J Sci Comput
DOI 10.1007/s10915-013-9771-3

A Posteriori Error Estimates for Weak Galerkin Finite Element Methods for Second Order Elliptic Problems

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Received: 29 January 2013 / Revised: 5 July 2013 / Accepted: 13 August 2013

Abstract A residual type a posteriori error estimator is presented and analyzed for Weak Galerkin finite element methods for second order elliptic problems. The error estimator is proved to be efficient and reliable through two estimates, one from below and the other from above, in terms of an H^1-equivalent norm for the exact error. Two numerical experiments are conducted to demonstrate the effectiveness of adaptive mesh refinement guided by this estimator.

Keywords Weak Galerkin · Finite element methods · Discrete weak gradient · Second-order elliptic problems · A posterior error estimate

Mathematics Subject Classification Primary: 65N15 , 65N30 · Secondary: 35J50

Long Chen: This research was supported in part by National Science Foundation Grant DMS-1115961 and in part by Department of Energy prime award # DE-SC0006903.

Junping Wang: The research of Wang was supported by the NSF IR/D program, while working at National Science Foundation. However, any opinion, finding, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

Xiu Ye: This research was supported in part by National Science Foundation Grant DMS-1115097.

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Published online: 29 August 2013

1 Introduction

In this paper, we are concerned with the development of residual type a posteriori error estimators for a recently introduced weak Galerkin finite element method for partial differential equations. For simplicity, we consider the following model second order elliptic problem

$$-\nabla\cdot(A\nabla u) = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a bounded polygonal or polyhedral domain in $\mathbb{R}^d$, $d = 2, 3$, and $A \in [L^\infty(\Omega)]^{d\times d}$ is a symmetric matrix-valued function on Ω. Assume that the matrix A satisfies the following property: there exist positive constants α and β such that

$$\alpha\xi^T\xi \le \xi^T A(x)\xi \le \beta\xi^T\xi \quad \text{for all } \xi\in\mathbb{R}^d, x\in\Omega. \tag{1.3}$$

Weak Galerkin (WG) refers to finite element techniques for partial differential equations in which differential operators are approximated by weak forms as distributions. Weak Galerkin methods were first introduced in [23] for simplicial grids and later on in [18,24] for shape regular polytopal meshes. The method has been successfully applied to elliptic interface problems [17], Helmholtz equations [20], and biharmonic equations [16,19]. However, the existing work on WG concerns only a priori error estimates for the corresponding numerical solutions. In this paper, we shall establish some theoretical and computational results for the a posteriori error analysis with application to adaptive grid refinement.

Computation with adaptive grid refinement has proved to be a useful and efficient tool in scientific computing over the last several decades. The key behind this technique is to design a good a posteriori error estimator that provides a guidance on how and where grids should be refined. The goal of this paper is to present an a posteriori error estimator, together with a theoretical upper and lower bound, for the weak Galerkin finite element solutions as developed in [18,23].

Briefly speaking, our a posteriori error estimator is of residual-type which is a combination of the standard conforming Galerkin finite elements and mixed finite elements. More precisely, if u_h is the WG approximation of the solution u to (1.1)–(1.2), then the a posterior error estimator, denoted by η, is given by

$$\eta^2 = \sum_{T\in\mathcal{T}_h}\eta_T^2 = \sum_{T\in\mathcal{T}_h}\left[\mathrm{osc}^2(f,T) + \eta_{c,T}^2 + \eta_{m,T}^2\right],$$

where

$$\eta_{c,T}^2 = h_T^2|A_T|^{-1}\|f_h + \nabla\cdot(A\nabla_w u_h)\|_T^2 + \frac{1}{2}\sum_{e\in\partial T} h_e|A_e^{\max}|^{-1}\int_e \mathbf{J}_e^2(A\nabla_w u_h\cdot\mathbf{n}),$$

$$\eta_{m,T}^2 = h_T^2|A_T|\|\nabla\times\nabla_w u_h\|_T^2 + \frac{1}{2}\sum_{e\in\partial T} h_e|A_e^{\min}|\int_e \mathbf{J}_e^2(\gamma_t(\nabla_w u_h)),$$

with h_T being the diameter of T, h_e the length of edge/face e, $\nabla_w u_h$ is the weak gradient of u_h, $\gamma_t(\cdot)$ is the tangential trace operator, and $\mathbf{J}_e$ denotes the jump across the edge/face e. The first term is also known as the data oscillation

$$\mathrm{osc}^2(f,\mathcal{T}_h) = \sum_{T\in\mathcal{T}_h}\mathrm{osc}^2(f,T) = \sum_{T\in\mathcal{T}_h} h_T^2|A_T|^{-1}\|f - f_h\|_T^2,$$

where f_h the projection of f to the weak Galerkin finite element space.

The part $\eta_{c,T}$ is similar to the error estimator of the conforming Galerkin element and $\eta_{m,T}$ is similar to the error estimator of the mixed element. Appropriate weight A_T, $A_e^{\max}$, and $A_e^{\min}$ are chosen to make the proposed estimator more robust to the jump of the diffusion coefficient.

The standard a posteriori error analysis for conforming Galerkin method is based on the orthogonality of the error to the finite element space. Unfortunately, such an orthogonality does not hold true for WG approximations. Since WG solution is more close to nonconforming and mixed finite element approximation, our a posteriori error estimates will be based on the following Helmholtz decomposition of the error: there exists $\psi \in H_0^1(\Omega)$ and $\phi \in H^1(\Omega)$ and $\int_\Omega \phi\, dx = 0$ such that

$$\nabla u - \nabla_w u_h = \nabla\psi + A^{-1}\nabla \times \phi.$$

In the above decomposition, $\psi \in H_0^1(\Omega)$ will lead to the error estimator $\eta_{c,T}$ which accounts for the conforming part whereas $\nabla \times \phi$ will introduce the error estimator $\eta_{m,T}$ which is the mixed element part.

Similar techniques have been employed in [8] for Crouzeix-Raviart non-conforming method and in [2,3] for Ravariat-Thomas and Brezzi, Douglas and Marini mixed finite element methods.

Using the Helmholtz type decomposition of the error and the partial orthogonality of the error to the conforming subspace, we are able to derive the following reliability

$$\|A^{\frac{1}{2}}(\nabla u - \nabla_w u_h)\| \le C_1\eta.$$

In addition, we will establish the following efficiency estimate

$$C_2\eta \le \|A^{\frac{1}{2}}(\nabla u - \nabla_w u_h)\| + \mathrm{osc}(f, \mathcal{T}_h),$$

by using the standard bubble function technique developed by Verfürth [22] (page 9).

The rest of this paper is organized as follows. In Sect. 2, we review the definition of weak gradient and its discrete approximation. In Sect. 3, we recall the weak Galerkin method and outline some key properties used in later sections. In Sect. 4, we present an a posteriori error estimator and establish a theory by deriving a reliability and efficiency estimate. In Sect. 5, we present two numerical experiments to show the effectiveness of our estimator.

