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DISTRIBUTIONAL FINITE ELEMENT CURL DIV COMPLEXES AND APPLICATION TO QUAD CURL PROBLEMS*

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Abstract. This paper addresses the challenge of constructing finite element curl div complexes in three dimensions. Tangential-normal continuity is introduced in order to develop distributional finite element curl div complexes. The spaces constructed are applied to discretize the quad curl problem, demonstrating optimal order of convergence. Furthermore, a hybridization technique is proposed, demonstrating its equivalence to nonconforming finite elements and weak Galerkin methods.

Key words. distributional finite element curl div complex, quad curl problem, error analysis, hybridization

MSC codes. 58J10, 65N12, 65N22, 65N30

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1. Introduction.

In this work, we will construct distributional finite element curl div complexes in three dimensions, and apply it to solve the fourth-order curl problem $-\operatorname{curl}\Delta\operatorname{curl}\boldsymbol{u}=\boldsymbol{f}, \operatorname{div}\boldsymbol{u}=0$ in a domain $\Omega\subset\mathbb{R}^3$ with boundary conditions $\boldsymbol{u}\times\boldsymbol{n}=\operatorname{curl}\boldsymbol{u}=0$ on $\partial\Omega$. Such a problem arises from multiphysics simulation such as modeling a magnetized plasma in magnetohydrodynamics [9].

We first give a brief literature review on distributional finite elements. The distributional finite element de Rham complexes are adopted to construct equilibrated residual error estimators in [6], which are then extended to discrete distributional differential forms in [32], discrete distributional elasticity complexes in [18], and discrete distributional Hessian and divdiv complexes in [26] with applications in cohomology groups. Recently, in [16], the distributional finite element divdiv element has been constructed and applied for solving the mixed formulation of the biharmonic equation in arbitrary dimensions. The distributional finite elements allow the use of piecewise polynomials with less smoothness, which is especially useful for high-order differential operators.

Let us use a more familiar 2nd order operator ∇^2 as an example to illustrate the motivation. The C^1-conforming finite element on tetrahedron meshes [25, 13, 15, 43] requires polynomials of degree 9 and above and possesses extra smoothness at vertices and edges. Therefore, it is hardly used in practice. Simple finite elements can be constructed if the differential operators are understood in the distribution sense.

For the discretization of the biharmonic equation in two dimensions, the so-called Hellan–Herrmann–Johnson (HHJ) mixed method [22, 23, 31] requires only normal-normal continuous finite elements for symmetric tensors and thus C^0-conforming

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Lagrange element, not C^1-conforming elements, can be used for displacement. This normal-normal continuous finite element is then employed to solve linear elasticity [36] and Reissner–Mindlin plates [37], and used to construct the first two-dimensional distributional finite element divdiv complexes in [11]. Recently, the distributional finite element divdiv element for solving the mixed formulation of the biharmonic equation has been extended to arbitrary dimensions in [16].

Now we move to the curl div operator. Introduce the space $H(\operatorname{curl}\operatorname{div},\Omega;\mathbb{T}) := \{\boldsymbol{\tau}\in L^2(\Omega;\mathbb{T}) : \operatorname{curl}\operatorname{div}\boldsymbol{\tau}\in L^2(\Omega;\mathbb{R}^3)\}$, where $\mathbb{T}$ is the space of traceless tensors. A mixed formulation of the quad-curl problem is to find $\boldsymbol{\sigma}\in H(\operatorname{curl}\operatorname{div},\Omega;\mathbb{T})$, $\boldsymbol{u}\in L^2(\Omega;\mathbb{R}^3)$, and $\phi\in H_0^1(\Omega)$ such that

$$\begin{aligned}(\boldsymbol{\sigma},\boldsymbol{\tau}) + b(\boldsymbol{\tau},\psi;\boldsymbol{u}) &= 0, && \forall\ \boldsymbol{\tau}\in H(\operatorname{curl}\operatorname{div},\Omega;\mathbb{T}), \psi\in H_0^1(\Omega),\\ b(\boldsymbol{\sigma},\phi;\boldsymbol{v}) &= -\langle \boldsymbol{f},\boldsymbol{v}\rangle, && \forall\ \boldsymbol{v}\in L^2(\Omega;\mathbb{R}^3),\end{aligned}$$

where the bilinear form $b(\boldsymbol{\tau},\psi;\boldsymbol{v}) := (\operatorname{curl}\operatorname{div}\boldsymbol{\tau},\boldsymbol{v}) + (\operatorname{grad}\psi,\boldsymbol{v})$. The term $(\operatorname{grad}\psi,\boldsymbol{u})$ is introduced to impose the divergence free condition $\operatorname{div}\boldsymbol{u}=0$.

