsymbol{x}) = \dim \mathbb{P}_k(\Omega;\mathbb{S}) = k^3+6k^2+11k+6.$$

We only need to prove that the sum is direct.

For any $\boldsymbol{\tau} = \nabla^2 q$ with $q\in\mathbb{P}_{k+2}(\Omega)$ satisfying $\boldsymbol{\tau}\in\operatorname{sym}(\mathbb{P}_{k-1}(\Omega;\mathbb{T})\times\boldsymbol{x})$, it follows $(\boldsymbol{x}\cdot\nabla)((\boldsymbol{x}\cdot\nabla)q - q) = \boldsymbol{x}^{\intercal}(\nabla^2 q)\boldsymbol{x} = 0$. Applying (34) and (33), we get $q\in\mathbb{P}_1(\Omega)$ and $\nabla^2 q = 0$. Thus the decomposition (41) holds. $\square$

Similarly for a two-dimensional domain $F\subset\mathbb{R}^2$, we have the following $\operatorname{div}\operatorname{div}$ polynomial complex and its Koszul complex

$$\boldsymbol{RT} \underset{\boldsymbol{\pi}_{RT}}{\overset{\subset}{\rightleftarrows}} \mathbb{P}_{k+1}(F;\mathbb{R}^2) \underset{\cdot\boldsymbol{x}^{\perp}}{\overset{\operatorname{sym}\operatorname{curl}_F}{\rightleftarrows}} \mathbb{P}_k(F;\mathbb{S}) \underset{\boldsymbol{x}\boldsymbol{x}^{\intercal}}{\overset{\operatorname{div}_F\operatorname{div}_F}{\rightleftarrows}} \mathbb{P}_{k-2}(F) \underset{\supset}{\rightleftarrows} 0\,, \tag{43}$$

where $\boldsymbol{\pi}_{RT}\boldsymbol{v} := \boldsymbol{v}(0,0) + \frac{1}{2}(\operatorname{div}\boldsymbol{v})(0,0)\boldsymbol{x}$, $\boldsymbol{x}^{\perp} = (x_2,-x_1)^{\intercal}$ is the rotation of $\boldsymbol{x} = (x_1,x_2)^{\intercal}$. A two-dimensional Hessian polynomial complex and its Koszul complex are

$$\mathbb{P}_1(F) \underset{\pi_1}{\overset{\subset}{\rightleftarrows}} \mathbb{P}_{k+2}(F) \underset{\boldsymbol{x}^{\intercal}\boldsymbol{\tau}\boldsymbol{x}}{\overset{\nabla_F^2}{\rightleftarrows}} \mathbb{P}_k(F;\mathbb{S}) \underset{\operatorname{sym}(\boldsymbol{x}^{\perp}\boldsymbol{v}^{\intercal})}{\overset{\operatorname{rot}_F}{\rightleftarrows}} \mathbb{P}_{k-1}(F) \underset{\supset}{\rightleftarrows} 0\,, \tag{44}$$

where $\pi_1 v := v(0,0) + \boldsymbol{x}^{\intercal}(\nabla v)(0,0)$. Verification of the exactness of these two complexes can be found in [6] which leads to the decompositions

$$\mathbb{P}_k(F;\mathbb{S}) = \operatorname{sym}\operatorname{curl}_F\, \mathbb{P}_{k+1}(F;\mathbb{R}^2) \oplus \boldsymbol{x}\boldsymbol{x}^{\intercal}\mathbb{P}_{k-2}(F),$$

$$\mathbb{P}_k(F;\mathbb{S}) = \nabla_F^2\mathbb{P}_{k+2}(F) \oplus \operatorname{sym}(\boldsymbol{x}^{\perp}\mathbb{P}_{k-1}(F;\mathbb{R}^2)).$$

4. Green's identities and traces

We first present a Green's identity based on which we can characterize two traces of $\boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$ on polyhedrons and give a sufficient continuity condition for a piecewise smooth function to be in $\boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$.

4.1. **Notation.** Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of polyhedral meshes of Ω. Our finite element spaces are constructed for tetrahedrons but some results, e.g., traces and Green's formula etc., hold for general polyhedrons. For each element $K\in\mathcal{T}_h$, denote by $\boldsymbol{n}_K$ the unit outward normal vector to ∂K, which will be abbreviated as $\boldsymbol{n}$ for simplicity. Let $\mathcal{F}_h$, $\mathcal{F}_h^i$, $\mathcal{E}_h$, $\mathcal{E}_h^i$, $\mathcal{V}_h$ and $\mathcal{V}_h^i$ be the union of all faces, interior faces, edges, interior edges, vertices and interior vertices of the partition $\mathcal{T}_h$, respectively. For any $F\in\mathcal{F}_h$, fix a unit normal vector $\boldsymbol{n}_F$ and two unit tangent vectors $\boldsymbol{t}_{F,1}$ and $\boldsymbol{t}_{F,2}$, which will be abbreviated as $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ without causing any confusions. For any $e\in\mathcal{E}_h$, fix a unit tangent vector $\boldsymbol{t}_e$ and two unit normal vectors $\boldsymbol{n}_{e,1}$ and

$\boldsymbol{n}_{e,2}$, which will be abbreviated as $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ without causing any confusions. For K being a polyhedron, denote by $\mathcal{F}(K)$, $\mathcal{E}(K)$ and $\mathcal{V}(K)$ the set of all faces, edges and vertices of K, respectively. For any $F \in \mathcal{F}_h$, let $\mathcal{E}(F)$ be the set of all edges of F. And for each $e \in \mathcal{E}(F)$, denote by $\boldsymbol{n}_{F,e}$ the unit vector being parallel to F and outward normal to ∂F. Furthermore, set

$$\mathcal{F}^i(K) := \mathcal{F}(K) \cap \mathcal{F}_h^i, \quad \mathcal{E}^i(F) := \mathcal{E}(F) \cap \mathcal{E}_h^i.$$

4.2. **Green's identities.** We first derive a Green's identity for smooth functions on polyhedrons.

Lemma 4.1 (Green's identity for div div operator in 3D)**.** *Let K be a polyhedron, and let $\boldsymbol{\tau} \in \mathcal{C}^2(K;\mathbb{S})$ and $v \in H^2(K)$. Then we have*

$$(\operatorname{div}\operatorname{div}\boldsymbol{\tau}, v)_K = (\boldsymbol{\tau}, \nabla^2 v)_K - \sum_{F\in\mathcal{F}(K)}\sum_{e\in\mathcal{E}(F)} (\boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}, v)_e$$

$$- \sum_{F\in\mathcal{F}(K)} \left[(\boldsymbol{n}_e^{\intercal}\boldsymbol{\tau}\boldsymbol{n}, \partial_n v)_F - (2\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}_e) + \partial_n(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}), v)_F\right]. \tag{45}$$

Proof. We start from the standard integration by parts

$$\begin{aligned}(\operatorname{div}\operatorname{div}\boldsymbol{\tau}, v)_K &= -(\operatorname{div}\boldsymbol{\tau}, \nabla v)_K + \sum_{F\in\mathcal{F}(K)} (\boldsymbol{n}^{\intercal}\operatorname{div}\boldsymbol{\tau}, v)_F \\ &= \left(\boldsymbol{\tau}, \nabla^2 v\right)_K - \sum_{F\in\mathcal{F}(K)} (\boldsymbol{\tau}\boldsymbol{n}, \nabla v)_F + \sum_{F\in\mathcal{F}(K)} (\boldsymbol{n}^{\intercal}\operatorname{div}\boldsymbol{\tau}, v)_F.\end{aligned}$$

We then decompose $\nabla v = \partial_n v\boldsymbol{n} + \nabla_F v$ and apply the Stokes theorem to get

$$\begin{aligned}(\boldsymbol{\tau}\boldsymbol{n}, \nabla v)_F &= (\boldsymbol{\tau}\boldsymbol{n}, \partial_n v\boldsymbol{n} + \nabla_F v)_F \\ &= (\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}, \partial_n v)_F - (\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}), v)_F + \sum_{e\in\mathcal{E}(F)} (\boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}, v)_e.\end{aligned}$$

Now we rewrite the term

$$(\boldsymbol{n}^{\intercal}\operatorname{div}\boldsymbol{\tau}, v)_F = (\operatorname{div}(\boldsymbol{\tau}\boldsymbol{n}), v)_F = (\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}), v)_F + (\partial_n(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}), v)_F.$$

Thus the Green's identity (45) follows by merging all terms. ☐

When the domain is smooth in the sense that $\mathcal{E}(K)$ is an empty set, the term $\sum_{F\in\mathcal{F}(K)}\sum_{e\in\mathcal{E}(F)}(\boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}, v)_e$ disappears. When v is continuous on edge e, this term will define a jump of the tensor $\boldsymbol{\tau}$.

A similar Green's identity in two dimensions is included here for later usage. To avoid confusion with the three-dimensional version, $\boldsymbol{n}_e$ is used to emphasize it is a normal vector of edge e of polygon F and differential operators with subscript F are used.

Lemma 4.2 (Green's identity for div div operator in 2D)**.** *Let F be a polygon, and let $\boldsymbol{\tau} \in \mathcal{C}^2(F;\mathbb{S})$ and $v \in H^2(F)$. Then we have*

$$\begin{aligned}(\operatorname{div}_F\operatorname{div}_F\boldsymbol{\tau}, v)_F &= (\boldsymbol{\tau}, \nabla_F^2 v)_F - \sum_{e\in\mathcal{E}(K)}\sum_{\delta\in\partial e} \operatorname{sign}_{e,\delta}(\boldsymbol{t}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_e)(\delta)v(\delta) \\ &\quad - \sum_{e\in\mathcal{E}(K)} \left[(\boldsymbol{n}_e^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_e, \partial_n v)_e - (2\partial_t(\boldsymbol{t}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_e) + \partial_n(\boldsymbol{n}_e^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_e), v)_e\right],\end{aligned}$$

where

$$\mathrm{sign}_{e,\delta} := \begin{cases} 1, & \text{if } \delta \text{ is the end point of } e, \\ -1, & \text{if } \delta \text{ is the start point of } e. \end{cases}$$

Here the trace $2\partial_t(\boldsymbol{t}^\intercal\boldsymbol{\tau}\boldsymbol{n}_e) + \partial_n(\boldsymbol{n}_e^\intercal\boldsymbol{\tau}\boldsymbol{n}_e) = \partial_t(\boldsymbol{t}^\intercal\boldsymbol{\tau}\boldsymbol{n}_e) + \boldsymbol{n}_e^\intercal \operatorname{div}\boldsymbol{\tau}$ is called the effective transverse shear force respectively for $\boldsymbol{\tau}$ being a moment and $\boldsymbol{n}_e^\intercal\boldsymbol{\tau}\boldsymbol{n}_e$ is the normal bending moment in the context of elastic mechanics [11].

4.3. **Traces and continuity across the boundary.** The Green's identity (45) motivates the definition of two trace operators for function $\boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{div}\operatorname{div}, K; \mathbb{S})$:

$$\begin{aligned} \operatorname{tr}_1(\boldsymbol{\tau}) &= \boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n}, \\ \operatorname{tr}_2(\boldsymbol{\tau}) &= 2\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}) + \partial_n(\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n}) = \operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}) + \boldsymbol{n}^\intercal\operatorname{div}\boldsymbol{\tau}. \end{aligned}$$

We first recall the trace of the space $\boldsymbol{H}(\operatorname{div}\operatorname{div}, K; \mathbb{S})$ on the boundary of polyhedron K (cf. [12, Lemma 3.2] and [20,23]). Let $H_{00}^{1/2}(F)$ be the closure of $\mathcal{C}_0^\infty(F)$ with respect to the norm $\|\cdot\|_{H^{1/2}(\partial K)}$, which includes all functions in $H^{1/2}(F)$ whose continuation to the whole boundary ∂K by zero belongs to $H^{1/2}(\partial K)$. Define the following trace spaces

$$\begin{aligned} H_{n,0}^{1/2}(\partial K) &:= \{\partial_n v|_{\partial K} : v \in H^2(K) \cap H_0^1(K)\} \\ &= \{g \in L^2(\partial K) : g|_F \in H_{00}^{1/2}(F) \ \ \forall\, F \in \mathcal{F}(K)\} \end{aligned}$$

with norm

$$\|g\|_{H_{n,0}^{1/2}(\partial K)} := \inf_{\substack{v \in H^2(K)\cap H_0^1(K) \\ \partial_n v = g}} \|v\|_2,$$

and

$$H_{t,0}^{3/2}(\partial K) := \{v|_{\partial K} : v \in H^2(K), \partial_n v|_{\partial K} = 0, v|_e = 0 \text{ for each edge } e \in \mathcal{E}(K)\}$$

with norm

$$\|g\|_{H_{t,0}^{3/2}(\partial K)} := \inf_{\substack{v \in H^2(K) \\ \partial_n v = 0, v = g}} \|v\|_2.$$

Let $H_n^{-1/2}(\partial K) := (H_{n,0}^{1/2}(\partial K))'$ for tr_1, and $H_t^{-3/2}(\partial K) := (H_{t,0}^{3/2}(\partial K))'$ for tr_2.

Lemma 4.3 (Lemma 3.2 in [12])**.** *For any $\boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{div}\operatorname{div}, K; \mathbb{S})$, it holds*

$$\|\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n}\|_{H_n^{-1/2}(\partial K)} + \|2\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}) + \partial_n(\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n})\|_{H_t^{-3/2}(\partial K)} \lesssim \|\boldsymbol{\tau}\|_{\boldsymbol{H}(\operatorname{div}\operatorname{div},K)}.$$

Conversely, for any $g_n \in H_n^{-1/2}(\partial K)$ and $g_t \in H_t^{-3/2}(\partial K)$, there exists some $\boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{div}\operatorname{div}, K; \mathbb{S})$ such that

$$\begin{gathered} \boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n}|_{\partial K} = g_n, \quad 2\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}) + \partial_n(\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n}) = g_t, \\ \|\boldsymbol{\tau}\|_{\boldsymbol{H}(\operatorname{div}\operatorname{div},K)} \lesssim \|g_n\|_{H_n^{-1/2}(\partial K)} + \|g_t\|_{H_t^{-3/2}(\partial K)}. \end{gathered}$$

The hidden constants depend only the shape of the domain K.

Notice that the term $(\boldsymbol{n}_{F,e}^\intercal\boldsymbol{\tau}\boldsymbol{n}, v)_e$ in the Green's identity (45) is not covered by Lemma 4.3. Indeed, the full characterization of the trace of $\boldsymbol{H}(\operatorname{div}\operatorname{div}, K; \mathbb{S})$ is defined by $(\operatorname{div}\operatorname{div}\boldsymbol{\tau}, v) - \left(\boldsymbol{\tau}, \nabla^2 v\right)_K$, which cannot be equivalently decoupled [12, Lemma 3.2]. It is possible, however, to face-wisely localize the trace if imposing additional smoothness.

We present a sufficient continuity condition for piecewise smooth functions to be in $\boldsymbol{H}(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S})$.