2 Weak Gradient and Discrete Weak Gradient

The natural Sobolev space for the model problem (1.1)–(1.2) is $H_0^1(\Omega)$. The classical Galerkin finite element method is to use a subspace of $H_0^1(\Omega)$ (i.e., conforming) for a numerical approximation. Consequently, continuous and piecewise polynomials are used in the conforming finite element methods. However, the construction of conforming finite element spaces for high order elements or more complicated problems such as fourth order problems are technically challenging and practically limited in high dimensions. To relax this difficulty, computational researchers started to develop finite element schemes that are nonconforming by using partially continuous or totally discontinuous functions. The use of discontinuous functions in the finite element approximation often provides the method with much needed flexibility in handling more complex practical problems.

Weak Galerkin makes use of discontinuous finite element functions in approximation. Unlike other nonconforming finite element methods where standard derivatives are taken on each element, the weak Galerkin finite element method relies on weak derivatives taken as approximate distributions for the functions in nonconforming finite element spaces. With appropriately defined weak gradient and its approximation, the weak Galerkin finite element formulation for solving (1.1)–(1.2) developed in [18,23] is symmetric, positive definite, and parameter free.

Weak functions and weak derivatives are defined as follows. Let K be any polygonal domain with boundary ∂K. A *weak function* on the region K refers to a function $v = \{v_0, v_b\}$ such that $v_0 \in L^2(K)$ and $v_b \in H^{\frac{1}{2}}(\partial K)$. The first component v_0 can be understood as the value of v in K, and the second component v_b represents v on the boundary of K. Note that v_b may not necessarily be related to the trace of v_0 on ∂K should a trace be well-defined. Denote by $W(K)$ the space of weak functions on K; i.e.,

$$W(K) := \{v = \{v_0, v_b\} : \ v_0 \in L^2(K), \ v_b \in H^{\frac{1}{2}}(\partial K)\}. \tag{2.1}$$

Note that $H_0^1(K) \subset W(K)$. Instead of approximating $H_0^1(\Omega)$ using its subspaces, the weak function space can be thought as an outer approximation of $H_0^1(\Omega)$. For such weak functions, their gradient always exists in the distribution sense. The weak gradient operator introduced in [23] is to project the abstract distribution into another appropriately chosen Sobolev space such that its approximation by polynomials is possible. More specifically, for a domain G, we define $H(\text{div}, G) = \{q : q \in L^2(G), \nabla \cdot q \in L^2(G)\}$ and define the weak Gradient as follows.

Definition 2.1 (*Weak Gradient*) The dual of $L^2(K)$ can be identified with itself by using the standard L^2 inner product as the action of linear functionals. With a similar interpretation, for any $v \in W(K)$, the *weak gradient* of v is defined as a linear functional $\nabla_w v$ in the dual space of $H(\text{div}, K)$ whose action on each $q \in H(\text{div}, K)$ is given by

$$(\nabla_w v, q)_K := -(v_0, \nabla \cdot q)_K + \langle v_b, q \cdot \mathbf{n}\rangle_{\partial K}, \tag{2.2}$$

where $\mathbf{n}$ is the unit outward normal direction to ∂K, $(v_0, \nabla \cdot q)_K = \int_K v_0(\nabla \cdot q) dK$ is the action of v_0 on $\nabla \cdot q$, and $\langle v_b, q \cdot \mathbf{n}\rangle_{\partial K}$ is the action of $q \cdot \mathbf{n}$ on $v_b \in H^{\frac{1}{2}}(\partial K)$.

By choosing a finite element subspace of $H(\text{div}, K)$, we obtain a discrete weak gradient. When K is a domain such as triangles, tetrahedron, rectangles and cubes, we chose Raviart-Thomas (RT) element or Brezzi, Douglas and Marini (BDM) element. For general polygons, we simply chose a polynomial space with suitable degree [18,24].

Let $P_r(K)$ be the set of polynomials on K with degree no more than r and $\widehat{P}_k(K)$ be the set of homogeneous polynomials of order k in the variable $\mathbf{x} = (x_1, \ldots, x_d)^T$. Let $G_k(K)$ be either $[P_k(K)]^d$ or $RT_k(K) = [P_k(K)]^d + \widehat{P}_k(K)\mathbf{x}$.

Definition 2.2 (*Discrete Weak Gradient*) The discrete weak gradient of v denoted by $\nabla_{w,k,K} v$ is defined as the unique polynomial $(\nabla_{w,k,K} v) \in G_k(K)$ satisfying the following equation

$$(\nabla_{w,k,K} v, q)_K = -(v_0, \nabla \cdot q)_K + \langle v_b, q \cdot \mathbf{n}\rangle_{\partial K} \quad \text{for all } q \in G_k(K). \tag{2.3}$$

Note that if $v \in H^1(K)$ and $\nabla v \in G_k(K)$, then $\nabla_{w,k,K} v = \nabla v$. For the weak function space $W(K)$, we discretize it by $W_{j,\ell}(K)$ given as follows

$$W_{j,\ell}(K) := \left\{v = \{v_0, v_b\} : \ v_0 \in P_j(K), v_b \in P_\ell(\partial K)\right\}.$$

Thus, different weak Galerkin finite element methods can be derived by choosing $W_{j,\ell}(K)$ and $G_k(K)$ with various combinations of the indices j, ℓ, and k. This paper shall mainly consider three pairs $W_{k,k}(T)-RT_k(T)$, $W_{k,k+1}(T)-\left[P_{k+1}(T)\right]^d$, and $W_{k+1,k}(T)-[P_k(T)]^d$ for integers $k \geq 0$ defined on simplices T.

3 Weak Galerkin Finite Element Method

Let $\mathcal{T}_h$ be a simplicial mesh for the weak Galerkin elements $W_{k,k}(T) - RT_k(T)$ and $W_{k,k+1}(T) - \left[P_{k+1}(T)\right]^d$. For the pair $W_{k+1,k}(T) - [P_k(T)]^d$, the partition $\mathcal{T}_h$ can be relaxed to general polygons in two dimensions or polyhedra in three dimensions satisfying a set of shape regularity conditions specified in [18,24], but our analysis can be only applied to simplicial meshes. Denote by $\mathcal{E}_h$ the set of all edges or faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior edges or faces. For a d-dimensional simplex S, let $h_S = |S|^{1/d}$ be the size of simplex S where $|S|$ is the d-dimensional Lebesgue measure of S. For a triangulation $\mathcal{T}_h$, the mesh size $h = \max_{T\in\mathcal{T}_h} h_T$.

Denote by $W_k(T) - G_k(T)$ a local weak Galerkin element that can be either $W_{k,k}(T) - RT_k(T)$, $W_{k,k+1}(T) - \left[P_{k+1}(T)\right]^d$, or $W_{k+1,k}(T) - [P_k(T)]^d$. Associated with $\mathcal{T}_h$ and a local element $W_k(T) - G_k(T)$, we define global weak Galerkin finite element spaces

$$
\begin{aligned}
V_h &:= \{v = \{v_0, v_b\} : \ \{v_0, v_b\}|_T \in W_k(T)\},\\
V_h^0 &:= \{v : \ v \in V_h, \ v_b = 0 \text{ on } \partial\Omega\}.
\end{aligned}
$$

The component v_0 is defined element-wise and totally discontinuous. The component v_b is defined on edges/faces which glues v_0 in different elements to be a reasonable approximation of a function in $H_0^1(\Omega)$.