Lemma 4.4 (cf. Proposition 3.6 in [12]). *Let* $\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega;\mathbb{S})$ *such that*

(i) $\boldsymbol{\tau}|_K \in \boldsymbol{H}(\operatorname{div}\operatorname{div}, K;\mathbb{S})$ *for each polyhedron* $K \in \mathcal{T}_h$*;*
(ii) $(2\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}_F) + \partial_{n_F}(\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n}))|_F \in L^2(F)$ *is single-valued for each* $F \in \mathcal{F}_h^i$*;*
(iii) $(\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n})|_F \in L^2(F)$ *is single-valued for each* $F \in \mathcal{F}_h^i$*;*
(iv) $(\boldsymbol{n}_i^\intercal\boldsymbol{\tau}\boldsymbol{n}_j)|_e \in L^2(e)$ *is single-valued for each* $e \in \mathcal{E}_h^i$*,* $i,j = 1,2$*,*

then $\boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$*.*

Proof. For any $v \in \mathcal{C}_0^\infty(\Omega)$, we get from the Green's identity (45) that

$$(\boldsymbol{\tau}, \nabla^2 v) = \sum_{K\in\mathcal{T}_h} (\operatorname{div}\operatorname{div}\boldsymbol{\tau}, v)_K + \sum_{K\in\mathcal{T}_h}\sum_{F\in\mathcal{F}^i(K)}\sum_{e\in\mathcal{E}^i(F)} (\boldsymbol{n}_{F,e}^\intercal\boldsymbol{\tau}\boldsymbol{n}, v)_e$$
$$+ \sum_{K\in\mathcal{T}_h}\sum_{F\in\mathcal{F}^i(K)} \left[(\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n}, \partial_n v)_F - (2\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}) + \partial_n(\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n}), v)_F\right].$$

Since the terms in (ii)-(iv) are single-valued and each interior face is repeated twice in the summation with opposite orientation, it follows

$$\langle \operatorname{div}\operatorname{div}\boldsymbol{\tau}, v\rangle = \sum_{K\in\mathcal{T}_h} (\operatorname{div}\operatorname{div}\boldsymbol{\tau}, v)_K.$$

Thus we have $\boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$ by the definition of derivatives of the distribution, and $(\operatorname{div}\operatorname{div}\boldsymbol{\tau})|_K = \operatorname{div}\operatorname{div}(\boldsymbol{\tau}|_K)$ for each $K \in \mathcal{T}_h$. ☐

For any piecewise smooth $\boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$, the single-valued term $(\boldsymbol{n}_i^\intercal\boldsymbol{\tau}\boldsymbol{n}_j)|_e$ in Lemma 4.4(iv) implies that there is some compatible condition for $\boldsymbol{\tau}$ at each vertex $\delta \in \mathcal{V}_h^i$. Indeed, for any $\delta \in \mathcal{V}_h^i$ and $F \in \mathcal{F}_h^i$ with δ being a vertex of F, let $\boldsymbol{n}_1 = \boldsymbol{t}_1 \times \boldsymbol{n}_F$ and $\boldsymbol{n}_2 = \boldsymbol{t}_2 \times \boldsymbol{n}_F$, where $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ are the unit tangential vectors of two edges of F sharing δ. Then by (iv) we have

$$[\![\boldsymbol{n}_1^\intercal\boldsymbol{\tau}\boldsymbol{n}_1]\!]_F(\delta) = [\![\boldsymbol{n}_2^\intercal\boldsymbol{\tau}\boldsymbol{n}_2]\!]_F(\delta) = [\![\boldsymbol{n}_F^\intercal\boldsymbol{\tau}\boldsymbol{n}_F]\!]_F(\delta) = [\![\boldsymbol{n}_1^\intercal\boldsymbol{\tau}\boldsymbol{n}_F]\!]_F(\delta) = [\![\boldsymbol{n}_2^\intercal\boldsymbol{\tau}\boldsymbol{n}_F]\!]_F(\delta) = 0,$$

where $[\![\cdot]\!]_F$ is the jump across F. Hence this suggests the tensor value at vertex as the degree of freedom when defining the finite element.

Continuity of $(\boldsymbol{n}_i^\intercal\boldsymbol{\tau}\boldsymbol{n}_j)|_e$ is a sufficient but not necessary condition for functions in $\boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$. Sufficient and necessary conditions are presented in [12, Proposition 3.6].

5. Divdiv conforming finite elements

In this section we construct conforming finite element space for $\boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$ and prove the unisolvence.

5.1. Finite element spaces for symmetric tensors.

Let K be a tetrahedron. Take the space of shape functions

$$\boldsymbol{\Sigma}_{\ell,k}(K) := \mathbb{C}_\ell(K;\mathbb{S}) \oplus \mathbb{C}_k^\oplus(K;\mathbb{S})$$

with $k \geq 3$ and $\ell \geq \max\{k-1, 3\}$. Recall that

$$\mathbb{C}_\ell(K;\mathbb{S}) = \operatorname{sym}\operatorname{curl}\mathbb{P}_{\ell+1}(K;\mathbb{T}), \quad \mathbb{C}_k^\oplus(K;\mathbb{S}) = \boldsymbol{x}\boldsymbol{x}^\intercal\mathbb{P}_{k-2}(K).$$

By Lemma 3.5, we have

$$\mathbb{P}_{\min\{\ell,k\}}(K;\mathbb{S}) \subseteq \boldsymbol{\Sigma}_{\ell,k}(K) \subseteq \mathbb{P}_{\max\{\ell,k\}}(K;\mathbb{S}) \quad \text{and} \quad \boldsymbol{\Sigma}_{k,k}(K) = \mathbb{P}_k(K;\mathbb{S}).$$

The most interesting cases are $\ell = k-1$ and $\ell = k$, which are analogous to RT (incomplete polynomial) and BDM (complete polynomial) $H(\operatorname{div})$-conforming elements for the vector functions, respectively.

For each edge, we choose two normal vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$. The degrees of freedom are given by

$$\boldsymbol{\tau}(\delta) \quad \forall\, \delta \in \mathcal{V}(K), \tag{46}$$

$$(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j, q)_e \quad \forall\, q \in \mathbb{P}_{\ell-2}(e), e \in \mathcal{E}(K),\ i,j = 1,2, \tag{47}$$

$$(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}, q)_F \quad \forall\, q \in \mathbb{P}_{\ell-3}(F), F \in \mathcal{F}(K), \tag{48}$$

$$(2\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}) + \partial_n(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}), q)_F \quad \forall\, q \in \mathbb{P}_{\ell-1}(F), F \in \mathcal{F}(K), \tag{49}$$

$$(\boldsymbol{\tau}, \boldsymbol{\varsigma})_K \quad \forall\, \boldsymbol{\varsigma} \in \nabla^2\mathbb{P}_{k-2}(K), \tag{50}$$

$$(\boldsymbol{\tau}, \boldsymbol{\varsigma})_K \quad \forall\, \boldsymbol{\varsigma} \in \operatorname{sym}(\mathbb{P}_{\ell-2}(K;\mathbb{T}) \times \boldsymbol{x}), \tag{51}$$

$$(\boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{n} \times \boldsymbol{x}q)_{F_1} \quad \forall\, q \in \mathbb{P}_{\ell-2}(F_1), \tag{52}$$

where $F_1 \in \mathcal{F}(K)$ is an arbitrary but fixed face. The DoF (52) is regarded as interior to the tetrahedron K, that is (52) will be double-valued if $F \in \mathcal{F}_h^i$ is selected in different elements.

Before we prove the unisolvence, we give a characterization of the space of shape functions restricted to edges and faces, and derive some consequences of vanishing degrees of freedom.

Lemma 5.1. *For any $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_{\ell,k}(K)$, we have*

$$\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j|_e \in \mathbb{P}_\ell(e), \quad \boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}|_F \in \mathbb{P}_\ell(F), \quad 2\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}) + \partial_n(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n})|_F \in \mathbb{P}_{\ell-1}(F)$$

for each edge $e \in \mathcal{E}(K)$, each face $F \in \mathcal{F}(K)$ and $i,j = 1,2$.

Proof. Take any $\boldsymbol{\tau} = \boldsymbol{x}\boldsymbol{x}^{\intercal}q \in \mathbb{C}_k^{\oplus}(K;\mathbb{S})$ with $q \in \mathbb{P}_{k-2}(K)$. Since $\boldsymbol{n}_i^{\intercal}\boldsymbol{x}$ is constant on each edge of K and $\boldsymbol{n}^{\intercal}\boldsymbol{x}$ is constant on each face of K,

$$\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j|_e = (\boldsymbol{n}_i^{\intercal}\boldsymbol{x})(\boldsymbol{n}_j^{\intercal}\boldsymbol{x})q \in \mathbb{P}_{k-2}(e), \quad \boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}|_F = (\boldsymbol{n}^{\intercal}\boldsymbol{x})^2 q \in \mathbb{P}_{k-2}(F),$$

and

$$\begin{aligned} 2\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}) + \partial_n(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}) &= (\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}) + \boldsymbol{n}^{\intercal}\operatorname{div}\boldsymbol{\tau})|_F \\ &= \boldsymbol{n}^{\intercal}\boldsymbol{x}(\operatorname{div}_F(\boldsymbol{x}q) + \operatorname{div}(\boldsymbol{x}q) + q) \in \mathbb{P}_{k-2}(F). \end{aligned}$$

Thus we conclude the results from the requirement $\ell \geq k-1$. ∎

Lemma 5.2. *For any $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_{\ell,k}(K)$ with the degrees of freedom* (46)-(51) *vanishing, we have*

$$\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j|_e = 0 \quad \forall\, e \in \mathcal{E}(K),\ i,j = 1,2, \tag{53}$$

$$\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}|_F = 0 \quad \forall\, F \in \mathcal{F}(K), \tag{54}$$

$$(2\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}) + \partial_n(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}))|_F = 0 \quad \forall\, F \in \mathcal{F}(K), \tag{55}$$

$$\operatorname{div}\operatorname{div}\boldsymbol{\tau} = 0,$$

$$(\boldsymbol{\tau}, \boldsymbol{\varsigma})_K = 0 \quad \forall\, \boldsymbol{\varsigma} \in \mathbb{P}_{\ell-1}(K;\mathbb{S}). \tag{56}$$

Proof. According to Lemma 5.1, we acquire (53)-(55) from the vanishing degrees of freedom (46)-(49) directly. The scalar function $\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}|_F$ is the standard Lagrange element and the vanishing function value $\boldsymbol{\tau}(\delta)$ at vertices is used to ensure (54).

Noting that $\operatorname{div}\operatorname{div}\boldsymbol{\tau} \in \mathbb{P}_{k-2}(K)$, we get from the Green's identity (45), (53)-(55) and the vanishing degrees of freedom (50) that $\operatorname{div}\operatorname{div}\boldsymbol{\tau} = 0$. Applying the

Green's identity (45) and (53)-(55), it follows

$$(\boldsymbol{\tau}, \nabla^2 v)_K = 0 \quad \forall\, v \in H^2(K),$$

which together with (51) and the decomposition (41) yields (56). □

With previous preparations, we prove the unisolvence as follows. For any $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_{\ell,k}(K)$ satisfying $\operatorname{div}\operatorname{div}\boldsymbol{\tau} = 0$, since $\operatorname{div}\operatorname{div} : \mathbb{C}_k^{\oplus}(K;\mathbb{S}) \to \mathbb{P}_{k-2}(K)$ is a bijection by Lemma 3.5, we have $\boldsymbol{\tau} \in \mathbb{C}_\ell(K;\mathbb{S}) \subseteq \mathbb{P}_\ell(K;\mathbb{S})$. By (56) the volume moments can only determine the polynomial of degree up to $\ell - 1$.

We then use the vanished trace. Similar to the RT and BDM elements [2], the vanishing normal-normal trace (54) implies the normal-normal part of $\boldsymbol{\tau}$ is zero. To determine the normal-tangential terms, further degrees of freedom are needed.

Unlike the traditional approach by transforming back to the reference element, we will choose an intrinsic coordinate. For ease of presentation, denote the four faces in $\mathcal{F}(K)$ by F_i, which is opposite to the ith vertex of K, and by $\boldsymbol{n}_i$ the outward unit normal vector of F_i for $i = 1, 2, 3, 4$. Let $\boldsymbol{t}_i$ be the unit tangential vector of the edge from vertex 4 to vertex i; see Fig. 2. The set of three vectors $\{\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3\}$ forms a basis of $\mathbb{R}^3$ although they may not be orthogonal in general. Consequently $\{\boldsymbol{t}_i \boldsymbol{t}_j^{\intercal}\}_{i,j=1}^3$ forms a basis of the second order tensor and $\boldsymbol{t}_i^{\intercal}\boldsymbol{n}_i \neq 0$ for $i = 1, 2, 3$. Let $\lambda_i(\boldsymbol{x})$ be the ith barycentric coordinate with respect to the tetrahedron K for

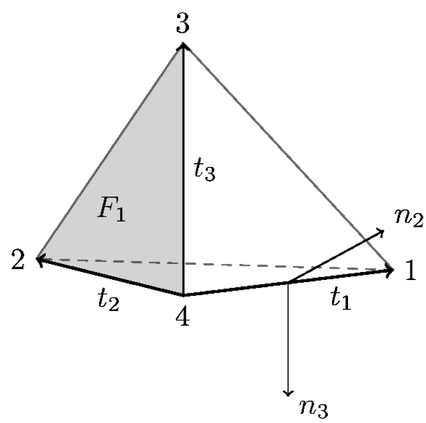

FIGURE 2. Local coordinate formed by three edge vectors

$i = 1, 2, 3, 4$. Then $\lambda_i|_{F_i} = 0$ and $\nabla\lambda_i = -c_i \boldsymbol{n}_i$ for some $c_i > 0$.

Theorem 5.3. *The degrees of freedom* (46)-(52) *are unisolvent for* $\boldsymbol{\Sigma}_{\ell,k}(K)$.

Proof. We first count the number of DoFs (46)-(52). Calculation of DoF (51) can be found in (42). The number of DoFs (46)-(52) is

$$\begin{aligned}&24 + 18(\ell - 1) + 2[(\ell - 1)(\ell - 2) + (\ell + 1)\ell] \\ &\quad + \frac{1}{6}(k^3 - k) - 4 + \frac{1}{6}\ell(\ell - 1)(5\ell + 14) + \frac{1}{2}\ell(\ell - 1) \\ =&\frac{1}{6}(5\ell^3 + 36\ell^2 + 67\ell + 36) + \frac{1}{6}(k^3 - k),\end{aligned}$$

which is the same as $\dim \boldsymbol{\Sigma}_{\ell,k}(K)$, cf. (38).

Take any $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_{\ell,k}(K)$ and suppose all the degrees of freedom (46)-(52) vanish. We are going to prove the function $\boldsymbol{\tau} = \mathbf{0}$. Using the local coordinate sketched in Fig. 2, we can expand $\boldsymbol{\tau}$ as

$$\boldsymbol{\tau} = \sum_{i,j=1}^{3} \tau_{ij} \boldsymbol{t}_i \boldsymbol{t}_j^{\intercal} \quad \text{with} \quad \tau_{ij} = \frac{\boldsymbol{n}_i^{\intercal} \boldsymbol{\tau} \boldsymbol{n}_j}{(\boldsymbol{t}_i^{\intercal} \boldsymbol{n}_i)(\boldsymbol{t}_j^{\intercal} \boldsymbol{n}_j)}.$$

Then $\boldsymbol{\tau}$ is represented as a matrix (τ_{ij}). As $\boldsymbol{\tau}$ is symmetric, $\tau_{ij} = \tau_{ji}$. By (54), it follows

$$\tau_{ii}|_{F_i} = \frac{1}{(\boldsymbol{t}_i^{\intercal} \boldsymbol{n}_i)^2} \boldsymbol{n}_i^{\intercal} \boldsymbol{\tau} \boldsymbol{n}_i|_{F_i} = 0, \quad i = 1, 2, 3.$$

Thus there exists $q_{\ell-1} \in \mathbb{P}_{\ell-1}(K)$ satisfying $\tau_{ii} = \lambda_i q_{\ell-1}$ for $i = 1, 2, 3$. Taking $\boldsymbol{\varsigma} = q_{\ell-1} \boldsymbol{n}_i \boldsymbol{n}_i^{\intercal}$ in (56) will produce

$$\tau_{ii} = 0, \quad i = 1, 2, 3. \tag{57}$$

Namely the diagonal of $\boldsymbol{\tau}$ is zero. So far, in the chosen coordinate, $\boldsymbol{n}_4^{\intercal} \boldsymbol{\tau} \boldsymbol{n}_4 = 0$ has no simple formulation and will be used later on.

On the other hand, from (53) we have $\Pi_{F_1}(\boldsymbol{\tau}\boldsymbol{n}_1) \in H_0(\operatorname{div}_{F_1}, F_1)$. As $\boldsymbol{n}_1^{\intercal} \boldsymbol{\tau} \boldsymbol{n}_1 = (\boldsymbol{t}_1^{\intercal} \boldsymbol{n}_1)^2 \tau_{11} = 0$ in K, cf. (57), it follows $\partial_{n_1}(\boldsymbol{n}_1^{\intercal} \boldsymbol{\tau} \boldsymbol{n}_1)|_{F_1} = 0$. Therefore (55) becomes

$$2 \operatorname{div}_{F_1}(\boldsymbol{\tau}\boldsymbol{n}_1)|_{F_1} = 0.$$

Hence there exists $q_{\ell-2} \in \mathbb{P}_{\ell-2}(F_1)$ such that $(\boldsymbol{n}_1 \times (\boldsymbol{\tau}\boldsymbol{n}_1))|_{F_1} = \nabla_{F_1}(b_{F_1} q_{\ell-2})$, where b_{F_1} is the cubic bubble function on face F_1. Together with (52) and the fact $\operatorname{div}_{F_1}(\boldsymbol{x}\mathbb{P}_{\ell-2}(F_1)) = \mathbb{P}_{\ell-2}(F_1)$, we get $(\boldsymbol{n}_1 \times (\boldsymbol{\tau}\boldsymbol{n}_1))|_{F_1} = \mathbf{0}$. Thus $(\boldsymbol{\tau}\boldsymbol{n}_1)|_{F_1} = \mathbf{0}$. Then there exists $\boldsymbol{q}_{\ell-1} \in \mathbb{P}_{\ell-1}(K; \mathbb{R}^3)$ such that $\boldsymbol{\tau}\boldsymbol{n}_1 = \lambda_1 \boldsymbol{q}_{\ell-1}$, combined with (56), yields $\boldsymbol{\tau}\boldsymbol{n}_1 = \mathbf{0}$. That is the first row of $\boldsymbol{\tau}$ is zero, i.e. $\tau_{11} = \tau_{12} = \tau_{13} = 0$.

By the symmetry, now $\boldsymbol{\tau} = 2\tau_{23} \operatorname{sym}(\boldsymbol{t}_2 \boldsymbol{t}_3^{\intercal})$. Multiplying $\boldsymbol{\tau}$ by $\boldsymbol{n}_4$ from both sides and restricting to F_4, we have

$$\tau_{23}|_{F_4} = \frac{1}{2} \frac{\boldsymbol{n}_4^{\intercal} \boldsymbol{\tau} \boldsymbol{n}_4}{(\boldsymbol{t}_2^{\intercal} \boldsymbol{n}_4)(\boldsymbol{t}_3^{\intercal} \boldsymbol{n}_4)}|_{F_4} = 0.$$

The denominator is non-zero as $\boldsymbol{t}_2, \boldsymbol{t}_3$ are non-tangential vectors of face F_4. Again there exists $q_{\ell-1} \in \mathbb{P}_{\ell-1}(K)$ satisfying $\tau_{23} = \lambda_4 q_{\ell-1}$. Taking $\boldsymbol{\varsigma} = \operatorname{sym}(\boldsymbol{t}_2 \boldsymbol{t}_3^{\intercal}) q_{\ell-1}$ in (56) gives $\tau_{23} = 0$. We thus have $\boldsymbol{\tau} = \mathbf{0}$ and consequently the uni-solvence. $\square$

Due to (49), it is arduous to figure out the explicit basis functions of $\boldsymbol{\Sigma}_{\ell,k}(K)$, which are dual to the degrees of freedom (46)-(52). Alternatively we can hybridize the degrees of freedom (49), and use the basis functions of the standard Lagrange element [6].

5.2. **Polynomial bubble function spaces.** Let

$$\mathbb{B}_{\ell,k}(\operatorname{div}\operatorname{div}, K; \mathbb{S}) := \{\boldsymbol{\tau} \in \boldsymbol{\Sigma}_{\ell,k}(K) : \text{all degrees of freedom (46)-(49) vanish}\}.$$

Together with vanishing (50), we can conclude that $\operatorname{div}\operatorname{div}\boldsymbol{\tau} = 0$. In view of Fig. 1 and Lemma 5.2, the last two sets of DoFs (51)-(52) can be replaced by

$$(\boldsymbol{\tau}, \boldsymbol{\varsigma})_K \quad \forall\, \boldsymbol{\varsigma} \in \mathbb{B}_{\ell,k}(\operatorname{div}\operatorname{div}, K; \mathbb{S}) \cap \ker(\operatorname{div}\operatorname{div}).$$

Next we give characterization of $\mathbb{B}_{\ell,k}(\operatorname{div}\operatorname{div}, K; \mathbb{S}) \cap \ker(\operatorname{div}\operatorname{div})$.

By the exactness of div div complex (20), if $\operatorname{div}\operatorname{div}\boldsymbol{\tau} = 0$ and $\operatorname{tr}(\boldsymbol{\tau}) = 0$, it is possible that $\boldsymbol{\tau} = \operatorname{sym}\operatorname{curl}\boldsymbol{\sigma}$ for some $\boldsymbol{\sigma} \in \mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K; \mathbb{T}) := \boldsymbol{H}_0(\operatorname{sym}\operatorname{curl}, K; \mathbb{T}) \cap \mathbb{P}_{\ell+1}(K; \mathbb{T})$. We will give an explicit characterization of $\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K; \mathbb{T})$, show $\mathbb{B}_{\ell,k}(\operatorname{div}\operatorname{div}, K;\mathbb{S}) \cap \ker(\operatorname{div}\operatorname{div}) = \operatorname{sym}\operatorname{curl}\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K;\mathbb{T})$, and consequently get a set of computable and symmetric DoFs.

We begin with a characterization of the trace of functions in $\boldsymbol{H}(\operatorname{sym}\operatorname{curl}, K;\mathbb{T})$.

Lemma 5.4 (Green's identity for sym curl operator)**.** *Let K be a polyhedron, and let $\boldsymbol{\tau} \in \boldsymbol{H}^1(K;\mathbb{M})$ and $\boldsymbol{\sigma} \in \boldsymbol{H}^1(K;\mathbb{S})$. Then we have*

$$(\operatorname{sym}\operatorname{curl}\boldsymbol{\tau}, \boldsymbol{\sigma})_K = (\boldsymbol{\tau}, \operatorname{curl}\boldsymbol{\sigma})_K - \sum_{F\in\mathcal{F}(K)} (\operatorname{sym}\Pi_F(\boldsymbol{\tau}\times\boldsymbol{n})\Pi_F, \Pi_F\boldsymbol{\sigma}\Pi_F)_F - \sum_{F\in\mathcal{F}(K)} (\boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{n}, \boldsymbol{n}\cdot\boldsymbol{\sigma}\Pi_F)_F.$$

Proof. As $\boldsymbol{\sigma}$ is symmetric,

$$(\operatorname{sym}\operatorname{curl}\boldsymbol{\tau}, \boldsymbol{\sigma})_K = (\operatorname{curl}\boldsymbol{\tau}, \boldsymbol{\sigma})_K = (\boldsymbol{\tau}, \operatorname{curl}\boldsymbol{\sigma})_K - (\boldsymbol{\tau}\times\boldsymbol{n}, \boldsymbol{\sigma})_{\partial K}.$$

On each face, we expand the boundary term

$$(\boldsymbol{\tau}\times\boldsymbol{n}, \boldsymbol{\sigma})_F = (\Pi_F(\boldsymbol{\tau}\times\boldsymbol{n})\Pi_F, \Pi_F\boldsymbol{\sigma}\Pi_F)_F + (\boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{n}, \boldsymbol{n}\cdot\boldsymbol{\sigma}\Pi_F)_F.$$

Then we use the fact $\Pi_F\boldsymbol{\sigma}\Pi_F$ is symmetric to arrive at the desired identity. ☐

Based on the Green's identity, we introduce the following trace operators for $\boldsymbol{H}(\operatorname{sym}\operatorname{curl})$ space

(1) $\operatorname{tr}_1(\boldsymbol{\tau}) := \Pi_F \operatorname{sym}(\boldsymbol{\tau}\times\boldsymbol{n})\Pi_F$,
(2) $\operatorname{tr}_1^{\perp}(\boldsymbol{\tau}) := \boldsymbol{n}\times\operatorname{sym}(\boldsymbol{\tau}\times\boldsymbol{n})\times\boldsymbol{n}$,
(3) $\operatorname{tr}_2(\boldsymbol{\tau}) := \boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{n}$.

Both $\operatorname{tr}_1(\boldsymbol{\tau})$ and $\operatorname{tr}_1^{\perp}(\boldsymbol{\tau})$ are symmetric tensors on each face and $\operatorname{tr}_2(\boldsymbol{\tau})$ is a vector function. Obviously $\operatorname{tr}_1(\boldsymbol{\tau}) = \boldsymbol{0}$ if and only if $\operatorname{tr}_1^{\perp}(\boldsymbol{\tau}) = \boldsymbol{0}$ as $\operatorname{tr}_1^{\perp}(\boldsymbol{\tau})$ is just a rotation of $\operatorname{tr}_1(\boldsymbol{\tau})$. Using the trace operators, $\boldsymbol{H}(\operatorname{sym}\operatorname{curl})$ polynomial bubble function space can be defined as

$$\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K;\mathbb{T}) := \{\boldsymbol{\tau}\in\mathbb{P}_{\ell+1}(K;\mathbb{T}) : (\boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{n})|_F = \boldsymbol{0}, \quad (\boldsymbol{n}\times\operatorname{sym}(\boldsymbol{\tau}\times\boldsymbol{n})\times\boldsymbol{n})|_F = \boldsymbol{0} \quad \forall\, F\in\mathcal{F}(K)\}.$$

We shall give an explicit characterization of $\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K;\mathbb{T})$.

Lemma 5.5. *Let $\boldsymbol{\tau}\in\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K;\mathbb{T})$. It holds*

$$\boldsymbol{\tau}|_e = \boldsymbol{0} \quad \forall\, e\in\mathcal{E}(K). \tag{58}$$

Proof. It is straightforward to verify (58) on the reference tetrahedron for which $\boldsymbol{e} = (1,0,0)$ and two normal vectors of the face containing $\boldsymbol{e}$ are $\boldsymbol{n}_1 = (1,0,0)$ and $\boldsymbol{n}_2 = (0,0,1)$. To avoid complicated transformation of trace operators, we provide a proof using an intrinsic basis of $\mathbb{T}$ on K.

Take any edge $e\in\mathcal{E}(K)$ with the tangential vector $\boldsymbol{t}$. Let $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ be the unit outward normal vectors of two faces sharing edge e. Set $\boldsymbol{s}_i := \boldsymbol{t}\times\boldsymbol{n}_i$ for $i = 1,2$. By a direction computation, we get on edge e for $i = 1,2$ that

$$\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{t} = (\boldsymbol{n}_i\cdot\boldsymbol{\tau}\times\boldsymbol{n}_i)\cdot\boldsymbol{s}_i = 0,$$

$$\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{s}_i = -(\boldsymbol{n}_i\cdot\boldsymbol{\tau}\times\boldsymbol{n}_i)\cdot\boldsymbol{t} = 0,$$

$$\boldsymbol{t}^{\intercal}\boldsymbol{\tau}\boldsymbol{t} - \boldsymbol{s}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{s}_i = 2\boldsymbol{t}\cdot\operatorname{sym}(\boldsymbol{\tau}\times\boldsymbol{n}_i)\cdot\boldsymbol{s}_i = 2\boldsymbol{s}_i\cdot(\boldsymbol{n}_i\times\operatorname{sym}(\boldsymbol{\tau}\times\boldsymbol{n}_i)\times\boldsymbol{n}_i)\cdot\boldsymbol{t} = 0,$$

$$\boldsymbol{t}^{\intercal}\boldsymbol{\tau}\boldsymbol{s}_i = -\boldsymbol{t}\cdot\operatorname{sym}(\boldsymbol{\tau}\times\boldsymbol{n}_i)\cdot\boldsymbol{t} = \boldsymbol{s}_i\cdot(\boldsymbol{n}_i\times\operatorname{sym}(\boldsymbol{\tau}\times\boldsymbol{n}_i)\times\boldsymbol{n}_i)\cdot\boldsymbol{s}_i = 0.$$

Both $\operatorname{span}\{\boldsymbol{s}_1, \boldsymbol{s}_2\}$ and $\operatorname{span}\{\boldsymbol{n}_1, \boldsymbol{n}_2\}$ form the same normal vector space of edge e; then the last identity implies

$$\boldsymbol{t}^\intercal \boldsymbol{\tau} \boldsymbol{n}_i = 0.$$

Then it is sufficient to prove the eight trace-free tensors

$$\boldsymbol{n}_1\boldsymbol{t}^\intercal,\ \boldsymbol{n}_2\boldsymbol{t}^\intercal,\ \boldsymbol{n}_1\boldsymbol{s}_1^\intercal,\ \boldsymbol{n}_2\boldsymbol{s}_2^\intercal,\ \boldsymbol{t}\boldsymbol{n}_1^\intercal,\ \boldsymbol{t}\boldsymbol{n}_2^\intercal,\ \boldsymbol{t}\boldsymbol{t}^\intercal - \boldsymbol{s}_1\boldsymbol{s}_1^\intercal,\ \boldsymbol{t}\boldsymbol{t}^\intercal - \boldsymbol{s}_2\boldsymbol{s}_2^\intercal \tag{59}$$

are linearly independent. Assume there exist $c_i \in \mathbb{R}$ for $i = 1, \dots, 8$ such that

$$\begin{aligned} c_1\boldsymbol{n}_1\boldsymbol{t}^\intercal + c_2\boldsymbol{n}_2\boldsymbol{t}^\intercal + c_3\boldsymbol{n}_1\boldsymbol{s}_1^\intercal + c_4\boldsymbol{n}_2\boldsymbol{s}_2^\intercal + c_5\boldsymbol{t}\boldsymbol{n}_1^\intercal + c_6\boldsymbol{t}\boldsymbol{n}_2^\intercal \\ + c_7(\boldsymbol{t}\boldsymbol{t}^\intercal - \boldsymbol{s}_1\boldsymbol{s}_1^\intercal) + c_8(\boldsymbol{t}\boldsymbol{t}^\intercal - \boldsymbol{s}_2\boldsymbol{s}_2^\intercal) = \boldsymbol{0}. \end{aligned}$$

Multiplying the last equation by $\boldsymbol{t}$ from the right and left respectively, we obtain

$$c_1\boldsymbol{n}_1 + c_2\boldsymbol{n}_2 + (c_7 + c_8)\boldsymbol{t} = \boldsymbol{0}, \quad c_5\boldsymbol{n}_1^\intercal + c_6\boldsymbol{n}_2^\intercal + (c_7 + c_8)\boldsymbol{t}^\intercal = \boldsymbol{0}.$$

Hence $c_1 = c_2 = c_5 = c_6 = c_7 + c_8 = 0$, which yields

$$c_3\boldsymbol{n}_1\boldsymbol{s}_1^\intercal + c_4\boldsymbol{n}_2\boldsymbol{s}_2^\intercal + c_7(\boldsymbol{s}_2\boldsymbol{s}_2^\intercal - \boldsymbol{s}_1\boldsymbol{s}_1^\intercal) = \boldsymbol{0}.$$

Multiplying the last equation by $\boldsymbol{n}_1$ from the right, it follows

$$(\boldsymbol{s}_2 \cdot \boldsymbol{n}_1)(c_4\boldsymbol{n}_2 + c_7\boldsymbol{s}_2) = \boldsymbol{0}.$$

As a result $c_4 = c_7 = 0$, and then $c_3 = 0$. $\square$

We write $\mathbb{P}_{\ell+1}(K;\mathbb{T})$ as $\mathbb{P}_{\ell+1}(K) \otimes \mathbb{T}$ and use the barycentric coordinate representation of a polynomial. That is a polynomial $p \in \mathbb{P}_{\ell+1}(K)$ which has a unique representation in terms of

$$p = \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}\lambda_4^{\alpha_4}, \quad \sum_{i=1}^{4}\alpha_i = \ell + 1, \alpha_i \in \mathbb{N}. \tag{60}$$

Lemma 5.5 implies that p must contain a face bubble $b_F = \lambda_i\lambda_j\lambda_k$ where (i, j, k) are three vertices of F. Otherwise, if $p = \lambda_i^{\alpha_i}\lambda_j^{\alpha_j}, \alpha_i + \alpha_j = \ell + 1$, then p is not zero on the edge (i, j).