Denote by $\nabla_{w,k}$ the discrete weak gradient operator on V_h computed by using (2.3) on each element T; i.e., $\nabla_{w,k,T}(v|_T) \in G_k(T)$,

$$
(\nabla_{w,k} v)|_T := \nabla_{w,k,T}(v|_T) \quad \text{for all } v \in V_h.
$$

For simplicity of notation, from now on we shall drop the subscript k in the notation $\nabla_{w,k}$ for the discrete weak gradient.

Define

$$
(A\nabla_w w, \ \nabla_w v) := \sum_{T\in\mathcal{T}_h} (A\nabla_w w, \ \nabla_w v)_T,
$$

and

$$
a(w, v) = \begin{cases} (A\nabla_w w, \ \nabla_w v), & \text{for } W_{k,k}(T) - RT_k(T) \text{ or } W_{k,k+1}(T) - \left[P_{k+1}(T)\right]^d, \\ (A\nabla_w w, \ \nabla_w v) + s(w, v), & \text{for } W_{k+1,k}(T) - [P_k(T)]^d, \end{cases}
$$

where

$$
s(w, v) := \sum_{T\in\mathcal{T}_h} h_T^{-1} \langle Q_b w_0 - w_b, Q_b v_0 - v_b\rangle_{\partial T}
$$

is a stabilization term. Here Q_b is the standard L^2 projection onto the polynomial space $P_k(e)$ for each flat edge/face e.

Weak Galerkin Algorithm 3.1 *A numerical approximation for (1.1) and (1.2) can be obtained by seeking $u_h = \{u_0, \ u_b\} \in V_h^0$ satisfying the following equation:*

$$
a(u_h, v) = (f, \ v_0) \quad \textit{for all } v = \{v_0, \ v_b\} \in V_h^0. \tag{3.1}
$$

3.1 A Priori Error Analysis

Denote by $Q_h u = \{Q_0 u,\ Q_b u\}$ the L^2 projection onto $W_k(T)$ and $\mathbb{Q}_h$ the L^2 projection onto $G_k(T)$. It is not hard to see the following operator identity:

$$\nabla_w(Q_h u) = \mathbb{Q}_h \nabla u. \tag{3.2}$$

In addition, the following a priori error estimates hold true [18,23].

Theorem 3.1 (A priori error estimates) *Let $u \in H^{k+2}(\Omega)$ and $u_h = \{u_0, u_b\} \in V_h^0$ be the solutions of (1.1)–(1.2) and (3.1). Then there exists a constant C such that for $k \geq 0$,*

$$\|\nabla_w(u_h - Q_h u)\| \leq Ch^{k+1}\|u\|_{k+2}.$$

Furthermore, assume that the usual H^2-regularity holds true for the model problem (1.1)–(1.2). Then, there exists a constant C such that

$$\|u_0 - Q_0 u\| \leq Ch^{k+2}\|u\|_{k+2}.$$

The key to a posteriori error analysis is to relax the smoothness assumption of the solution and the regularity assumption for the operator. There are two main difficulties in adapting the standard a posteriori error analysis for the conforming finite element method to the WG method, which are

1. Inconsistency: The weak Galerkin discretization scheme (3.1) is not consistent in the sense that for the solution u of (1.1)–(1.2), one has $a(u,\ v) \neq (f, v_0)$ for some $v = \{v_0,\ v_b\} \in V_h^0$.
2. Non-orthogonality: As a consequence of the inconsistency, the usual orthogonality property for the conforming Galerkin does not hold true for the weak Galerkin approximation; i.e.,

$$a(u - u_h,\ v) \neq 0 \ \text{ for some } v = \{v_0,\ v_b\} \in V_h^0.$$

In the coming subsection, we shall demonstrate that a certain orthogonality-like equation is essential for the a posterior error analysis.

3.2 Properties of Weak Galerkin Approximations

Define $S_1 = \{v \in H_0^1(\Omega), v|_T \in P_1(T)\}$. For the weak Galerkin finite element space V_h with polynomial degree of $k \geq 1$, we can naturally embed S_1 into V_h by choosing v_b as the trace of v on edges or faces. We then have the following important partial orthogonality result.

Lemma 3.2 (Partial orthogonality) *Let V_h be a weak Galerkin finite element space with polynomial degree $k \geq 1$. Let $u_h \in V_h^0$ and u be the solutions of (3.1) and (1.1)-(1.2) respectively. Then*

$$(A(\nabla u - \nabla_w u_h), \nabla_w v) = 0, \ \ \textit{for all } v \in S_1 \subset V_h^0. \tag{3.3}$$

Proof For $v \in S_1 \subset H_0^1(\Omega)$, it is easy to see $\nabla_w v = \nabla v$. It follows from $v \in H_0^1(\Omega)$ that

$$(A\nabla u, \nabla v) = (f,\ v).$$

By using (3.1) and $\nabla_w v = \nabla v$, we obtain

$$(A\nabla_w u_h, \nabla v) = a(u_h, v) = (f,\ v).$$

The difference of two equations above yields (3.3). □

The partial orthogonality (3.3) does not hold true for the lowest order WG method (i.e., $k = 0$). For this case, we shall make use of the conservation property of the WG approximation.

Lemma 3.3 (Conservation) *Let u be the solution of (1.1)-(1.2) and $u_h = \{u_0, u_b\} \in V_h^0$ be the solution of (3.1). Then we have $\mathbb{Q}(A\nabla_w u_h) \in H(\mathrm{div}, \Omega)$ and*

$$-\nabla \cdot \mathbb{Q}(A\nabla_w u_h) = f_h. \tag{3.4}$$

Proof By letting $v = \{0, v_b\}$ in (3.1), we get $\mathbb{Q}(A\nabla_w u_h) \in H(\mathrm{div}, \Omega)$. This means that $\mathbb{Q}(A\nabla_w u_h) \cdot \mathbf{n}$ is continuous across each edge/face. By choosing $v = \{v_0, 0\}$, we obtain (3.4). □

4 A Posteriori Error Analysis for WG Methods

In this section, an a posteriori error estimator will be described and analyzed for the weak Galerkin finite element formulation (3.1). To this end, let e be an interior edge or face common to elements T_1 and T_2 in $\mathcal{T}_h$, and let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the unit normal vectors on e exterior to T_1 and T_2, respectively. In two dimensions, let $\mathbf{t}_1$ and $\mathbf{t}_2$ be the unit tangential vectors on e obtained by a 90° rotation of $\mathbf{n}_1$ and $\mathbf{n}_2$ counterclockwise respectively. We define the tangential trace of a vector function $\mathbf{w}$ in T_i as $\gamma_{t,\partial T_i}(\mathbf{w}) = \mathbf{w} \cdot \mathbf{t}_i$ in two dimensions and $\gamma_{t,\partial T_i}(\mathbf{w}) = \mathbf{w} \times \mathbf{n}_i$ in three dimensions, for $i = 1, 2$.

For a vector function $\mathbf{w}$, we define normal jump $[\mathbf{w} \cdot \mathbf{n}]_e$ and tangential jump $[\gamma_t(\mathbf{w})]_e$ across e, as

$$[\mathbf{w} \cdot \mathbf{n}]_e = \mathbf{w}|_{\partial T_1} \cdot \mathbf{n}_1 + \mathbf{w}|_{\partial T_2} \cdot \mathbf{n}_2, \quad [\gamma_t(\mathbf{w})]_e = \gamma_{t,\partial T_1}(\mathbf{w}) + \gamma_{t,\partial T_2}(\mathbf{w}).$$

Next, we define

$$\mathbf{J}_e(A\nabla_w u_h \cdot \mathbf{n}) = \begin{cases} [A\nabla_w u_h \cdot \mathbf{n}] & \text{if } \ e \in \mathcal{E}_h^0 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathbf{J}_e(\gamma_t(\nabla_w u_h)) = \begin{cases} [\gamma_t(\nabla_w u_h)]_e & \text{if } \ e \in \mathcal{E}_h^0 \\ 2\gamma_t(\nabla_w u_h) & \text{otherwise.} \end{cases}$$

In two dimensions, define two differential operators for a scalar function f and a vector function $\mathbf{v} = (v_1, v_2)$,

$$\nabla^{\perp} f = \left(-\frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_1}\right)^T, \quad \nabla \times \mathbf{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.$$

For $e \in \mathcal{E}_h^0$, denote by $\omega_e = T_1 \cup T_2$ the macro-element associated with e, where T_1 and T_2 are two elements in $\mathcal{T}_h$ sharing e as a common edge/face. Similarly, for a vertex x, $\omega_x = \{T' \in \mathcal{T}_h, x \in T'\}$ and for an element $T \in \mathcal{T}_h$, $\omega_T = \{T' \in \mathcal{T}_h, T' \cap T \neq \varnothing\}$. With a slight abuse of notation, for the matrix function A, we use $|A|$ to denote its determinant. Note that $0 < \alpha \leq |A| \leq \beta$ by assumption (1.3). Let A_T be the average of A on T, $|A_e^{\max}| = \max_{T\in\omega_e} |A_T|$, and $|A_e^{\min}| = \min_{T\in\omega_e} |A_T|$.

Let f_h be the L^2 projection of f to V_h. Define the data oscillation for the load function f on $\mathcal{T}_h$ as

$$\mathrm{osc}^2(f, \mathcal{T}_h) = \sum_{T\in\mathcal{T}_h} \mathrm{osc}^2(f, T) = \sum_{T\in\mathcal{T}_h} h_T^2 |A_T|^{-1} \|f - f_h\|_T^2.$$

Define a global error estimator as

$$\eta^2 = \sum_{T\in\mathcal{T}_h} \eta_T^2 = \sum_{T\in\mathcal{T}_h} \left[\mathrm{osc}^2(f,T) + \eta_{c,T}^2 + \eta_{m,T}^2\right], \tag{4.1}$$

where

$$\eta_{c,T}^2 = h_T^2|A_T|^{-1}\|f_h + \nabla\cdot(A\nabla_w u_h)\|_T^2 + \frac{1}{2}\sum_{e\in\partial T} h_e|A_e^{\max}|^{-1}\int_e \mathbf{J}_e^2(A\nabla_w u_h\cdot\mathbf{n}).$$

$$\eta_{m,T}^2 = h_T^2|A_T|\|\nabla\times\nabla_w u_h\|_T^2 + \frac{1}{2}\sum_{e\in\partial T} h_e|A_e^{\min}|\int_e \mathbf{J}_e^2(\gamma_t(\nabla_w u_h)).$$

The part $\eta_{c,T}$ is similar to the error estimator of the conforming Galerkin element and $\eta_{m,T}$ is similar to that of the mixed finite element.

Remark 4.1 When A is piecewise constant, for the element $W_{k,k}(T) - RT_k(T)$ or $W_{k,k+1}(T)-\left[P_{k+1}(T)\right]^d$, by Lemma 3.3, we have $f_h+\nabla\cdot(A\nabla_w u_h) = 0$ and $\mathbf{J}(A\nabla_w u_h\cdot\mathbf{n}) = 0$. Thus, the above error estimator is a variant of that for the mixed finite element approximation using $H(\mathrm{div},\Omega)$ elements [2,3,5,11] with a more appropriate weight. In fact, one can verify that when A is piecewise constant, the flux $A\nabla_w u_h$ is the same as that obtained from the corresponding mixed finite element method.

For the lowest order WG element i.e., when choosing $G_0(T) = RT_0(T)$, the term $\nabla\times\nabla_w u_h$ also vanishes.

4.1 Reliability

This subsection is devoted to a study of reliability for the error estimator η defined in (4.1). For simplicity of notation, results shall be presented in two dimensions and comments will be made on the notation change for three dimensional problems.

Let K be an element with e as an edge. It is well known that there exists a constant C such that for any function $g\in H^1(K)$

$$\|g\|_e^2 \le C\left(h_K^{-1}\|g\|_K^2 + h_K\|\nabla g\|_K^2\right). \tag{4.2}$$

The following Helmholtz decomposition of an L^2 function is well known; see, for example [11] (page 31) or [1]. For completeness we include a short proof here.

Lemma 4.2 *For $\nabla u - \nabla_w u_h \in L^2(\Omega)$, there exist $\psi\in H_0^1(\Omega)$ and $\phi\in H^1(\Omega)$ and $\int_\Omega \phi\,dx = 0$ such that*

$$\nabla u - \nabla_w u_h = \nabla\psi + A^{-1}\nabla^\perp\phi \tag{4.3}$$

and that

$$\|A^{\frac12}(\nabla u - \nabla_w u_h)\|^2 = \|A^{\frac12}\nabla\psi\|^2 + \|A^{-\frac12}\nabla^\perp\phi\|^2. \tag{4.4}$$

Proof The function $\psi\in H_0^1(\Omega)$ is obtained by solving the elliptic equation

$$(A\nabla\psi,\nabla v) = (A(\nabla u - \nabla_w u_h),\nabla v) \quad \text{for all } v\in H_0^1(\Omega),$$

whose well posedness is from the Lax-Millgram theorem. Then

$$\mathrm{div}(A(\nabla u - \nabla_w u_h) - A\nabla\psi) = 0.$$

Recall we assume from the very beginning Ω is a polygon. So Ω is simply connected and consequently there exists $\phi \in H^1(\Omega)$ such that $A(\nabla u - \nabla_w u_h) - A\nabla\psi = \nabla^\perp\phi$. Further, as $\nabla^\perp(\text{constant}) = 0$, we can chose ϕ such that the average is zero.

The orthogonality $(\nabla\psi, \nabla^\perp\phi) = -(\psi, \nabla\cdot\nabla^\perp\phi) = 0$ implies the identity (4.4). □

Remark 4.3 In three dimensions, the vector potential ψ satisfying $\nabla\times\psi = A(\nabla u - \nabla_w u_h) - A\nabla\psi$ can be chosen in $H^1(\Omega)$; see [12] (page 45).

In the decomposition (4.3), $\psi \in H_0^1(\Omega)$ will lead to the error estimator $\eta_{c,T}$ which accounts for the conforming part whereas ϕ will introduce the error estimator $\eta_{m,T}$ which is the mixed element part.