We consider the subspace $b_F\mathbb{P}_{\ell-2}(K)\otimes\mathbb{T}$ and identify its intersection with $\ker(\operatorname{tr})$. Due to the face bubble b_F, the polynomial is zero on the other faces. So we only need to consider the trace on face F. Without loss of generality, we can choose the coordinate s.t. $\boldsymbol{n}_F = (0, 0, 1)$. Choose the canonical basis of $\mathbb{T}$ associated to this coordinate. Then a direct calculation to find out $\ker(\operatorname{tr}) \cap \mathbb{T}$ consists of

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Switching to an intrinsic basis, we obtain the following explicit characterization of $\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K; \mathbb{T})$.

Lemma 5.6. *For each face F, we choose two unit tangent vectors $\boldsymbol{t}_1, \boldsymbol{t}_2$ s.t. $(\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{n}_F)$ forms an orthonormal basis of $\mathbb{R}^3$. Then*

$$\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K; \mathbb{T}) = \operatorname{span}\{p b_F \psi_i^F, p \in \mathbb{P}_{\ell-2}(K), F \in \mathcal{F}(K), i = 1, 2, 3\}, \tag{61}$$

where the three trace-free tensors are:

$$\psi_1^F = \boldsymbol{t}_1\boldsymbol{n}_F^\intercal, \quad \psi_2^F = \boldsymbol{t}_2\boldsymbol{n}_F^\intercal, \quad \psi_3^F = \boldsymbol{t}_1\boldsymbol{t}_1^\intercal + \boldsymbol{t}_2\boldsymbol{t}_2^\intercal - 2\boldsymbol{n}_F\boldsymbol{n}_F^\intercal.$$

Proof. Using the formulae (10)-(11), by the direct calculation, we can easily show $\psi_i^F \in \ker(\mathrm{tr}_F) \cap \mathbb{T}$ for each face F and $i = 1, 2, 3$, where tr_F denotes the trace operators $(\mathrm{tr}_1, \mathrm{tr}_2)$ restricted to F. As $\dim \ker(\mathrm{tr}_F) \cap \mathbb{T} = 3$, we conclude that

$$\ker(\mathrm{tr}_F) \cap (b_F \mathbb{P}_{\ell-2}(K) \otimes \mathbb{T}) = \mathrm{span}\{p b_F \psi_i^F, p \in \mathbb{P}_{\ell-2}(K), i = 1, 2, 3\}.$$

By Lemma 5.5 we know that

$$\ker(\mathrm{tr}) \cap (\mathbb{P}_{\ell+1} \otimes \mathbb{T}) = \cup_F \ker(\mathrm{tr}_F) \cap (b_F \mathbb{P}_{\ell-2}(K) \otimes \mathbb{T})$$

and thus (61) follows. □

We only give a generating set of the bubble function space as the 12 constant matrices $\{\psi_1^F, \psi_2^F, \psi_3^F, F \in \mathcal{F}(K)\}$ are not linearly independent. Next we find out a basis from this generating set.

Lemma 5.7. *Let (i, j, k) be three vertices of face F and $\mathbb{P}_{\ell-2}(F) = \{\lambda_i^{\alpha_1}\lambda_j^{\alpha_2}\lambda_k^{\alpha_3}, \alpha_1 + \alpha_2 + \alpha_3 = \ell - 2, \alpha_i \in \mathbb{N}, i = 1, 2, 3\}$. Define $\mathbb{B}_{F,\ell+1} := b_F \mathbb{P}_{\ell-2}(F) \otimes \mathrm{span}\{\psi_1^F, \psi_2^F, \psi_3^F\}$ and $\mathbb{B}_{K,\ell+1} = b_K \mathbb{P}_{\ell-3}(K) \otimes \mathrm{span}\{\psi_1^F, \psi_2^F, F \in \mathcal{F}(K)\}$. Then*

$$\mathbb{B}_{\ell+1}(\mathrm{sym\,curl}, K; \mathbb{T}) = \oplus_{F \in \mathcal{F}(K)} \mathbb{B}_{F,\ell+1} \oplus \mathbb{B}_{K,\ell+1}, \tag{62}$$

and consequently

$$\dim \mathbb{B}_{\ell+1}(\mathrm{sym\,curl}, K; \mathbb{T}) = \frac{2}{3}\ell(\ell - 1)(2\ell + 5) = \frac{1}{3}(4\ell^3 + 6\ell^2 - 10\ell).$$

Proof. The 12 constant matrices $\{\psi_1^F, \psi_2^F, \psi_3^F, F \in \mathcal{F}(K)\}$ are not linearly independent as $\dim \mathbb{T} = 8$. Among them, $\{\psi_1^F, \psi_2^F, F \in \mathcal{F}(K)\}$ forms a basis of $\mathbb{T}$ which can be proved as verifying the linear independence of (59) in Lemma 5.5 or see [15].

For each pb_F, with $p \in \mathbb{P}_{\ell-2}(K)$, we can group into either $b_K \mathbb{P}_{\ell-3}(K)$ or $b_F \mathbb{P}_{\ell-2}(F)$ depending on if the polynomial $p|_F$ is zero or not, respectively. That is, for one fixed face F:

$$b_F \mathbb{P}_{\ell-2}(K) = b_F \mathbb{P}_{\ell-2}(F) \oplus b_K \mathbb{P}_{\ell-3}(K).$$

The sum is direct in view of the barycentric representation (60) of a polynomial. Then coupled with $\{\psi_i^F\}$, we get the basis (62) of the bubble function space.

The dimension of $\mathbb{B}_{\ell+1}(\mathrm{sym\,curl}, K; \mathbb{T})$ is

$$4 \cdot 3 \cdot \dim \mathbb{P}_{\ell-2}(F) + 8 \dim \mathbb{P}_{\ell-3}(K) = \frac{1}{3}(4\ell^3 + 6\ell^2 - 10\ell),$$

as required. □

We then verify $\mathrm{sym\,curl}\, \mathbb{B}_{\ell+1}(\mathrm{sym\,curl}, K; \mathbb{T}) \subseteq \mathbb{B}_{\ell,k}(\mathrm{div\,div}, K; \mathbb{S})$ by verifying all boundary DoFs vanish.

Lemma 5.8. *Let $\boldsymbol{\tau} \in \mathbb{B}_{\ell+1}(\mathrm{sym\,curl}, K; \mathbb{T})$. Assume edge $e \in \mathcal{E}(K)$ is shared by faces F_i and F_j. It holds $\boldsymbol{n}_i^{\intercal}(\mathrm{sym\,curl}\,\boldsymbol{\tau})\boldsymbol{n}_j|_e = 0$.*

Proof. For the ease of notation, let $\boldsymbol{\sigma} = \mathrm{sym\,curl}\,\boldsymbol{\tau}$. Suppose

$$\boldsymbol{\tau} = \sum_{F \in \mathcal{F}(K)} \sum_{l=1}^{3} q_{F,l} b_F \psi_l^F$$

with $q_{F,l} \in \mathbb{P}_{\ell-2}(K)$. By $b_F|_e = 0$, we get

$$\boldsymbol{n}_i^{\intercal}\boldsymbol{\sigma}\boldsymbol{n}_j|_e = \sum_{F \in \mathcal{F}(K)} \sum_{l=1}^{3} q_{F,l}|_e (\boldsymbol{n}_i^{\intercal}\, \mathrm{sym\,curl}(b_F \psi_l^F)\boldsymbol{n}_j)|_e.$$

Since $\lambda_i|_e = \lambda_j|_e = 0$, we can see that $(\boldsymbol{n}_i \times \boldsymbol{n}_F \cdot \nabla b_F)|_e = (\boldsymbol{n}_j \times \boldsymbol{n}_F \cdot \nabla b_F)|_e = 0$. Thus for $l = 1, 2$,

$$
\begin{aligned}
& 2(\boldsymbol{n}_i^\intercal \operatorname{sym}\operatorname{curl}(b_F \psi_l^F)\boldsymbol{n}_j)|_e \\
&= -(\boldsymbol{n}_i \cdot (b_F \boldsymbol{t}_l \boldsymbol{n}_F) \times \nabla \cdot \boldsymbol{n}_j)|_e - (\boldsymbol{n}_j \cdot (b_F \boldsymbol{t}_l \boldsymbol{n}_F) \times \nabla \cdot \boldsymbol{n}_i)|_e \\
&= \boldsymbol{n}_i \cdot \boldsymbol{t}_l (\boldsymbol{n}_F \times \boldsymbol{n}_j \cdot \nabla b_F)|_e + \boldsymbol{n}_j \cdot \boldsymbol{t}_l (\boldsymbol{n}_F \times \boldsymbol{n}_i \cdot \nabla b_F)|_e \\
&= 0.
\end{aligned}
$$

Next consider $l = 3$. When $F \neq F_j$, the face bubble b_F has a factor λ_j, which implies $(\boldsymbol{n}_j \times \nabla b_F)|_e = \boldsymbol{0}$. Thus

$$(\boldsymbol{n}_i^\intercal \operatorname{curl}(b_F \psi_3^F)\boldsymbol{n}_j)|_e = -(\boldsymbol{n}_i \cdot (b_F \psi_3^F) \times \nabla \cdot \boldsymbol{n}_j)|_e = (\boldsymbol{n}_i \cdot \psi_3^F \cdot (\boldsymbol{n}_j \times \nabla b_F))|_e = 0.$$

When $F = F_j$, the face bubble b_F has a factor λ_i. By the fact that $(\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{n}_j)$ forms an orthonormal basis of $\mathbb{R}^3$,

$$
\begin{aligned}
\boldsymbol{n}_i \cdot \boldsymbol{t}_2 (\boldsymbol{t}_2 \times \boldsymbol{n}_j \cdot \nabla \lambda_i) = \boldsymbol{n}_i \cdot (\boldsymbol{n}_j \times \boldsymbol{t}_1)(\boldsymbol{t}_1 \cdot \nabla \lambda_i) &= -(\boldsymbol{t}_1 \cdot \nabla \lambda_i)(\boldsymbol{n}_j \times \boldsymbol{n}_i \cdot \boldsymbol{t}_1) \\
&= -\boldsymbol{n}_i \cdot \boldsymbol{t}_1 (\boldsymbol{n}_j \times \nabla \lambda_i \cdot \boldsymbol{t}_1),
\end{aligned}
$$

which implies

$$\boldsymbol{n}_i \cdot \boldsymbol{t}_1 (\boldsymbol{n}_j \times \nabla \lambda_i \cdot \boldsymbol{t}_1) + \boldsymbol{n}_i \cdot \boldsymbol{t}_2 (\boldsymbol{n}_j \times \nabla \lambda_i \cdot \boldsymbol{t}_2) = 0.$$

As a result,

$$(\boldsymbol{n}_i^\intercal \operatorname{curl}(b_F \psi_3^F)\boldsymbol{n}_j)|_e = \boldsymbol{n}_i \cdot \boldsymbol{t}_1 (\boldsymbol{n}_j \times \nabla b_F \cdot \boldsymbol{t}_1)|_e + \boldsymbol{n}_i \cdot \boldsymbol{t}_2 (\boldsymbol{n}_j \times \nabla b_F \cdot \boldsymbol{t}_2)|_e = 0.$$

Similarly $(\boldsymbol{n}_j^\intercal \operatorname{curl}(b_F \psi_3^F)\boldsymbol{n}_i)|_e = 0$ holds. Hence $(\boldsymbol{n}_i^\intercal \operatorname{sym}\operatorname{curl}(b_F \psi_3^F)\boldsymbol{n}_j)|_e = 0$.

Therefore $\boldsymbol{n}_i^\intercal \boldsymbol{\sigma} \boldsymbol{n}_j|_e = 0$. ∎

Next we show the two traces $\operatorname{tr}_2(\boldsymbol{\tau})$ is in $H(\operatorname{div}_F)$ and $\operatorname{tr}_1(\boldsymbol{\tau})$ in $H(\operatorname{div}_F \operatorname{div}_F)$.

Lemma 5.9. *When $\boldsymbol{\sigma} = \operatorname{sym}\operatorname{curl} \boldsymbol{\tau}$ with $\boldsymbol{\tau} \in \boldsymbol{H}^2(K; \mathbb{M})$, we can express the trace in terms of the differential operators on surface F of K*

$$\boldsymbol{n}^\intercal \boldsymbol{\sigma} \boldsymbol{n} = \operatorname{div}_F (\boldsymbol{n} \cdot \boldsymbol{\tau} \times \boldsymbol{n}), \tag{63}$$

$$
\begin{aligned}
\nabla_F^\perp \cdot (\boldsymbol{n} \times \boldsymbol{\sigma} \cdot \boldsymbol{n}) + \boldsymbol{n}^\intercal \operatorname{div} \boldsymbol{\sigma} &= -\operatorname{rot}_F \operatorname{rot}_F (\boldsymbol{n} \times \operatorname{sym}(\boldsymbol{\tau} \times \boldsymbol{n}) \times \boldsymbol{n}) \\
&= \operatorname{div}_F \operatorname{div}_F (\Pi_F \operatorname{sym}(\boldsymbol{\tau} \times \boldsymbol{n}) \Pi_F).
\end{aligned} \tag{64}
$$

Proof. By

$$\boldsymbol{n}^\intercal \boldsymbol{\sigma} \boldsymbol{n} = \frac{1}{2} \boldsymbol{n} \cdot (\nabla \times (\boldsymbol{\tau}^\intercal) - \boldsymbol{\tau} \times \nabla) \cdot \boldsymbol{n} = \frac{1}{2} \nabla_F^\perp \cdot (\boldsymbol{\tau}^\intercal) \cdot \boldsymbol{n} + \frac{1}{2} \boldsymbol{n} \cdot \boldsymbol{\tau} \cdot \nabla_F^\perp$$

and the fact $\nabla_F^\perp \cdot (\boldsymbol{\tau}^\intercal) \cdot \boldsymbol{n} = \boldsymbol{n} \cdot \boldsymbol{\tau} \cdot \nabla_F^\perp$, we get

$$\boldsymbol{n}^\intercal \boldsymbol{\sigma} \boldsymbol{n} = \boldsymbol{n} \cdot \boldsymbol{\tau} \cdot \nabla_F^\perp = \operatorname{rot}_F (\boldsymbol{n} \cdot \boldsymbol{\tau} \Pi_F).$$

Then the identity (63) holds from (18).