Theorem 4.4 (Upper bound) *Let u be the solution of (1.1)–(1.2) and $u_h = \{u_0, u_b\} \in V_h^0$ be the solution of (3.1). Then, for $k \geq 0$, there exists a positive constant C_1 such that*

$$\|A^{\frac{1}{2}}(\nabla u - \nabla_w u_h)\| \leq C_1\eta. \tag{4.5}$$

Proof It follows from (4.3) that

$$\begin{aligned}\|A^{\frac{1}{2}}(\nabla u - \nabla_w u_h)\|^2 &= (A(\nabla u - \nabla_w u_h), \nabla u - \nabla_w u_h)\\ &= (A(\nabla u - \nabla_w u_h), \nabla\psi) + (A(\nabla u - \nabla_w u_h), A^{-1}\nabla^\perp\phi).\end{aligned} \tag{4.6}$$

We chose the robust interpolant ψ_h introduced in [10,21] satisfying the following two estimates:

$$|A_T|^{1/2}\|\psi - \psi_h\|_{0,T} \leq Ch_T\|A^{1/2}\nabla\psi\|_{\omega_T} \tag{4.7}$$

$$|A_e^{\max}|^{1/2}\|\psi - \psi_h\|_{0,e} \leq Ch_e^{1/2}\|A^{1/2}\nabla\psi\|_{\omega_e}. \tag{4.8}$$

We split the first term on the right hand side of (4.6) as

$$(A(\nabla u - \nabla_w u_h), \nabla\psi - \nabla\psi_h) + (A(\nabla u - \nabla_w u_h), \nabla\psi_h) = I_1 + I_2.$$

For I_1, using the integration by parts, (4.7)-(4.8), the triangle inequality, and (1.3), we have

$$\begin{aligned}|I_1| &= \left|\sum_{T\in\mathcal{T}_h}(f + \nabla\cdot(A\nabla_w u_h), \psi - \psi_h)_T + \sum_{e\in\mathcal{E}_h^0}\int_e([A\nabla_w u_h\cdot\mathbf{n}], \psi - \psi_h)\right|\\ &\leq \sum_{T\in\mathcal{T}_h} h_T|A_T|^{-1/2}\left(\|f_h + \nabla\cdot(A\nabla_w u_h)\| + \|f - f_h\|\right)h_T^{-1}|A_T|^{1/2}\|\psi - \psi_h\|_{0,T}\\ &\quad + \sum_{e\in\mathcal{E}_h^0} h_e^{1/2}|A_e^{\max}|^{-1/2}\|\mathbf{J}_e(A\nabla_w u_h\cdot\mathbf{n})\|_e h_e^{-1/2}|A_e^{\max}|^{1/2}\|\psi - \psi_h\|_{0,e}\\ &\leq C\sum_T \eta_{c,T}\|A^{1/2}\nabla\psi\|_{\omega_T} \leq C\eta\|A^{\frac{1}{2}}\nabla\psi\| \leq C\eta\|A^{1/2}(\nabla u - \nabla_w u_h)\|.\end{aligned}$$

When $k \geq 1$, it follows from (3.3) that $I_2 = 0$. When $k = 0$, the continuous P_1 finite element space S_1 is no longer a subspace of V_h. We may use the conservation property (3.4) to estimate I_2 as follows. Using $\nabla\psi_h|_T \in G_k(T)$, $\mathbb{Q}(A\nabla_w u_h) \in H(\text{div}, \Omega)$ and (3.4), we have

$$\begin{aligned}(A(\nabla u - \nabla_w u_h), \nabla\psi_h) &= (A\nabla u - \mathbb{Q}(A\nabla_w u_h), \nabla\psi_h) = \sum_{T\in\mathcal{T}_h}(f - f_h, \psi_h)_T\\ &= \sum_{T\in\mathcal{T}_h}(f - f_h, \psi_h - \bar{\psi})_T = \sum_{T\in\mathcal{T}_h}(f - f_h, \psi_h - \psi)_T + (f - f_h, \psi - \bar{\psi})_T,\end{aligned}$$

where $\bar{\psi}$ is the average of ψ on T. Thus,

$$
\begin{aligned}
|(A(\nabla u - \nabla_w u_h), \nabla \psi_h)| &\le C \sum_{T\in\mathcal{T}_h} \|f - f_h\|_T \left(\|\psi_h - \psi\|_T + \|\psi - \bar{\psi}\|_T\right) \\
&\le C \sum_{T\in\mathcal{T}_h} h_T |A_T|^{-1/2} \|f - f_h\|_T |A_T|^{1/2} \|\nabla\psi\|_T \\
&\le C\mathrm{osc}(f, \mathcal{T}_h)\|A^{1/2}\nabla\psi\|.
\end{aligned}
$$

Combining the estimates for I_1 and I_2, we arrive at

$$
|(A(\nabla u - \nabla_w u_h), \nabla\psi)| \le C\eta\|A^{\frac{1}{2}}\nabla\psi\| \le C\eta\|A^{1/2}(\nabla u - \nabla_w u_h)\|. \tag{4.9}
$$

To estimate the second term on the right hand side of (4.6), we apply the robust interpolant corresponding to the weight A^{-1}. Namely let $\phi_h \in S_1$ be a robust interpolant of ϕ satisfying

$$
|A_T|^{-1/2}\|\psi - \psi_h\|_{0,T} \le Ch_T\|A^{-1/2}\nabla\psi\|_{\omega_T} \tag{4.10}
$$

$$
|A_e^{\min}|^{-1/2}\|\psi - \psi_h\|_{0,e} \le Ch_e^{1/2}\|A^{-1/2}\nabla\psi\|_{\omega_e}. \tag{4.11}
$$

Here $|A_e^{\min}|$ is used since $\max_{T\in\omega_e}|A_T^{-1}| = (\min_{T\in\omega_e}|A_T|)^{-1}$. Since $\nabla^\perp\phi \in H(\mathrm{div}, \Omega)$ and $u \in H_0^1(\Omega)$, we have

$$
(\nabla u, \nabla^\perp\phi) = -(u, \nabla\cdot\nabla^\perp\phi) = 0. \tag{4.12}
$$

It follows from $\nabla^\perp\phi_h|_T \in G_k(T)$, $\nabla^\perp\phi_h \in H(\mathrm{div}, \Omega)$ and (2.3) that

$$
(\nabla_w u_h, \nabla^\perp\phi_h) = \sum_{T\in\mathcal{T}_h}\Big(-(u_0, \nabla\cdot\nabla^\perp\phi_h)_T + \langle u_b, \nabla^\perp\phi_h\cdot\mathbf{n}\rangle_{\partial T}\Big) = 0. \tag{4.13}
$$

Using (4.12), (4.13), integration by parts, (4.2), (1.3) and (4.4), we have

$$
\begin{aligned}
&|(\nabla u - \nabla_w u_h, \nabla^\perp\phi)| = |(\nabla_w u_h, \nabla^\perp(\phi - \phi_h))| \\
&= \left|\sum_{T\in\mathcal{T}_h}(\nabla\times\nabla_w u_h, \phi - \phi_h)_T + \sum_{e\in\mathcal{E}_h}\int_e[\nabla_w u_h\cdot\mathbf{t}](\phi - \phi_h)\right| \\
&\le C\sum_{T\in\mathcal{T}_h} h_T|A_T|^{1/2}\|\nabla\times\nabla_w u_h\|_T\, h_T^{-1}|A_T|^{-1/2}\|\phi - \phi_h\|_T \qquad (4.14)\\
&+C\sum_{e\in\mathcal{E}_h} h_e^{1/2}|A_e^{\min}|^{1/2}\|[\nabla_w u_h\cdot\mathbf{t}]\|_e\, h_e^{-1/2}|A_e^{\min}|^{-1/2}\|\phi - \phi_h\|_e \\
&\le C\eta\ \|A^{\frac{1}{2}}(\nabla u - \nabla_w u_h)\|. \qquad (4.15)
\end{aligned}
$$

Applying the bounds derived in (4.9) and (4.15) to (4.6), we arrive at

$$
\|A^{\frac{1}{2}}(\nabla u - \nabla_w u_h)\| \le C\eta.
$$

This completes the proof. □

Remark 4.5 The constant C in the upper bound estimate does depend on the condition number of the coefficient A, i.e., $\kappa(A) \le \beta/\alpha$. Following [7,21], one may improve the constant to be independent on the condition number $\kappa(A)$ if the distribution of A in ω_x satisfies the so-called *quasi-monotone* condition.