Next we prove (64). Employing (17) with $\boldsymbol{v} = \boldsymbol{\tau}^\intercal \cdot \boldsymbol{n}$,

$$
\begin{aligned}
\nabla_F^\perp \cdot (\boldsymbol{n} \times \boldsymbol{\sigma} \cdot \boldsymbol{n}) &= \frac{1}{2} \nabla_F^\perp \cdot \big(\boldsymbol{n} \times (\nabla \times (\boldsymbol{\tau}^\intercal) - \boldsymbol{\tau} \times \nabla) \cdot \boldsymbol{n}\big) \\
&= \frac{1}{2} \nabla_F^\perp \cdot \big(\boldsymbol{n} \times (\nabla \times (\boldsymbol{\tau}^\intercal \cdot \boldsymbol{n}))\big) + \frac{1}{2} \nabla_F^\perp \cdot (\boldsymbol{n} \times \boldsymbol{\tau}) \cdot \nabla_F^\perp \\
&= \frac{1}{2} \nabla_F^\perp \cdot \big(\nabla (\boldsymbol{n} \cdot \boldsymbol{\tau}^\intercal \cdot \boldsymbol{n}) - \partial_n (\boldsymbol{\tau}^\intercal \cdot \boldsymbol{n})\big) + \frac{1}{2} \nabla_F^\perp \cdot (\boldsymbol{n} \times \boldsymbol{\tau}) \cdot \nabla_F^\perp \\
&= -\frac{1}{2} \nabla_F^\perp \cdot \big(\partial_n (\boldsymbol{\tau}^\intercal \cdot \boldsymbol{n})\big) + \frac{1}{2} \nabla_F^\perp \cdot (\boldsymbol{n} \times \boldsymbol{\tau}) \cdot \nabla_F^\perp.
\end{aligned}
$$

On the other side, we have

$$\begin{aligned}\boldsymbol{n}\cdot\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot\nabla &= \frac{1}{2}\boldsymbol{n}\cdot(\nabla\times(\boldsymbol{\tau}^{\intercal}))\cdot\nabla = \frac{1}{2}\nabla_F^{\perp}\cdot(\boldsymbol{\tau}^{\intercal})\cdot\nabla \\ &= \frac{1}{2}\nabla_F^{\perp}\cdot(\boldsymbol{\tau}^{\intercal})\cdot(\boldsymbol{n}\partial_n+\nabla_F) = \frac{1}{2}\nabla_F^{\perp}\cdot\big(\partial_n(\boldsymbol{\tau}^{\intercal}\cdot\boldsymbol{n})\big) + \frac{1}{2}\nabla_F^{\perp}\cdot(\boldsymbol{\tau}^{\intercal})\cdot\nabla_F \\ &= \frac{1}{2}\nabla_F^{\perp}\cdot\big(\partial_n(\boldsymbol{\tau}^{\intercal}\cdot\boldsymbol{n})\big) - \frac{1}{2}\nabla_F^{\perp}\cdot(\boldsymbol{\tau}^{\intercal}\times\boldsymbol{n})\cdot\nabla_F^{\perp}.\end{aligned}$$

The sum of the last two identities gives

$$\nabla_F^{\perp}\cdot(\boldsymbol{n}\times\boldsymbol{\sigma}\cdot\boldsymbol{n}) + \boldsymbol{n}\cdot\operatorname{div}\boldsymbol{\sigma} = \nabla_F^{\perp}\cdot\operatorname{sym}(\boldsymbol{n}\times\boldsymbol{\tau}\Pi_F)\cdot\nabla_F^{\perp}.$$

Therefore (64) follows from $\operatorname{sym}(\boldsymbol{n}\times\boldsymbol{\tau}\Pi_F) = -\boldsymbol{n}\times\operatorname{sym}(\boldsymbol{\tau}\times\boldsymbol{n})\times\boldsymbol{n}$. ☐

Note that $\nabla_F^{\perp}\cdot(\boldsymbol{n}\times\boldsymbol{\sigma}\cdot\boldsymbol{n}) + \boldsymbol{n}^{\intercal}\operatorname{div}\boldsymbol{\sigma}$ is an equivalent formulation of the second trace of $\boldsymbol{\sigma}$. Lemma 5.9 implies the following trace complexes

$$\begin{array}{ccccccccc} \boldsymbol{RT} & \xrightarrow{\subset} & \boldsymbol{v} & \xrightarrow{\operatorname{dev}\operatorname{grad}} & \boldsymbol{\tau} & \xrightarrow{\operatorname{sym}\operatorname{curl}} & \boldsymbol{\sigma} & \xrightarrow{\operatorname{div}\operatorname{div}} & p \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{R} & \xrightarrow{\subset} & \boldsymbol{v}\cdot\boldsymbol{n} & \xrightarrow{-\operatorname{curl}_F} & \boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{n} & \xrightarrow{\operatorname{div}_F} & \boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n} & \longrightarrow & 0 \end{array},$$

and

$$\begin{array}{ccccccccc} \boldsymbol{RT} & \xrightarrow{\subset} & \boldsymbol{v} & \xrightarrow{\operatorname{dev}\operatorname{grad}} & \boldsymbol{\tau} & \xrightarrow{\operatorname{sym}\operatorname{curl}} & \boldsymbol{\sigma} & \xrightarrow{\operatorname{div}\operatorname{div}} & p \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \boldsymbol{RT}_F & \xrightarrow{\subset} & \Pi_F\boldsymbol{v} & \xrightarrow{-\operatorname{sym}\operatorname{curl}_F} & \Pi_F\operatorname{sym}(\boldsymbol{\tau}\times\boldsymbol{n})\Pi_F & \xrightarrow{\operatorname{div}_F\operatorname{div}_F} & \operatorname{tr}_2(\boldsymbol{\sigma}) & \longrightarrow & 0 \end{array}.$$

Those trace complexes will guide the design of edge and face degrees of freedom to ensure the required continuity.

5.3. **The bubble complex.** Combining Lemmas 5.8 and 5.9 gives the following result.

Lemma 5.10. *It holds*

$$\operatorname{sym}\operatorname{curl}\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl},K;\mathbb{T}) \subseteq (\mathbb{B}_{\ell,k}(\operatorname{div}\operatorname{div},K;\mathbb{S})\cap\ker(\operatorname{div}\operatorname{div})). \tag{65}$$

Proof. For $\boldsymbol{\tau}\in\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl},K;\mathbb{T})$, by construction, $\boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{n}=\boldsymbol{0}$ and $\boldsymbol{n}\times\operatorname{sym}(\boldsymbol{\tau}\times\boldsymbol{n})\times\boldsymbol{n}=\boldsymbol{0}$ on ∂K. Let $\boldsymbol{\sigma}=\operatorname{sym}\operatorname{curl}\boldsymbol{\tau}$. Then by Lemma 5.9, DoFs (48)-(49) vanish. By Lemma 5.8, (47) vanishes. As $\boldsymbol{\tau}$ contains a face bubble, $\boldsymbol{\sigma}$ will have an edge bubble function which means $\boldsymbol{\sigma}(\delta)=\boldsymbol{0}$ for all $\delta\in\mathcal{V}(K)$. Therefore $\operatorname{sym}\operatorname{curl}\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl},K;\mathbb{T})\subseteq\mathbb{B}_{\ell,k}(\operatorname{div}\operatorname{div},K;\mathbb{S})$. The property $\operatorname{div}\operatorname{div}(\operatorname{sym}\operatorname{curl}\boldsymbol{\tau}) = 0$ is from the div div complex. ☐

Indeed the "$\subseteq$" in (65) can be changed to "$=$". This will be clear after we present a bubble complex. In the sequel, we denote by $\mathbb{P}_{k-2,1}^{\perp}(K)$ the L^2-orthogonal complement space of $\mathbb{P}_1(K)$ in $\mathbb{P}_{k-2}(K)$ with respect to the inner product $(\cdot,\cdot)_K$.

Lemma 5.11. *For each $K\in\mathcal{T}_h$, it holds*

$$\operatorname{div}\operatorname{div}\mathbb{B}_{\ell,k}(\operatorname{div}\operatorname{div},K;\mathbb{S}) = \mathbb{P}_{k-2,1}^{\perp}(K). \tag{66}$$

Consequently

$$\dim(\mathbb{B}_{\ell,k}(\operatorname{div}\operatorname{div},K;\mathbb{S})\cap\ker(\operatorname{div}\operatorname{div})) = \frac{1}{6}\ell(\ell-1)(5\ell+17). \tag{67}$$

Proof. From the integration by parts, it is obviously true that

$$\operatorname{div}\operatorname{div}\mathbb{B}_{\ell,k}(\operatorname{div}\operatorname{div},K;\mathbb{S})\subseteq\mathbb{P}_{k-2,1}^{\perp}(K).$$

On the other side, for any $v\in\mathbb{P}_{k-2,1}^{\perp}(K)$, due to the fact that $\operatorname{div}\operatorname{div}\boldsymbol{H}_0^2(K;\mathbb{S})=L^2(K)\cap\mathbb{P}_1^{\perp}(K)$ [10], where $\mathbb{P}_1^{\perp}(K)$ is a subspace of $L^2(K)$ being orthogonal to $\mathbb{P}_1(K)$ with respect to the L^2-inner product $(\cdot,\cdot)_K$, there exists $\widetilde{\boldsymbol{\tau}}\in\boldsymbol{H}_0^2(K;\mathbb{S})$ such that

$$\operatorname{div}\operatorname{div}\widetilde{\boldsymbol{\tau}}=v.$$

Then take $\boldsymbol{\tau}\in\mathbb{B}_{\ell,k}(\operatorname{div}\operatorname{div},K;\mathbb{S})$ with the rest DoFs

$$\begin{aligned}(\boldsymbol{\tau}-\widetilde{\boldsymbol{\tau}},\boldsymbol{\varsigma})_K&=0\quad\forall\,\boldsymbol{\varsigma}\in\nabla^2\mathbb{P}_{k-2}(K)\oplus\operatorname{sym}(\mathbb{P}_{\ell-2}(K;\mathbb{T})\times\boldsymbol{x}),\\((\boldsymbol{\tau}-\widetilde{\boldsymbol{\tau}})\boldsymbol{n},\boldsymbol{n}\times\boldsymbol{x}q)_{F_1}&=0\quad\forall\,q\in\mathbb{P}_{\ell-2}(F_1).\end{aligned}$$

Applying the Green's identity (45), we get

$$(\operatorname{div}\operatorname{div}(\boldsymbol{\tau}-\widetilde{\boldsymbol{\tau}}),q)_K=0\quad\forall\,q\in\mathbb{P}_{k-2}(K).$$

This implies $\operatorname{div}\operatorname{div}\boldsymbol{\tau}=\operatorname{div}\operatorname{div}\widetilde{\boldsymbol{\tau}}=v$. Namely (66) holds.

An immediate result of (66) is

$$\begin{aligned}\dim(\mathbb{B}_{\ell,k}(\operatorname{div}\operatorname{div},K;\mathbb{S})\cap\ker(\operatorname{div}\operatorname{div}))&=\dim\mathbb{B}_{\ell,k}(\operatorname{div}\operatorname{div},K;\mathbb{S})-\dim\mathbb{P}_{k-2}(K)+4\\&=\frac{1}{6}\ell(\ell-1)(5\ell+14)+\frac{1}{2}\ell(\ell-1)\\&=\frac{1}{6}\ell(\ell-1)(5\ell+17).\end{aligned}$$

□

Define

$$\mathbb{B}_{\ell+2}(\operatorname{grad},K;\mathbb{R}^3):=\{\boldsymbol{v}\in\mathbb{P}_{\ell+2}(K;\mathbb{R}^3):\boldsymbol{v}|_{\partial K}=\boldsymbol{0}\}=b_K\mathbb{P}_{\ell-2}(K;\mathbb{R}^3).$$

Now we are in the position to present the so-called bubble complex.

Theorem 5.12. *The bubble function spaces for the* div div *complex*

$$\begin{aligned}&0\to\mathbb{B}_{\ell+2}(\operatorname{grad},K;\mathbb{R}^3)\xrightarrow{\operatorname{dev}\operatorname{grad}}\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl},K;\mathbb{T})\xrightarrow{\operatorname{sym}\operatorname{curl}}\mathbb{B}_{\ell,k}(\operatorname{div}\operatorname{div},K;\mathbb{S})\\&\xrightarrow{\operatorname{div}\operatorname{div}}\mathbb{P}_{k-2,1}^{\perp}(K)\to 0\end{aligned}\tag{68}$$

form an exact sequence.

Proof. Take any $\boldsymbol{v}\in\mathbb{B}_{\ell+2}(\operatorname{grad},K;\mathbb{R}^3)$ with $\boldsymbol{v}|_{\partial K}=\boldsymbol{0}$. We have on each face $F\in\mathcal{F}(K)$,

$$\boldsymbol{n}\cdot(\operatorname{dev}\operatorname{grad}\boldsymbol{v})\times\boldsymbol{n}=\boldsymbol{n}\cdot(\operatorname{grad}\boldsymbol{v})\times\boldsymbol{n}=-(\boldsymbol{n}\times\nabla)(\boldsymbol{v}\cdot\boldsymbol{n})=\boldsymbol{0},\tag{69}$$

and

$$\begin{aligned}\boldsymbol{n}\times\operatorname{sym}((\operatorname{dev}\operatorname{grad}\boldsymbol{v})\times\boldsymbol{n})\times\boldsymbol{n}&=\boldsymbol{n}\times\operatorname{sym}((\operatorname{grad}\boldsymbol{v})\times\boldsymbol{n})\times\boldsymbol{n}\\&=-\boldsymbol{n}\times\operatorname{sym}(\boldsymbol{v}\nabla_F^{\perp})\times\boldsymbol{n}\\&=-\boldsymbol{n}\times\operatorname{sym}((\Pi_F\boldsymbol{v})\nabla_F^{\perp})\times\boldsymbol{n}=\boldsymbol{0}.\end{aligned}\tag{70}$$

Hence $\operatorname{dev}\operatorname{grad}\mathbb{B}_{\ell+2}(\operatorname{grad},K;\mathbb{R}^3)\subseteq\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl},K;\mathbb{T})\cap\ker(\operatorname{sym}\operatorname{curl})$. Thanks to Lemma 5.10 and (66), we conclude that (68) is a complex.

We then verify the exactness from left to right.

(1) $\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K;\mathbb{T}) \cap \ker(\operatorname{sym}\operatorname{curl}) = \operatorname{dev}\operatorname{grad}\mathbb{B}_{\ell+2}(\operatorname{grad}, K;\mathbb{R}^3)$, *i.e. if* $\operatorname{sym}\operatorname{curl}\boldsymbol{\tau} = \mathbf{0}$ *and* $\boldsymbol{\tau} \in \mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K;\mathbb{T})$*, then there exists a* $\boldsymbol{v} \in \mathbb{B}_{\ell+2}(\operatorname{grad}, K;\mathbb{R}^3)$*, s.t.* $\boldsymbol{\tau} = \operatorname{dev}\operatorname{grad}\boldsymbol{v}$.

Firstly, by the exactness of the polynomial div div complex (25), there exists $\boldsymbol{v} \in \mathbb{P}_{\ell+2}(K;\mathbb{R}^3)$ such that $\boldsymbol{\tau} = \operatorname{dev}\operatorname{grad}\boldsymbol{v}$. As $\boldsymbol{RT} = \ker(\operatorname{dev}\operatorname{grad})$, we can further impose constraint $\int_F \boldsymbol{v}\cdot\boldsymbol{n} = 0$ for each $F \in \mathcal{F}(K)$. By (69), we get $\boldsymbol{v}\cdot\boldsymbol{n}\,|_F \in \mathbb{P}_0(F)$. Hence $\boldsymbol{v}\cdot\boldsymbol{n}|_F = 0$, which indicates $\boldsymbol{v}(\delta) = \mathbf{0}$ for each vertex $\delta \in \mathcal{V}(K)$. By (70), we obtain $\operatorname{sym}((\Pi_F\boldsymbol{v})\nabla_F^{\perp}) = \mathbf{0}$, i.e. $\Pi_F\boldsymbol{v} \in \mathbb{P}_0(F;\mathbb{R}^2) + (\Pi_F\boldsymbol{x})\mathbb{P}_0(F)$. This combined with $\boldsymbol{v}(\delta) = \mathbf{0}$ for each vertex $\delta \in \mathcal{V}(F)$ means $\Pi_F\boldsymbol{v} = \mathbf{0}$, and then $\boldsymbol{v}|_F = \mathbf{0}$ for each $F \in \mathcal{F}(K)$. Thus $\boldsymbol{v} \in \mathbb{B}_{\ell+2}(\operatorname{grad}, K;\mathbb{R}^3)$.

(2) $\operatorname{sym}\operatorname{curl}\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K;\mathbb{T}) = \mathbb{B}_{\ell,k}(\operatorname{div}\operatorname{div}, K;\mathbb{S}) \cap \ker(\operatorname{div}\operatorname{div})$.

By step (1), we acquire

$$
\begin{aligned}
&\dim\operatorname{sym}\operatorname{curl}\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K;\mathbb{T})\\
&= \dim\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K;\mathbb{T}) - \dim\mathbb{B}_{\ell+2}(\operatorname{grad}, K;\mathbb{R}^3)\\
&= \dim\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K;\mathbb{T}) - \dim\mathbb{P}_{\ell-2}(K;\mathbb{R}^3)\\
&= \frac{1}{6}\ell(\ell-1)(5\ell+17),
\end{aligned}
\tag{71}
$$

which together with (67) indicates

$$
\dim\operatorname{sym}\operatorname{curl}\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K;\mathbb{T}) = \dim(\mathbb{B}_{\ell,k}(\operatorname{div}\operatorname{div}, K;\mathbb{S}) \cap \ker(\operatorname{div}\operatorname{div})).
$$

Together with (65) implies $\operatorname{sym}\operatorname{curl}\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K;\mathbb{T}) = \mathbb{B}_{\ell,k}(\operatorname{div}\operatorname{div}, K;\mathbb{S}) \cap \ker(\operatorname{div}\operatorname{div})$.

(3) $\operatorname{div}\operatorname{div}\mathbb{B}_{\ell,k}(\operatorname{div}\operatorname{div}, K;\mathbb{S}) = \mathbb{P}_{k-2,1}^{\perp}(K)$. This is (66) proved in Lemma 5.11.

Therefore complex (68) is exact. □

As a result of complex (68), we can replace the degrees of freedom (51)-(52) by

$$
(\boldsymbol{\tau}, \boldsymbol{\varsigma})_K \quad \forall\, \boldsymbol{\varsigma} \in \operatorname{sym}\operatorname{curl}\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K;\mathbb{T}).
\tag{72}
$$

The dimension of (72) is counted in (71), which also matches the sum of (51)-(52).

Below we summarize the unisolvence for space $\boldsymbol{\Sigma}_{\ell,k}(K)$ with different DoFs.

Corollary 5.13. *The degrees of freedom* (46)-(50) *and* (72) *are unisolvent for* $\boldsymbol{\Sigma}_{\ell,k}(K)$.

Notice that although $\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K;\mathbb{T})$ is in a symmetric form, cf. (62), the degree of freedom (72) is indeed not simpler than (51)-(52) in computation as $\operatorname{sym}\operatorname{curl}\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K;\mathbb{T})$ is much more complicated than polynomials on a face.

5.4. **Two-dimensional** div div **conforming finite elements.** Recently we have constructed div div conforming finite elements in two dimensions in [6]. Here we briefly review the results and compare to the three-dimensional case.

Let F be a triangle. Take the space of shape functions

$$
\boldsymbol{\Sigma}_{\ell,k}(F) := \mathbb{C}_{\ell}(F;\mathbb{S}) \oplus \mathbb{C}_k^{\oplus}(F;\mathbb{S})
\tag{73}
$$

with $k \geq 3$ and $\ell \geq \max\{k-1, 3\}$ and

$$
\mathbb{C}_{\ell}(F;\mathbb{S}) = \operatorname{sym}\operatorname{curl}_F\,\mathbb{P}_{\ell+1}(F;\mathbb{R}^2), \quad \mathbb{C}_k^{\oplus}(F;\mathbb{S}) = \boldsymbol{x}\boldsymbol{x}^{\intercal}\mathbb{P}_{k-2}(F).
$$

Here the polynomial space for $\boldsymbol{H}(\operatorname{sym}\operatorname{curl}, F;\mathbb{R}^2)$ is the vector space not a tensor space, which simplifies the construction significantly.

The degrees of freedom are given by

$$\boldsymbol{\tau}(\delta) \quad \forall\, \delta \in \mathcal{V}(F), \tag{74}$$

$$(\boldsymbol{n}_e^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_e, q)_e \quad \forall\, q \in \mathbb{P}_{\ell-2}(e), e \in \mathcal{E}(F), \tag{75}$$

$$(\partial_t(\boldsymbol{t}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_e) + \boldsymbol{n}_e^{\intercal}\operatorname{div}_F\boldsymbol{\tau}, q)_e \quad \forall\, q \in \mathbb{P}_{\ell-1}(e), e \in \mathcal{E}(F), \tag{76}$$

$$(\boldsymbol{\tau}, \boldsymbol{\varsigma})_F \quad \forall\, \boldsymbol{\varsigma} \in \nabla_F^2\mathbb{P}_{k-2}(F), \tag{77}$$

$$(\boldsymbol{\tau}, \boldsymbol{\varsigma})_F \quad \forall\, \boldsymbol{\varsigma} \in \operatorname{sym}(\boldsymbol{x}^{\perp}\mathbb{P}_{\ell-2}(F;\mathbb{R}^2)). \tag{78}$$

Here to avoid confusion with the three-dimensional version, we use $\boldsymbol{n}_e$ to emphasize it is a normal vector of edge vector e.

The unisolvence is again better understood with the help of Fig. 1. By the vanishing degrees of freedom (74)-(76), the trace vanishes. Then together with the vanishing DoF (77), $\operatorname{div}\operatorname{div}\boldsymbol{\tau} = 0$. The DoF (78) is to identify the intersection of the bubble space and the kernel of div div. Define

$$\mathbb{B}_{\ell,k}(\operatorname{div}_F\operatorname{div}_F, F) := \{\boldsymbol{\tau} \in \boldsymbol{\Sigma}_{\ell,k}(F) : \text{all degrees of freedom (74)-(76) vanish}\}.$$

It turns out the space $\mathbb{B}_{\ell,k}(\operatorname{div}_F\operatorname{div}_F, F) \cap \ker(\operatorname{div}_F\operatorname{div}_F)$ is much simpler in two dimensions.

The key is the following formula on the trace tr_2.

Lemma 5.14. *When $\boldsymbol{\tau} = \operatorname{sym}\operatorname{curl}_F \boldsymbol{v}$, we have*

$$\partial_t(\boldsymbol{t}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_e) + \boldsymbol{n}_e^{\intercal}\operatorname{div}_F\boldsymbol{\tau} = \partial_t(\boldsymbol{t}^{\intercal}\partial_t\boldsymbol{v}). \tag{79}$$

Proof. Since $\operatorname{div}_F\operatorname{curl}_F \boldsymbol{v} = 0$, we have

$$\boldsymbol{n}_e^{\intercal}\operatorname{div}_F\boldsymbol{\tau} = \frac{1}{2}\boldsymbol{n}_e^{\intercal}\operatorname{div}_F(\operatorname{curl}_F \boldsymbol{v})^{\intercal} = \frac{1}{2}\boldsymbol{n}_e^{\intercal}\operatorname{curl}_F\operatorname{div}_F\boldsymbol{v} = \frac{1}{2}\partial_t\operatorname{div}_F\boldsymbol{v}.$$

As $\operatorname{div}_F\boldsymbol{v} = \operatorname{trace}(\nabla_F\boldsymbol{v})$ is invariant to the rotation, we can write it as

$$\operatorname{div}_F\boldsymbol{v} = \boldsymbol{t}^{\intercal}\nabla_F\boldsymbol{v}\boldsymbol{t} + \boldsymbol{n}_e^{\intercal}\nabla_F\boldsymbol{v}\boldsymbol{n}_e = \boldsymbol{t}^{\intercal}\partial_t\boldsymbol{v} + \boldsymbol{n}_e^{\intercal}\partial_n\boldsymbol{v}.$$

Then

$$\partial_t(\boldsymbol{t}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_e) + \boldsymbol{n}_e^{\intercal}\operatorname{div}_F\boldsymbol{\tau} = \frac{1}{2}\partial_t[\boldsymbol{t}^{\intercal}\partial_t\boldsymbol{v} - \boldsymbol{n}_e^{\intercal}\partial_n\boldsymbol{v} + \operatorname{div}_F\boldsymbol{v}] = \partial_t(\boldsymbol{t}^{\intercal}\partial_t\boldsymbol{v}),$$

i.e. (79) holds. □

Lemma 5.15. *The following bubble complex*

$$\boldsymbol{0} \xrightarrow{\subset} b_F\mathbb{P}_{\ell-2}(F;\mathbb{R}^2) \xrightarrow{\operatorname{sym}\operatorname{curl}_F} \mathbb{B}_{\ell,k}(\operatorname{div}_F\operatorname{div}_F, F) \xrightarrow{\operatorname{div}_F\operatorname{div}_F} \mathbb{P}_{k-2,1}^{\perp}(F) \to 0$$

is exact.

Proof. The fact that $\operatorname{div}_F\operatorname{div}_F : \mathbb{B}_{\ell,k}(\operatorname{div}_F\operatorname{div}_F, F) \to \mathbb{P}_{k-2,1}^{\perp}(F)$ is surjective can be proved similarly to Lemma 5.11.

For $\boldsymbol{\tau} \in \mathbb{B}_{\ell,k}(\operatorname{div}_F\operatorname{div}_F, F) \cap \ker(\operatorname{div}_F\operatorname{div}_F)$, from the complex (43), we can find $\boldsymbol{v} \in \mathbb{P}_{\ell+1}(F)$ s.t. $\operatorname{sym}\operatorname{curl}_F\boldsymbol{v} = \boldsymbol{\tau}$. We will prove $\boldsymbol{v}|_{\partial F} = \boldsymbol{0}$.

Since $\boldsymbol{RT} = \ker(\operatorname{sym}\operatorname{curl}_F)$, we can further impose constraint $\int_e \boldsymbol{v}\cdot\boldsymbol{n}_e = 0$ for each $e \in \mathcal{E}(F)$. The fact $(\boldsymbol{n}_e^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_e)|_{\partial F} = 0$ implies

$$\partial_t(\boldsymbol{n}_e^{\intercal}\boldsymbol{v})|_{\partial F} = (\boldsymbol{n}_e^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_e)|_{\partial F} = 0.$$

Hence $\boldsymbol{n}_e^{\intercal}\boldsymbol{v}|_{\partial F} = 0$. This also means $\boldsymbol{v}(\delta) = \boldsymbol{0}$ for each $\delta \in \mathcal{V}(F)$.

By Lemma 5.14, since

$$\partial_t(\boldsymbol{t}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_e) + \boldsymbol{n}_e^{\intercal}\operatorname{div}_F\boldsymbol{\tau} = \partial_t(\boldsymbol{t}^{\intercal}\partial_t\boldsymbol{v})$$

and $(\partial_t(\boldsymbol{t}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_e) + \boldsymbol{n}_e^{\intercal}\operatorname{div}_F\boldsymbol{\tau})|_{\partial F} = 0$, we acquire

$$\partial_{tt}(\boldsymbol{t}^{\intercal}\boldsymbol{v})|_{\partial F} = 0.$$

That is $\boldsymbol{t}^{\intercal}\boldsymbol{v}|_e \in \mathbb{P}_1(e)$ on each edge $e \in \mathcal{E}(F)$. Noting that $\boldsymbol{v}(\delta) = \boldsymbol{0}$ for each $\delta \in \mathcal{V}(F)$, we get $\boldsymbol{t}^{\intercal}\boldsymbol{v}|_{\partial F} = 0$ and consequently $\boldsymbol{v}|_{\partial F} = \boldsymbol{0}$, i.e.,

$$\boldsymbol{v} = b_F\psi_{\ell-2}, \quad \text{for some } \psi_{\ell-2} \in \mathbb{P}_{\ell-2}(F;\mathbb{R}^2).$$

□

We now prove the unisolvence as follows.

Theorem 5.16. *The degrees of freedom* (74)-(78) *are unisolvent for* $\boldsymbol{\Sigma}_{\ell,k}(F)$ (73).

Proof. We first count the number of DoFs (74)-(78) and the dimension of the space, i.e., $\dim\boldsymbol{\Sigma}_{\ell,k}(K)$. Both of them are

$$\ell^2 + 5\ell + 3 + \frac{1}{2}k(k-1).$$

Then suppose all the degrees of freedom (74)-(78) applied to $\boldsymbol{\tau}$ vanish. We are going to prove the function $\boldsymbol{\tau} = \boldsymbol{0}$.

By the vanishing degrees of freedom (74)-(76), the two traces are vanished. Together with (77), the Green's identity implies $\operatorname{div}_F\operatorname{div}_F\boldsymbol{\tau} = 0$. Then

$$\boldsymbol{\tau} = \operatorname{sym}\operatorname{curl}_F(b_F\psi_{\ell-2}), \quad \text{for some } \psi_{\ell-2} \in \mathbb{P}_{\ell-2}(F;\mathbb{R}^2).$$

We then use the fact $\operatorname{rot}_F : \operatorname{sym}(\boldsymbol{x}^{\perp}\mathbb{P}_{\ell-2}(F;\mathbb{R}^2)) \to \mathbb{P}_{\ell-2}(F;\mathbb{R}^2)$ is bijection, cf. the complex (44), to find $\phi_{\ell-2}$ s.t. $\operatorname{rot}_F(\operatorname{sym}(\boldsymbol{x}^{\perp}\phi_{\ell-2})) = \psi_{\ell-2}$. Finally we finish the unisolvence proof by choosing $\boldsymbol{\varsigma} = \operatorname{sym}(\boldsymbol{x}^{\perp}\phi_{\ell-2})$ in (78). The fact

$$(\boldsymbol{\tau},\boldsymbol{\varsigma})_F = (\operatorname{sym}\operatorname{curl}_F(b_F\psi_{\ell-2}), \operatorname{sym}(\boldsymbol{x}^{\perp}\phi_{\ell-2}))_F = (b_F\psi_{\ell-2}, \psi_{\ell-2})_F = 0$$

will imply $\psi_{\ell-2} = \boldsymbol{0}$ and consequently $\boldsymbol{\tau} = \boldsymbol{0}$. □

As finite element spaces for $\boldsymbol{H}^1$ are relatively mature and the bubble function space of $\mathbb{P}_{\ell+1}(F;\mathbb{R}^2)\cap\boldsymbol{H}_0^1(F;\mathbb{R}^2) = b_F\mathbb{P}_{\ell-2}(F;\mathbb{R}^2)$, the design of div div conforming finite elements in two dimensions is relatively easy. By rotation, we can construct finite elements for the strain space $\boldsymbol{H}(\operatorname{rot}_F\operatorname{rot}_F, F;\mathbb{S})$; see [6, Section 3.4].

6. Finite elements for sym curl-conforming trace-free tensors

In this section we construct conforming finite element spaces for $\boldsymbol{H}(\operatorname{sym}\operatorname{curl},\Omega;\mathbb{T})$.

6.1. **A finite element space.** Let K be a tetrahedron. For each edge e, we set a direction vector $\boldsymbol{t}$ and then choose two orthonormal vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ being orthogonal to e such that $\boldsymbol{n}_2 = \boldsymbol{t}\times\boldsymbol{n}_1$ and $\boldsymbol{n}_1 = \boldsymbol{n}_2\times\boldsymbol{t}$. Take the space of shape functions as $\mathbb{P}_{\ell+1}(K;\mathbb{T})$. The degrees of freedom $\mathcal{N}_{\ell+1}(K)$ are given by

(80) $$\boldsymbol{\tau}(\delta) \quad \forall\,\delta\in\mathcal{V}(K),$$

(81) $$(\operatorname{sym}\operatorname{curl}\boldsymbol{\tau})(\delta) \quad \forall\,\delta\in\mathcal{V}(K),$$

(82) $$(\boldsymbol{n}_i^{\intercal}(\operatorname{sym}\operatorname{curl}\boldsymbol{\tau})\boldsymbol{n}_j, q)_e \quad \forall\,q\in\mathbb{P}_{\ell-2}(e), e\in\mathcal{E}(K), i,j=1,2,$$

(83) $$(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{t}, q)_e \quad \forall\,q\in\mathbb{P}_{\ell-1}(e), e\in\mathcal{E}(K), i=1,2,$$

(84) $$(\boldsymbol{n}_2^{\intercal}(\operatorname{curl}\boldsymbol{\tau})\boldsymbol{n}_1 + \partial_t(\boldsymbol{t}^{\intercal}\boldsymbol{\tau}\boldsymbol{t}), q)_e \quad \forall\,q\in\mathbb{P}_{\ell}(e), e\in\mathcal{E}(K),$$

(85) $$(\boldsymbol{n}\times\operatorname{sym}(\boldsymbol{\tau}\times\boldsymbol{n})\times\boldsymbol{n}, \boldsymbol{\varsigma})_F \quad \forall\,\boldsymbol{\varsigma}\in(\nabla_F^{\perp})^2\,\mathbb{P}_{\ell-1}(F)\oplus\operatorname{sym}(\boldsymbol{x}\otimes\mathbb{P}_{\ell-1}(F;\mathbb{R}^2)),$$

(86) $$(\boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{n}, \boldsymbol{q})_F \quad \forall\,\boldsymbol{q}\in\nabla_F\mathbb{P}_{\ell-3}(F)\oplus\boldsymbol{x}^{\perp}\mathbb{P}_{\ell-1}(F), F\in\mathcal{F}(K),$$

(87) $$(\boldsymbol{\tau},\boldsymbol{q})_K \quad \forall\,\boldsymbol{q}\in\mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K;\mathbb{T}).$$

The degrees of freedom (81), (82), and (87) are motivated by (46), (47), and (72), respectively, as $\operatorname{sym}\operatorname{curl}\boldsymbol{\tau} \in \boldsymbol{H}(\operatorname{div}\operatorname{div}, K; \mathbb{S})$. Recall that $\operatorname{tr}_2(\boldsymbol{\tau}) \in H(\operatorname{div}_F)$ and $\operatorname{tr}_1(\boldsymbol{\tau}) \in H(\operatorname{div}_F \operatorname{div}_F)$, cf. Lemma 5.9. Let $\boldsymbol{n}_{F,e} = \boldsymbol{t} \times \boldsymbol{n}$ be the norm vector of e sitting on the face F. For div_F elements on face F, the normal trace becomes

$$(\boldsymbol{n} \cdot \boldsymbol{\tau} \times \boldsymbol{n}) \cdot \boldsymbol{n}_{F,e} = \boldsymbol{n}^{\intercal} \boldsymbol{\tau} \boldsymbol{t},$$

which motivates (83). Together with (86), $\boldsymbol{n} \cdot \boldsymbol{\tau} \times \boldsymbol{n}$ can be determined. For the $\operatorname{div}_F \operatorname{div}_F$ element, the normal-normal trace becomes

$$\boldsymbol{n}_{F,e}^{\intercal}(\Pi_F \operatorname{sym}(\boldsymbol{\tau} \times \boldsymbol{n})\Pi_F)\boldsymbol{n}_{F,e} = \boldsymbol{n}_{F,e}^{\intercal} \operatorname{sym}(\boldsymbol{\tau} \times \boldsymbol{n})\boldsymbol{n}_{F,e} = \boldsymbol{n}_{F,e}^{\intercal} \boldsymbol{\tau} \boldsymbol{t}, \tag{88}$$

which can be also determined by (83). Notice that for each edge e, there are two $\boldsymbol{n}_{F,e}$ inside one tetrahedron. In (83), the two normal vectors $\boldsymbol{n}_1, \boldsymbol{n}_2$ are chosen independent of elements and (83) can determine the projection of vector $\boldsymbol{\tau t}$ to the plane orthogonal to edge e including $\boldsymbol{n}_{F,e}^{\intercal} \boldsymbol{\tau} \boldsymbol{t}$.