For a singular vertex x, i.e., A in ω_x is not quasi-monotone, following [7], we include the weight

$$\Lambda_T := \max_{T' \in \omega_T} \left(\frac{|A_T|}{|A_{T'}|} \right), \quad \bar{\Lambda}_T := \max_{T' \in \omega_T} \left(\frac{|A_{T'}|}{|A_T|} \right)$$

for the conforming part $\eta_{c,T}$ and mixed part $\eta_{m,T}$ for each element T in ω_x, respectively.

4.2 Efficiency

We use the standard bubble function technique [22] (page 9) to derive an efficiency estimate. To this end, let T_1 and T_2 be two elements in $\mathcal{T}_h$ which share e as a common edge. Denote by $\omega_e = T_1 \cup T_2$ the macro-element associated with e.

Lemma 4.6 (Local lower bound) *There exists a constant* $C > 0$ *such that*

$$\begin{aligned} h_T \| f_h + \nabla \cdot (A \nabla_w u_h) \|_T &\le C(\| A^{\frac{1}{2}} (\nabla u - \nabla_w u_h) \|_T + h_T \| f - f_h \|_T), && (4.16) \\ \| \nabla \times (\nabla_w u_h) \|_T &\le C \| A^{\frac{1}{2}} (\nabla u - \nabla_w u_h) \|_T, && (4.17) \\ h_e^{\frac{1}{2}} \| [\nabla_w u_h \cdot \mathbf{t}] \|_e &\le C \| A^{\frac{1}{2}} (\nabla u - \nabla_w u_h) \|_{\omega_e}, && (4.18) \\ h_e^{\frac{1}{2}} \| [A \nabla_w u_h \cdot \mathbf{n}] \|_e &\le C(\mathrm{osc}(f, \omega_e) + \| A^{\frac{1}{2}} (\nabla u - \nabla_w u_h) \|_{\omega_e}). && (4.19) \end{aligned}$$

Proof Let $w_T = (f_h + \nabla \cdot (A \nabla_w u_h)) \phi_T(\mathbf{x})$ where $\phi_T(\mathbf{x}) = 27 \lambda_1 \lambda_2 \lambda_3$ is a bubble function defined on T (see [22] page 9, (1.5)). Then we have

$$(f, \ w_T)_T = (A \nabla u, \ \nabla w_T)_T.$$

Subtracting and adding $(A \nabla_w u_h, \ \nabla w_T)_T$ and $(f_h, \ w_T)_T$ from both sides of the equation give

$$(f - f_h, \ w_T)_T + (f_h, \ w_T)_T - (A \nabla_w u_h, \ \nabla w_T)_T = (A(\nabla u - \nabla_w u_h), \ \nabla w_T)_T.$$

Using the integration by parts, the above equation becomes

$$(f_h + \nabla \cdot (A \nabla_w u_h), \ w_T)_T = (A(\nabla u - \nabla_w u_h), \ \nabla w_T)_T - (f - f_h, \ w_T)_T.$$

The properties of the bubble function $\phi_T(\mathbf{x})$ implies

$$\begin{aligned} &\| f_h + \nabla \cdot (A \nabla_w u_h) \|_T^2 \\ &\quad \le C(\| A^{\frac{1}{2}} (\nabla u - \nabla_w u_h) \|_T + h_T \| f - f_h \|_T) h_T^{-1} \| f_h + \nabla \cdot (A \nabla_w u_h) \|_T, \end{aligned}$$

which leads to the following estimate

$$h_T \| f_h + \nabla \cdot (A \nabla_w u_h) \|_T \le C(\| A^{\frac{1}{2}} (\nabla u - \nabla_w u_h) \|_T + h_T \| f - f_h \|_T). \qquad (4.20)$$

Let $\omega_T = \nabla \times (\nabla_w u_h) \phi_T(\mathbf{x})$. It follows from the integration by parts that

$$\begin{aligned} 0 = (\nabla u, \ \nabla^{\perp} \omega_T)_T &= (\nabla u - \nabla_w u_h, \ \nabla^{\perp} \omega_T)_T + (\nabla_w u_h, \ \nabla^{\perp} \omega_T)_T \\ &= (\nabla u - \nabla_w u_h, \ \nabla^{\perp} \omega_T)_T - (\nabla \times (\nabla_w u_h), \ \omega_T)_T. \end{aligned}$$

Using the properties of area bubble and (1.3), we have

$$\begin{aligned} \| \nabla \times (\nabla_w u_h) \|_T^2 &\le \| \nabla u - \nabla_w u_h \|_T \| \nabla^{\perp} w_T \| \\ &\le C \| A^{\frac{1}{2}} (\nabla u - \nabla_w u_h) \|_T h_T^{-1} \| \nabla \times (\nabla_w u_h) \|_T, \end{aligned}$$

which implies (4.17).

Let $w_e = [\nabla u_h \cdot \mathbf{t}]_e \phi_e(\mathbf{x})$, where $\phi_e(\mathbf{x})$ is an edge bubble defined on e. It follows from the integration by parts that

$$\begin{aligned}
0 &= \sum_{T\in\omega_e} (\nabla u,\ \nabla^{\perp} w_e)_T = \sum_{T\in\omega_e} \Big((\nabla u - \nabla_w u_h,\ \nabla^{\perp} w_e)_T + (\nabla_w u_h,\ \nabla^{\perp} w_e)_T \Big) \\
&= \sum_{T\in\omega_e} \Big((\nabla u - \nabla_w u_h,\ \nabla^{\perp} w_e)_T - (\nabla \times (\nabla_w u_h),\ w_e)_T \Big) + \int_e [\nabla_w u_h \cdot \mathbf{t}]^2 \phi_e(\mathbf{x}) ds.
\end{aligned}$$

Using the properties of the edge bubble function $\phi_e(\mathbf{x})$, (1.3) and (4.16), we arrive at

$$\begin{aligned}
h_e^{\frac{1}{2}} \|[\nabla_w u_h \cdot \mathbf{t}]\|_e &\le C(\|A^{\frac{1}{2}}(\nabla u - \nabla_w u_h)\|_{\omega_e} + h_T \|\nabla \times (\nabla_w u_h)\|_{\omega_e}) \\
&\le C\|A^{\frac{1}{2}}(\nabla u - \nabla_w u_h)\|_{\omega_e}.
\end{aligned}$$

This verifies the validity of (4.18).