The other trace of a $\operatorname{div}_F \operatorname{div}_F$ element will be determined by (82) and (84), which is less obvious. Lemma 6.1 is borrowed from [16, Lemma 9 and Remark 8].

Lemma 6.1. *Let $F \in \mathcal{F}(K)$ with a normal vector $\boldsymbol{n}_F$. For an edge $e \in \mathcal{E}(F)$, we fix a direction vector $\boldsymbol{t}$ for e and choose two orthonormal vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ being orthogonal to e such that $\boldsymbol{n}_2 = \boldsymbol{t} \times \boldsymbol{n}_1$ and $\boldsymbol{n}_1 = \boldsymbol{n}_2 \times \boldsymbol{t}$. Let $\boldsymbol{n}_{F,e} = \boldsymbol{t} \times \boldsymbol{n}_F$. For any sufficiently smooth tensor $\boldsymbol{\tau}$, we have*

$$\begin{aligned}\boldsymbol{n}_{F,e}^{\intercal}(\operatorname{curl}\boldsymbol{\tau})\boldsymbol{n}_F = {}& (\boldsymbol{n}_F \cdot \boldsymbol{n}_1)(\boldsymbol{n}_F \cdot \boldsymbol{n}_2)\left[\boldsymbol{n}_2^{\intercal}(\operatorname{sym}\operatorname{curl}\boldsymbol{\tau})\boldsymbol{n}_2 - \boldsymbol{n}_1^{\intercal}(\operatorname{sym}\operatorname{curl}\boldsymbol{\tau})\boldsymbol{n}_1\right] \\ & - 2(\boldsymbol{n}_F \cdot \boldsymbol{n}_2)^2 \boldsymbol{n}_1^{\intercal}(\operatorname{sym}\operatorname{curl}\boldsymbol{\tau})\boldsymbol{n}_2 + \boldsymbol{n}_2^{\intercal}(\operatorname{curl}\boldsymbol{\tau})\boldsymbol{n}_1.\end{aligned} \tag{89}$$

For $\operatorname{tr}_1(\boldsymbol{\tau}) = \Pi_F \operatorname{sym}(\boldsymbol{\tau} \times \boldsymbol{n}_F)\Pi_F$, we have

$$\partial_t(\boldsymbol{t}^{\intercal} \operatorname{tr}_1(\boldsymbol{\tau})\boldsymbol{n}_{F,e}) + \boldsymbol{n}_{F,e}^{\intercal} \operatorname{div}_F(\operatorname{tr}_1(\boldsymbol{\tau})) = \boldsymbol{n}_{F,e}^{\intercal}(\operatorname{curl}\boldsymbol{\tau})\boldsymbol{n}_F + \partial_t(\boldsymbol{t}^{\intercal}\boldsymbol{\tau}\boldsymbol{t}). \tag{90}$$

Consequently it can be determined by DoFs (82) *and* (84)*.*

Proof. On the plane orthogonal to e, the vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ form an orthonormal basis. We expand $\boldsymbol{n}_F = c_1\boldsymbol{n}_1 + c_2\boldsymbol{n}_2$ in this coordinate, with $c_i = \boldsymbol{n}_F \cdot \boldsymbol{n}_i$ for $i = 1, 2$. Then $\boldsymbol{n}_{F,e} = \boldsymbol{t} \times \boldsymbol{n}_F = c_1\boldsymbol{n}_2 - c_2\boldsymbol{n}_1$. Then in this coordinate

$$\begin{aligned}\boldsymbol{n}_{F,e}^{\intercal}(\operatorname{curl}\boldsymbol{\tau})\boldsymbol{n}_F &= (c_1\boldsymbol{n}_2 - c_2\boldsymbol{n}_1)^{\intercal}(\operatorname{curl}\boldsymbol{\tau})(c_1\boldsymbol{n}_1 + c_2\boldsymbol{n}_2) \\ &= c_1c_2(\boldsymbol{n}_2^{\intercal}(\operatorname{curl}\boldsymbol{\tau})\boldsymbol{n}_2 - \boldsymbol{n}_1^{\intercal}(\operatorname{curl}\boldsymbol{\tau})\boldsymbol{n}_1) \\ &\quad + c_1^2\boldsymbol{n}_2^{\intercal}(\operatorname{curl}\boldsymbol{\tau})\boldsymbol{n}_1 - c_2^2\boldsymbol{n}_1^{\intercal}(\operatorname{curl}\boldsymbol{\tau})\boldsymbol{n}_2.\end{aligned}$$

Thus we acquire (89) from the fact $c_1^2 + c_2^2 = 1$.

On the other hand, by the fact $\nabla_F = \boldsymbol{t}\partial_t + \boldsymbol{n}_{F,e}\partial_{n_{F,e}}$, we obtain

$$\begin{aligned}&\partial_t(\boldsymbol{t}^{\intercal} \operatorname{tr}_1(\boldsymbol{\tau})\boldsymbol{n}_{F,e}) + \boldsymbol{n}_{F,e}^{\intercal} \operatorname{div}_F(\operatorname{tr}_1(\boldsymbol{\tau})) \\ =&2\partial_t(\boldsymbol{t}^{\intercal} \operatorname{tr}_1(\boldsymbol{\tau})\boldsymbol{n}_{F,e}) + \partial_{n_{F,e}}(\boldsymbol{n}_{F,e}^{\intercal} \operatorname{tr}_1(\boldsymbol{\tau})\boldsymbol{n}_{F,e}) \\ =&2\partial_t(\boldsymbol{t}^{\intercal} \operatorname{sym}(\boldsymbol{\tau} \times \boldsymbol{n}_F)\boldsymbol{n}_{F,e}) + \partial_{n_{F,e}}(\boldsymbol{n}_{F,e}^{\intercal} \operatorname{sym}(\boldsymbol{\tau} \times \boldsymbol{n}_F)\boldsymbol{n}_{F,e}) \\ =&\partial_t(\boldsymbol{t}^{\intercal}\boldsymbol{\tau}\boldsymbol{t} - \boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_{F,e}) + \partial_{n_{F,e}}(\boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\tau}\boldsymbol{t}),\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{n}_{F,e}^{\intercal}(\operatorname{curl}\boldsymbol{\tau})\boldsymbol{n}_F &= (\boldsymbol{n}_F \times \nabla) \cdot (\boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\tau}) = (\boldsymbol{n}_F \times \nabla) \cdot (\boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\tau}\boldsymbol{t}\boldsymbol{t} + \boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_{F,e}\boldsymbol{n}_{F,e}) \\ &= \partial_{n_{F,e}}(\boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\tau}\boldsymbol{t}) - \partial_t(\boldsymbol{n}_{F,e}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_{F,e}).\end{aligned}$$

Therefore (90) is true. ∎

The trace $\partial_t(\boldsymbol{t}^\intercal \operatorname{tr}_1(\boldsymbol{\tau})\boldsymbol{n}_{F,e}) + \boldsymbol{n}_{F,e}^\intercal \operatorname{div}_F(\operatorname{tr}_1(\boldsymbol{\tau}))$ depends on F. For one edge e in a tetrahedron K, there are two such traces. Lemma 6.1 shows that these two traces are linearly dependent and only one DoF (84) is needed.

Lemma 6.2. *Let $F \in \mathcal{F}(K)$ and $\boldsymbol{\tau} \in \mathbb{P}_{\ell+1}(K;\mathbb{T})$. If all the degrees of freedom* (80)-(86) *vanish, then $\boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{n} = \boldsymbol{0}$ and $\boldsymbol{n}\times \operatorname{sym}(\boldsymbol{\tau}\times\boldsymbol{n})\times\boldsymbol{n} = \boldsymbol{0}$ on face F.*

Proof. It follows from (63), (83) and the first part of (86) that

$$(\boldsymbol{n}^\intercal(\operatorname{sym}\operatorname{curl}\boldsymbol{\tau})\boldsymbol{n}, q)_F = (\operatorname{div}_F(\boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{n}), q)_F = 0 \quad \forall\, q \in \mathbb{P}_{\ell-3}(F).$$

This combined with (81)-(82) yields $\boldsymbol{n}^\intercal(\operatorname{sym}\operatorname{curl}\boldsymbol{\tau})\boldsymbol{n}|_F = 0$, i.e. $\operatorname{div}_F(\boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{n}|_F) = 0$. Thanks to the unisolvence of BDM element, we achieve $\boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{n}|_F = \boldsymbol{0}$ from (83) and the second part of (86).

Let $\boldsymbol{\sigma} = \Pi_F \operatorname{sym}(\boldsymbol{\tau}\times\boldsymbol{n}_F)\Pi_F$ for simplicity. Thanks to (88), we get from (83) that $\boldsymbol{n}_{F,e}^\intercal\boldsymbol{\sigma}\boldsymbol{n}_{F,e} = 0$ on each edge $e \in \mathcal{E}(F)$. By (89)-(90), it follows from (81)-(82) and (84) that $(\partial_t(\boldsymbol{t}^\intercal\boldsymbol{\sigma}\boldsymbol{n}_{F,e}) + \boldsymbol{n}_{F,e}^\intercal \operatorname{div}_F \boldsymbol{\sigma})|_e = 0$, which together with (85) and the unisolvence of div div element in two dimensions, i.e. Theorem 5.16, implies that $\boldsymbol{\sigma}|_F = \boldsymbol{0}$. ☐

We are in the position to prove the unisolvence.

Theorem 6.3. *The degrees of freedom* (80)-(87) *are unisolvent for $\mathbb{P}_{\ell+1}(K;\mathbb{T})$.*

Proof. It is easy to see that

$$\begin{aligned}\#\mathcal{N}_{\ell+1}(K) &= 56 + 6(6\ell-2) + 4\left(2\ell(\ell+1) + \frac{1}{2}(\ell-1)(\ell-2) - 4\right)\\ &\quad + \frac{1}{3}(4\ell^3 + 6\ell^2 - 10\ell) = \frac{4}{3}(\ell+4)(\ell+3)(\ell+2)\\ &= \dim \mathbb{P}_{\ell+1}(K;\mathbb{T}).\end{aligned}$$

Take any $\boldsymbol{\tau} \in \mathbb{P}_{\ell+1}(K;\mathbb{T})$ and suppose all the degrees of freedom (80)-(87) vanish. Then by Lemma 6.2, $\boldsymbol{\tau} \in \mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K;\mathbb{T})$. Then taking $\boldsymbol{q} = \boldsymbol{\tau}$ in (87), we conclude $\boldsymbol{\tau} = \boldsymbol{0}$. ☐

6.2. **Lagrange-type degrees of freedom.** The DoF $\mathcal{N}_{\ell+1}$ is designed to form a finite element div div complex. If the exactness of the sequence is not the concern, we can construct simpler degrees of freedom. Below is the Lagrange-type $\boldsymbol{H}(\operatorname{sym}\operatorname{curl})$-conforming finite elements for trace-free tensors. Take the space of shape functions as $\mathbb{P}_{\ell+1}(K;\mathbb{T})$. The degrees of freedom are given by

$$\boldsymbol{\tau}(\delta) \quad \forall\, \delta \in \mathcal{V}(K), \tag{91}$$

$$(\boldsymbol{\tau}, \boldsymbol{q})_e \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{\ell-1}(e;\mathbb{T}), e \in \mathcal{E}(K), \tag{92}$$

$$(\boldsymbol{n}\times\operatorname{sym}(\boldsymbol{\tau}\times\boldsymbol{n})\times\boldsymbol{n}, \boldsymbol{q})_F \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{\ell-2}(F;\mathbb{S}), F \in \mathcal{F}(K), \tag{93}$$

$$(\boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{n}, \boldsymbol{q})_F \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{\ell-2}(F;\mathbb{R}^2), F \in \mathcal{F}(K), \tag{94}$$

$$(\boldsymbol{\tau}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K;\mathbb{T}). \tag{95}$$

It is straightforward to verify the unisolvence of (91)-(95) due to the characterization of trace operators and bubble functions.

We can also take another set of degrees of freedom

$$
\begin{aligned}
&\boldsymbol{\tau}(\delta) && \forall\, \delta \in \mathcal{V}(K),\\
&(\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{t}, q)_e && \forall\, q \in \mathbb{P}_{\ell-1}(e), e \in \mathcal{E}(K), i=1,2,\\
&(\boldsymbol{n}\times \operatorname{sym}(\boldsymbol{\tau}\times\boldsymbol{n})\times\boldsymbol{n}, \boldsymbol{q})_F && \forall\, \boldsymbol{q} \in \mathring{\mathbb{P}}_{\ell}(F;\mathbb{S}), F\in\mathcal{F}(K),\\
&(\boldsymbol{n}\cdot\boldsymbol{\tau}\times\boldsymbol{n}, \boldsymbol{q})_F && \forall\, \boldsymbol{q} \in \nabla_F\mathbb{P}_{\ell}(F)\oplus \boldsymbol{x}^{\perp}\mathbb{P}_{\ell-1}(F), F\in\mathcal{F}(K),\\
&(\boldsymbol{\tau}, \boldsymbol{q})_K && \forall\, \boldsymbol{q} \in \mathbb{B}_{\ell+1}(\operatorname{sym}\operatorname{curl}, K;\mathbb{T}),
\end{aligned} \tag{96}
$$

where

$$
\mathring{\mathbb{P}}_{\ell}(F;\mathbb{S}) := \{\boldsymbol{q}\in\mathbb{P}_{\ell}(F;\mathbb{S}) : (\boldsymbol{t}_1^{\intercal}\boldsymbol{q}\boldsymbol{t}_2)(\delta) = 0 \text{ for each } \delta\in\mathcal{V}(K)\}
$$

with $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ being the unit tangential vectors of two edges of F sharing δ. The degree of freedom (96) is motivated by the Hellan-Herrmann-Johnson mixed method for the Kirchhoff plate bending problems [13, 14, 18] in two dimensions.

7. A finite element div div complex in three dimensions

In this section, we collect finite element spaces defined before to form a finite element div div complex. We assume $\mathcal{T}_h$ is a triangulation of a topological trivial domain Ω.