As to (4.19), let $v_e = [A\nabla_w u_h \cdot \mathbf{n}]_e \phi_e(\mathbf{x})$. It is easy to see that

$$\sum_{T\in\omega_e} ((f,\ v_e)_T - (A\nabla_w u_h, \nabla v_e)_T) = \sum_{T\in\omega_e} (A(\nabla u - \nabla_w u_h), \nabla v_e)_T.$$

Using the integration by parts for the term $(A\nabla_w u_h, \nabla v_e)_T$ and the properties of the edge bubble function $\phi_e(\mathbf{x})$, we obtain

$$\begin{aligned}
&\|[A\nabla_w u_h \cdot \mathbf{n}]\|_e^2 \\
&\le \left| \sum_{T\in\omega_e} ((f - f_h,\ v_e)_T + (f_h + \nabla\cdot(A\nabla_w u_h),\ v_e)_T - (A(\nabla u - \nabla_w u_h), \nabla v_e)_T) \right| \\
&\le C(h_e^{\frac{1}{2}} \|f - f_h\|_{\omega_e} + h_e^{\frac{1}{2}} \|f_h + \nabla\cdot(A\nabla_w u_h)\|_{\omega_e} + h_e^{-\frac{1}{2}} \|A^{\frac{1}{2}}(\nabla u - \nabla_w u_h)\|_{\omega_e}) \\
&\|[A\nabla_w u_h \cdot \mathbf{n}]\|_e.
\end{aligned}$$

Using (4.16), we have

$$h_e^{\frac{1}{2}} \|[A\nabla_w u_h \cdot \mathbf{n}]\|_e \le C(\mathrm{osc}(f, \omega_e) + \|A^{\frac{1}{2}}(\nabla u - \nabla_w u_h)\|_{\omega_e}). \tag{4.21}$$

This verifies (4.19), and hence completes the proof. □

Summing over $e \in \mathcal{E}_e$ and $T \in \mathcal{T}_h$, we obtain the following lower bound for the error estimator.

Theorem 4.7 (Lower bound) *There exists a constant $C_2 > 0$ such that*

$$C_2 \eta \le \|A^{\frac{1}{2}}(\nabla u - \nabla_w u_h)\| + \mathrm{osc}(f, \mathcal{T}_h).$$

Remark 4.8 Again, the constant C_2 in the lower bound estimate does also depend on the condition number of the coefficient A, i.e. $\kappa(A) \le \beta/\alpha$ and this is unavoidable by an example constructed in [7].

5 Numerical Experiments

We consider the following adaptive algorithm applied to WG.

```
[u_J, T_J] = AFEM_WG (T_0, f, tol, θ)

  AFEM compute an approximation u_J by adaptive finite
  element methods.
  Input:  T_0 initial triangulation; f data; tol stopping
          criteria; θ ∈ (0,1) marking parameter.
  Output: T_J a triangulation; u_J WG finite element
          approximation on T_J.
  η = 1; k = 0;
  while η ≥ tol
     SOLVE equation (3.1) on T_k to get the solution u_k;
     ESTIMATE the error by η = η(u_k, f, T_k);
     MARK a set M_k ⊂ T_k with minimum number such that
                           η²(u_k, f, M_k) ≥ θ η²(u_k, f, T_k);
     REFINE element in M_k and necessary elements to a
            conforming triangulation T_{k+1};
     k = k + 1;
  end
  u_J = u_k; T_J = T_k;
```

To confirm the theoretical results established in the previous sections, numerical experiments are carried out for two test examples. The simulation is implemented using the MATLAB software package *i*FEM [4]. Multigrid solvers developed in [6] is used for solving the linear algebraic system. The tolerance for the iterative solvers is 10^{-8} which is small enough and does not affect the approximation error.

5.1 Example: Kellogg problem

We show the efficiency of our residual-based a posteriori error estimator with a discontinuous coefficient problem. The bulk marking strategy by Dörfler [9] with $\theta = 0.2$ is adopted in our simulation for marking. Marked elements are refined by the newest vertex bisection [15].

We employ a test example designed by Kellogg [13]. Consider the partial differential Eq. (1.1) with $\Omega = (-1, 1)^2$ and the coefficient matrix A is piecewise constant: in the first and third quadrants, $A = a_1\boldsymbol{I}$; in the second and fourth quadrants, $A = a_2\boldsymbol{I}$. For $f = 0$, the exact solution in polar coordinates has been chosen to be $u(r, \theta) = r^\gamma \mu(\theta)$, where

$$\mu(\theta) = \begin{cases} \cos\left((\frac{\pi}{2} - \sigma)\gamma\right) \cos\left((\theta - \frac{\pi}{2} + \rho)\gamma\right) & \text{if } 0 \le \theta \le \frac{\pi}{2}, \\ \cos(\rho\gamma)\cos((\theta - \pi + \sigma)\gamma) & \text{if } \frac{\pi}{2} \le \theta \le \pi, \\ \cos(\sigma\gamma)\cos((\theta - \pi - \rho)\gamma) & \text{if } \pi \le \theta \le \frac{3\pi}{2}, \\ \cos\left((\frac{\pi}{2} - \rho)\gamma\right) \cos\left((\theta - \frac{3\pi}{2} - \sigma)\gamma\right) & \text{if } \frac{3\pi}{2} \le \theta \le 2\pi, \end{cases}$$

and the constants

$\gamma = 0.1,\ \rho = \pi/4,\ \sigma = -14.9225565104455152,\ a_1 = 161.4476387975881,\ a_2 = 1.$ The solution u is barely in $H^1(\Omega)$. Indeed, $u \in H^{1+\gamma}(\Omega)$.

We use the lowest order WG method, i.e., $W_{0,0} - RT_0$ pairs and expect the first order convergence of the energy error $\|A^{\frac{1}{2}}(\nabla u - \nabla_{w,h} u_h)\| \le CN^{-\frac{1}{2}}$. Here we change the conventional measurement h to the number of degree of freedom N. For quasi-uniform mesh in two dimensions, $h = \mathcal{O}(N^{-\frac{1}{2}})$. Since A is piecewise constant on $\mathcal{T}_0$, $f = 0$, and RT_0 is used, the error estimator contains simply the tangential jump of the weak derivative of u_h.

We present an adaptive grid generated by our algorithm in Fig. 1a and plot the decay rate of the error as well as the estimator η in Fig. 1b. The approximated rate r is obtained by finding

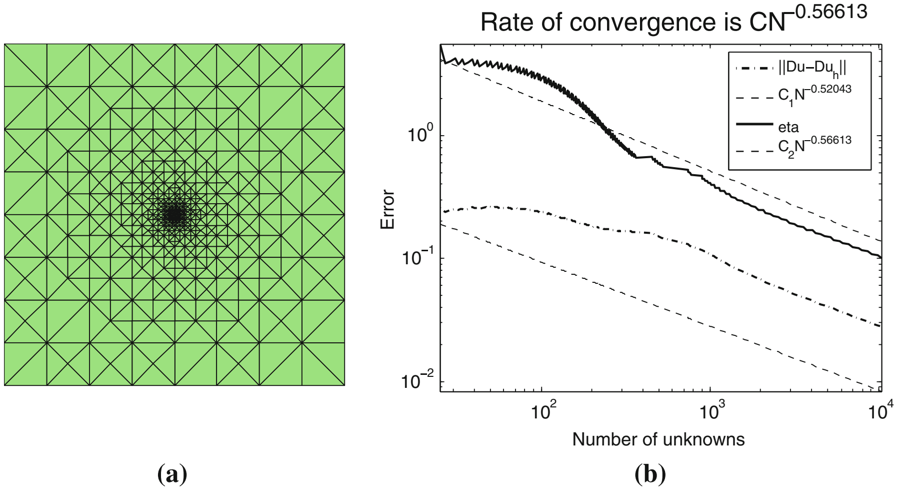


Fig. 1 An adaptive grid and the error history for the Kellogg problem. **a** An adaptive grid with 10,376 nodes generated by `AFEM_WG`. **b** Decay of the error and estimator

Table 1 Error table of Kellogg problem

N	e	η	N	e	η
200	1.83099e−01	1.3987e+00	4,037	4.52260e−02	1.68851e−01
455	1.60243e−01	6.70852e−01	5,162	3.98050e−02	1.48580e−01
743	1.30391e−01	5.22967e−01	6,158	3.63765e−02	1.32420e−01
894	1.17721e−01	4.67049e−01	7,591	3.26328e−02	1.21772e−01
1, 009	1.07411e−01	4.09971e−01	8,654	3.05242e−02	1.13932e−01
1, 210	9.30829e−02	3.52864e−01	9,096	2.97702e−02	1.08385e−01
2, 062	6.59239e−02	2.46943e−01	10,376	2.78425e−02	1.01355e−01

N degree of freedom, $e = \|A^{1/2}(\nabla u - \nabla_{w,h} u_h)\|$, and η the a posteriori estimator defined in (4.1)

a least square fitting in the logarithmic scale, i.e. $\log \|A^{1/2}(\nabla u - \nabla_w u_h)\| \approx r \log N + c$. From Fig. 1b, it is clear that our adaptive algorithm achieves almost first order convergence. The estimator also decays in the optimal order.

We also show the results using a modified error estimator $\tilde{\eta}$ without the weight $|A_e^{\min}|$ in the tangential jump. It can be seen that the use of the weight indeed improves the numerical approximation considerably. For example, to achieve the same accuracy around 4.6×10^{-2}, using the weighted estimator saves half of degree of freedom; see Tables 1 and 2.

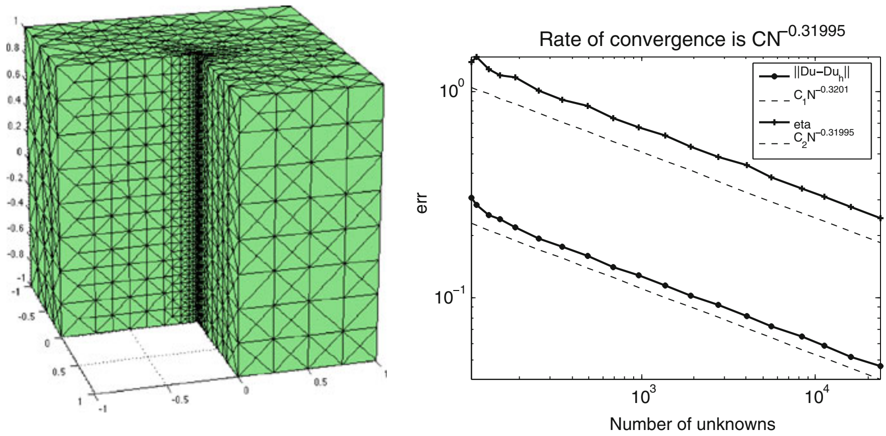
5.2 Example: L-shaped Problem in Three Dimensions

We choose an L-shaped domain $\Omega = (-1, 1)^3/(0, 1) \times (0, 1) \times (-1, -1)$. An initial mesh $\mathcal{T}_0$ is obtained by partitioning x-axes, y-axes, and z-axes into 4 equally distributed subintervals, then dividing one cube into six tetrahedron. The weak finite element space is $V_h = W_{0,0} - RT_0$. We use the longest-edge bisection method; i.e., we always choose the longest edge as the refinement edge, which is equivalent to the Kossaczký bisection method [14] for the initial triangulation we chose.

Table 2 Error table for Kellogg problem

N	e	$\tilde{\eta}$	N	e	$\tilde{\eta}$
289	2.54850e−01	3.88835e−01	4,048	6.50579e−02	9.36614e−02
423	1.98394e−01	3.57294e−01	4,591	6.31056e−02	8.74870e−02
607	1.53291e−01	2.77124e−01	5,458	5.83596e−02	7.81750e−02
802	1.24548e−01	2.32392e−01	6,666	5.44374e−02	7.17687e−02
916	1.14234e−01	2.10528e−01	9,092	4.87604e−02	6.01197e−02
1,149	1.00426e−01	1.85962e−01	9,825	4.72077e−02	5.88379e−02
2,011	8.01611e−02	1.39326e−01	10,351	4.64879e−02	5.62424e−02

N degree of freedom, $e = \|A^{1/2}(\nabla u - \nabla_{w,h} u_h)\|$, and $\tilde{\eta}$ a variant of estimator (4.1) without weight



(a) **(b)**

Fig. 2 An adaptive grid and the error history for the L-shape problem in three dimensions. **a** An adaptive grid generated by `AFEM_WG`. **b** Decay of the error and estimator

Table 3 Error table of the L-shape problem in three dimensions

N	e	η	N	e	η
113	2.80997e−01	1.47589e+00	1,374	1.15270e−01	6.13495e−01
133	2.50932e−01	1.27877e+00	1,914	1.02298e−01	5.40045e−01
154	2.39392e−01	1.19578e+00	2,763	9.25623e−02	4.80238e−01
188	2.19841e−01	1.16370e+00	4,052	8.17900e−02	4.37118e−01
258	1.92732e−01	1.01342e+00	5,573	7.27413e−02	3.84206e−01
351	1.76494e−01	9.14528e−01	8,375	6.52928e−02	3.38291e−01
489	1.59540e−01	8.50567e−01	11,281	5.86513e−02	3.09448e−01
687	1.40592e−01	7.45588e−01	15,960	5.20083e−02	2.74171e−01
965	1.28852e−01	6.68246e−01	23,419	4.64729e−02	2.43616e−01

N degree of freedom, $e = \|A^{1/2}(\nabla u - \nabla_{w,h} u_h)\|$, and η the a posteriori estimator defined in (4.1)

We set $A = \mathbf{I}$ and chose the Dirichlet boundary condition and the source $f = 0$ so that the exact solution is $u = r^{\frac{2}{3}} \sin(\frac{2}{3}\theta)$ in the cylindrical coordinate. The function u contains an edge-type singularity. Again, for this example, only the tangential jump $\nabla \times \nabla_w u_h$ across faces contributes to the error estimator η.

For quasi-uniform grids in three dimensions, $h = \mathcal{O}(N^{-\frac{1}{3}})$. Therefore we expect the error to satisfy $\|A^{\frac{1}{2}}(\nabla u - \nabla_{w,h} u_h)\| \leq CN^{-\frac{1}{3}}$ which is indeed the case shown in Fig. 2b. An adaptive grids with correctly refined along the singular edge is also presented in Fig. 2a. Selected error and estimator is summarized in Table 3.

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