7.1. A finite element divdiv complex.

We start from the vectorial Hermite element space in three dimensions [9]

$$
\begin{aligned}
\boldsymbol{V}_h := \{\boldsymbol{v}_h \in \boldsymbol{H}^1(\Omega;\mathbb{R}^3) : &\boldsymbol{v}_h|_K \in \mathbb{P}_{\ell+2}(K;\mathbb{R}^3) \text{ for each } K\in\mathcal{T}_h,\\
&\nabla\boldsymbol{v}_h(\delta) \text{ is single-valued at each vertex } \delta\in\mathcal{V}_h\}.
\end{aligned}
$$

The local degrees of freedom for $\boldsymbol{V}_h(K) := \boldsymbol{V}_h|_K$ are

$$
\begin{aligned}
\boldsymbol{v}(\delta), \nabla\boldsymbol{v}(\delta) &\quad \forall\, \delta\in\mathcal{V}(K),\\
(\boldsymbol{v}, \boldsymbol{q})_e &\quad \forall\, \boldsymbol{q}\in\mathbb{P}_{\ell-2}(e;\mathbb{R}^3), e\in\mathcal{E}(K),\\
(\boldsymbol{v}, \boldsymbol{q})_F &\quad \forall\, \boldsymbol{q}\in\mathbb{P}_{\ell-1}(F;\mathbb{R}^3), F\in\mathcal{F}(K),\\
(\boldsymbol{v}, \boldsymbol{q})_K &\quad \forall\, \boldsymbol{q}\in\mathbb{P}_{\ell-2}(K;\mathbb{R}^3).
\end{aligned}
$$

The unisolvence for $\boldsymbol{V}_h(K)$ is trivial. And

$$
\dim \boldsymbol{V}_h = 12\#\mathcal{V}_h + 3(\ell-1)\#\mathcal{E}_h + \frac{3}{2}(\ell+1)\ell\#\mathcal{F}_h + \frac{1}{2}(\ell^3-\ell)\#\mathcal{T}_h.
$$

Let

$$
\begin{aligned}
\boldsymbol{\Sigma}_h^{\mathbb{T}} := \{\boldsymbol{\tau}_h\in\boldsymbol{L}^2(\Omega;\mathbb{T}) : &\boldsymbol{\tau}_h|_K\in\mathbb{P}_{\ell+1}(K;\mathbb{T}) \text{ for each } K\in\mathcal{T}_h, \text{ all the}\\
&\text{degrees of freedom (80)-(86) are single-valued}\},
\end{aligned}
$$

then

$$
\begin{aligned}
\dim\boldsymbol{\Sigma}_h^{\mathbb{T}} = 14\#\mathcal{V}_h &+ (6\ell-2)\#\mathcal{E}_h + \left(2\ell(\ell+1) + \frac{1}{2}(\ell-1)(\ell-2) - 4\right)\#\mathcal{F}_h\\
&+\frac{1}{3}(4\ell^3+6\ell^2-10\ell)\#\mathcal{T}_h.
\end{aligned}
$$

Clearly Lemma 6.2 ensures $\boldsymbol{\Sigma}_h^{\mathbb{T}}\subset\boldsymbol{H}(\operatorname{sym}\operatorname{curl},\Omega;\mathbb{T})$. Let

$$
\begin{aligned}
\boldsymbol{\Sigma}_h^{\mathbb{S}} := \{\boldsymbol{\tau}_h\in\boldsymbol{L}^2(\Omega;\mathbb{S}) : &\boldsymbol{\tau}_h|_K\in\boldsymbol{\Sigma}_{\ell,k}(K) \text{ for each } K\in\mathcal{T}_h, \text{ all the}\\
&\text{degrees of freedom (46)-(49) are single-valued}\},
\end{aligned}
$$

then

$$\dim \boldsymbol{\Sigma}_h^{\mathbb{S}} = 6\#\mathcal{V}_h + 3(\ell-1)\#\mathcal{E}_h + (\ell^2-\ell+1)\#\mathcal{F}_h + \left(\frac{1}{2}\ell(\ell-1) + \frac{1}{6}(\ell-1)\ell(5\ell+14) + \frac{1}{6}(k^3-k) - 4\right)\#\mathcal{T}_h.$$

The proof of Lemma 5.2 ensures $\boldsymbol{\Sigma}_h^{\mathbb{S}} \subset \boldsymbol{H}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})$. Let

$$\mathcal{Q}_h := \mathbb{P}_{k-2}(\mathcal{T}_h) = \{q_h \in L^2(\Omega) : q_h|_K \in \mathbb{P}_{k-2}(K) \text{ for each } K \in \mathcal{T}_h\}$$

be the discontinuous polynomial space. Obviously

$$\dim \mathcal{Q}_h = \frac{1}{6}(k^3-k)\#\mathcal{T}_h.$$

Lemma 7.1. *It holds*

$$\operatorname{div}\operatorname{div}\boldsymbol{\Sigma}_h^{\mathbb{S}} = \mathcal{Q}_h.$$

Proof. Apparently $\operatorname{div}\operatorname{div}\boldsymbol{\Sigma}_h^{\mathbb{S}} \subseteq \mathcal{Q}_h$. Then we focus on $\mathcal{Q}_h \subseteq \operatorname{div}\operatorname{div}\boldsymbol{\Sigma}_h^{\mathbb{S}}$.

Take any $v_h \in \mathcal{Q}_h$. By the fact $\operatorname{div}\operatorname{div}\boldsymbol{H}^2(\Omega;\mathbb{S}) = L^2(\Omega)$ [10], there exists $\boldsymbol{\tau} \in \boldsymbol{H}^2(\Omega;\mathbb{S})$ such that

$$\operatorname{div}\operatorname{div}\boldsymbol{\tau} = v_h.$$

Let $I_h\boldsymbol{\tau} \in \boldsymbol{\Sigma}_h^{\mathbb{S}}$ be determined by

$$\boldsymbol{N}(I_h\boldsymbol{\tau}) = \boldsymbol{N}(\boldsymbol{\tau})$$

for all DoFs $\boldsymbol{N}$ from (46) to (52). Note that for functions in $H^2(K)$, the integrals on edge and pointwise value are well-defined. Since $\ell \geq 3$, it follows from the Green's identity (45) that

$$(\operatorname{div}\operatorname{div}(\boldsymbol{\tau} - I_h\boldsymbol{\tau}), q)_K = 0 \quad \forall\, q \in \mathbb{P}_1(K),\ K \in \mathcal{T}_h.$$

Hence $(v_h - \operatorname{div}\operatorname{div} I_h\boldsymbol{\tau})|_K = \operatorname{div}\operatorname{div}(\boldsymbol{\tau} - I_h\boldsymbol{\tau})|_K \in \mathbb{P}_{k-2,1}^{\perp}(K)$. Applying (66), there exists $\boldsymbol{\tau}_b \in \boldsymbol{\Sigma}_h^{\mathbb{S}}$ such that $\boldsymbol{\tau}_b|_K \in \mathbb{B}_{\ell,k}(\operatorname{div}\operatorname{div},K;\mathbb{S})$ for each $K \in \mathcal{T}_h$, and

$$v_h - \operatorname{div}\operatorname{div} I_h\boldsymbol{\tau} = \operatorname{div}\operatorname{div}\boldsymbol{\tau}_b.$$

Therefore $v_h = \operatorname{div}\operatorname{div}(I_h\boldsymbol{\tau} + \boldsymbol{\tau}_b)$, where $I_h\boldsymbol{\tau} + \boldsymbol{\tau}_b \in \boldsymbol{\Sigma}_h^{\mathbb{S}}$, as required. $\square$

Theorem 7.2. *Assume Ω is a bounded and topologically trivial Lipschitz domain in $\mathbb{R}^3$. The finite element* div div *complex*

$$\boldsymbol{RT} \xrightarrow{\subset} \boldsymbol{V}_h \xrightarrow{\operatorname{dev}\operatorname{grad}} \boldsymbol{\Sigma}_h^{\mathbb{T}} \xrightarrow{\operatorname{sym}\operatorname{curl}} \boldsymbol{\Sigma}_h^{\mathbb{S}} \xrightarrow{\operatorname{div}\operatorname{div}} \mathcal{Q}_h \to 0 \tag{97}$$

is exact.

Proof. For any sufficient vector function $\boldsymbol{v}$ and $e \in \mathcal{E}(K)$, we have from $\boldsymbol{t} = \boldsymbol{n}_1 \times \boldsymbol{n}_2$ that

$$\begin{aligned}
&\boldsymbol{n}_2^{\intercal}(\operatorname{curl}(\operatorname{dev}\operatorname{grad}\boldsymbol{v}))\boldsymbol{n}_1 + \partial_t(\boldsymbol{t}^{\intercal}(\operatorname{dev}\operatorname{grad}\boldsymbol{v})\boldsymbol{t})\\
&= -\frac{1}{3}\boldsymbol{n}_1 \cdot \operatorname{curl}(\boldsymbol{n}_2 \operatorname{div}\boldsymbol{v}) + \partial_{tt}(\boldsymbol{v}\cdot\boldsymbol{t}) - \frac{1}{3}\partial_t(\operatorname{div}\boldsymbol{v})\\
&= \frac{1}{3}(\boldsymbol{n}_1 \times \boldsymbol{n}_2)\cdot\nabla(\operatorname{div}\boldsymbol{v}) + \partial_{tt}(\boldsymbol{v}\cdot\boldsymbol{t}) - \frac{1}{3}\partial_t(\operatorname{div}\boldsymbol{v}) = \partial_{tt}(\boldsymbol{v}\cdot\boldsymbol{t}).
\end{aligned}$$

Hence by (69)-(70) it is easy to see that $\operatorname{dev}\operatorname{grad}\boldsymbol{V}_h \subset \boldsymbol{\Sigma}_h^{\mathbb{T}}$. It holds from Lemma 6.2 and the degrees of freedom (81)-(82) that

$$\operatorname{sym}\operatorname{curl}\boldsymbol{\Sigma}_h^{\mathbb{T}} \subset \boldsymbol{\Sigma}_h^{\mathbb{S}}. \tag{98}$$

Thus we get from Lemma 7.1 that (97) is a complex.

We then verify the exactness.

(1) $\boldsymbol{V}_h \cap \ker(\operatorname{dev}\operatorname{grad}) = \boldsymbol{RT}$. By the exactness of the complex (20),

$$\boldsymbol{RT} \subseteq \boldsymbol{V}_h \cap \ker(\operatorname{dev}\operatorname{grad}) \subseteq \boldsymbol{H}^1(\Omega;\mathbb{R}^3) \cap \ker(\operatorname{dev}\operatorname{grad}) = \boldsymbol{RT}.$$

(2) $\boldsymbol{\Sigma}_h^{\mathbb{T}} \cap \ker(\operatorname{sym}\operatorname{curl}) = \operatorname{dev}\operatorname{grad}\boldsymbol{V}_h$, *i.e. if* $\operatorname{sym}\operatorname{curl}\boldsymbol{\tau} = \boldsymbol{0}$ *and* $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_h^{\mathbb{T}}$, *then there exists a* $\boldsymbol{v} \in \boldsymbol{V}_h$, *s.t.* $\boldsymbol{\tau} = \operatorname{dev}\operatorname{grad}\boldsymbol{v}$.

Since $\operatorname{sym}\operatorname{curl}\boldsymbol{\tau} = \boldsymbol{0}$, by the div div complex (20) and the polynomial div div complex (25), there exists $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega;\mathbb{R}^3)$ such that $\boldsymbol{\tau} = \operatorname{dev}\operatorname{grad}\boldsymbol{v}$ and $\boldsymbol{v}|_K \in \mathbb{P}_{\ell+2}(K;\mathbb{R}^3)$ for each $K \in \mathcal{T}_h$. To show $\boldsymbol{v} \in \boldsymbol{V}_h$, it suffices to prove $\operatorname{div}\boldsymbol{v}$ is single-valued at each vertex in $\mathcal{V}_h$, since $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega;\mathbb{R}^3)$ and $\operatorname{dev}\operatorname{grad}\boldsymbol{v} = \boldsymbol{\tau}$ is single-valued at each vertex in $\mathcal{V}_h$. To this end, take a tetrahedron $K \in \mathcal{T}_h$, a vertex $\delta \in \mathcal{V}(K)$ and an edge $e \in \mathcal{E}(K)$ such that δ is an endpoint of e. By the fact $\operatorname{grad}\boldsymbol{v} = \operatorname{dev}\operatorname{grad}\boldsymbol{v} + \frac{1}{3}(\operatorname{div}\boldsymbol{v})\boldsymbol{I}$, we get

$$(\operatorname{div}\boldsymbol{v}|_K)(\delta) = 3\big(\partial_{\boldsymbol{t}}(\boldsymbol{v}\cdot\boldsymbol{t})\big)(\delta) - 3\boldsymbol{t}^{\intercal}\boldsymbol{\tau}(\delta)\boldsymbol{t},$$

where $\boldsymbol{t}$ is the unit tangential vector of e. This implies $\operatorname{div}\boldsymbol{v}$ is single-valued at each vertex in $\mathcal{V}_h$. And then $\boldsymbol{\Sigma}_h^{\mathbb{T}} \cap \ker(\operatorname{sym}\operatorname{curl}) \subseteq \operatorname{dev}\operatorname{grad}\boldsymbol{V}_h$.

(3) $\operatorname{div}\operatorname{div}\boldsymbol{\Sigma}_h^{\mathbb{S}} = \mathcal{Q}_h$. This is Lemma 7.1.

(4) $\boldsymbol{\Sigma}_h^{\mathbb{S}} \cap \ker(\operatorname{div}\operatorname{div}) = \operatorname{sym}\operatorname{curl}\boldsymbol{\Sigma}_h^{\mathbb{T}}$.

We verify this identity by dimension count. By Lemma 7.1,

$$\begin{aligned}\dim(\boldsymbol{\Sigma}_h^{\mathbb{S}} \cap \ker(\operatorname{div}\operatorname{div})) &= \dim\boldsymbol{\Sigma}_h^{\mathbb{S}} - \dim\mathcal{Q}_h \\ &= 6\#\mathcal{V}_h + 3(\ell-1)\#\mathcal{E}_h + (\ell^2-\ell+1)\#\mathcal{F}_h \\ &\quad + \left(\frac{1}{6}(\ell-1)\ell(5\ell+17) - 4\right)\#\mathcal{T}_h.\end{aligned} \tag{99}$$

As a result of step (2),

$$\begin{aligned}\dim\operatorname{sym}\operatorname{curl}\boldsymbol{\Sigma}_h^{\mathbb{T}} = \dim\boldsymbol{\Sigma}_h^{\mathbb{T}} - \dim\operatorname{dev}\operatorname{grad}\boldsymbol{V}_h &= \dim\boldsymbol{\Sigma}_h^{\mathbb{T}} - \dim\boldsymbol{V}_h + 4 \\ &= 2\#\mathcal{V}_h + (3\ell+1)\#\mathcal{E}_h + (\ell^2-\ell-3)\#\mathcal{F}_h \\ &\quad + \frac{1}{6}(\ell-1)\ell(5\ell+17)\#\mathcal{T}_h + 4.\end{aligned}$$

Applying the Euler's formula $\#\mathcal{V}_h - \#\mathcal{E}_h + \#\mathcal{F}_h - \#\mathcal{T}_h = 1$, we get from (99) that $\dim\operatorname{sym}\operatorname{curl}\boldsymbol{\Sigma}_h^{\mathbb{T}} = \dim(\boldsymbol{\Sigma}_h^{\mathbb{S}} \cap \ker(\operatorname{div}\operatorname{div}))$. Then the result follows from (98).

Therefore the finite element div div complex (97) is exact. ☐

For the completeness, we present a two-dimensional finite element div div complex but restricted to one element. A global version of (100) as well as a commutative diagram involving quasi-interpolation operators from Sobolev spaces to finite element spaces can be found in [6].

Let $\boldsymbol{V}_{\ell+1}(F) := \mathbb{P}_{\ell+1}(F;\mathbb{R}^2)$ with $\ell \geq 2$ be the vectorial Hermite element [3,9].

Lemma 7.3. *For any triangle F, the polynomial complex*

$$\boldsymbol{RT} \xrightarrow{\subset} \boldsymbol{V}_{\ell+1}(F) \xrightarrow{\operatorname{sym}\operatorname{curl}_F} \boldsymbol{\Sigma}_{\ell,k}(F) \xrightarrow{\operatorname{div}_F\operatorname{div}_F} \mathbb{P}_{k-2}(F) \to 0 \tag{100}$$

is exact.

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