Finite element spaces conforming to $H(\operatorname{curl}\operatorname{div},\Omega;\mathbb{T})$ are relatively complicated due to the smoothness requirement $\operatorname{curl}\operatorname{div}\boldsymbol{\tau}\in L^2(\Omega;\mathbb{R}^3)$. In the distributional sense

$$\langle \operatorname{curl}\operatorname{div}\boldsymbol{\tau},\boldsymbol{v}\rangle = -(\boldsymbol{\tau},\operatorname{grad}\operatorname{curl}\boldsymbol{v}), \quad \boldsymbol{v}\in H(\operatorname{grad}\operatorname{curl},\Omega),$$

the smoothness can be shifted to the test function $\boldsymbol{v}$, where $H(\operatorname{grad}\operatorname{curl},\Omega) := \{\boldsymbol{u}\in L^2(\Omega;\mathbb{R}^3) : \operatorname{curl}\boldsymbol{u}\in H^1(\Omega;\mathbb{R}^3)\}$. Of course, $H(\operatorname{grad}\operatorname{curl})$-conforming finite elements are not easy to construct either. For example, the $H(\operatorname{grad}\operatorname{curl})$-conforming finite elements are constructed in [42, 27, 40, 14, 15], which requires polynomial of degree at least 7 and dimension of shape function space at least 315.

The key idea is to strike a balance of the smoothness of the trial function $\boldsymbol{\tau}$ and the test function $\boldsymbol{v}$. Given a mesh $\mathcal{T}_h$, let $H^s(\mathcal{T}_h)$ be the space of piecewise H^s function. Introduce the traceless tensor space with tangential-normal continuity

$$\Sigma^{\mathrm{tn}} := \{\boldsymbol{\tau}\in H^1(\mathcal{T}_h;\mathbb{T}) : [\![\boldsymbol{n}\times\boldsymbol{\tau}\boldsymbol{n}]\!]|_F = 0 \text{ for each } F\in\mathring{\mathcal{F}}_h\},$$

where $[\![\boldsymbol{n}\times\boldsymbol{\tau}\boldsymbol{n}]\!]$ is the jump of $\boldsymbol{n}\times\boldsymbol{\tau}\boldsymbol{n}$ across all interior faces F. While for the test space, we use space

$$V_0^{\mathrm{curl}} := H_0(\operatorname{curl},\Omega)\cap H^1(\operatorname{curl},\mathcal{T}_h),$$

where $H^1(\operatorname{curl},\mathcal{T}_h) := \{\boldsymbol{v}\in H^1(\mathcal{T}_h;\mathbb{R}^3) : \operatorname{curl}\boldsymbol{v}\in H^1(\mathcal{T}_h;\mathbb{R}^3)\}$. Define a weak operator $(\operatorname{curl}\operatorname{div})_w : \Sigma^{\mathrm{tn}} \to (V_0^{\mathrm{curl}})'$ by

$$\langle (\operatorname{curl}\operatorname{div})_w\boldsymbol{\tau},\boldsymbol{v}\rangle := \sum_{T\in\mathcal{T}_h}(\operatorname{div}\boldsymbol{\tau},\operatorname{curl}\boldsymbol{v})_T - \sum_{F\in\mathring{\mathcal{F}}_h}([\![\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}]\!],\boldsymbol{n}_F\cdot\operatorname{curl}\boldsymbol{v})_F, \tag{1.1}$$

which is analog to the weak divdiv operator in HHJ mixed method [22, 23, 31]. Now the function $\boldsymbol{\tau}$ is tangential-normally continuous and $\boldsymbol{v}$ is tangentially continuous so that $\boldsymbol{n}_F\cdot\operatorname{curl}\boldsymbol{v} = \operatorname{rot}_F\boldsymbol{v}$ is continuous on face F. One can easily show $(\operatorname{curl}\operatorname{div})_w\boldsymbol{\tau} = \operatorname{curl}\operatorname{div}\boldsymbol{\tau}$ in the distribution sense by taking $\boldsymbol{v}\in C_0^\infty(\Omega;\mathbb{R}^3)$ in (1.1).

We will use the tangential-normal continuous finite element constructed in [21] for the discretization of Σ^{tn}. For an integer $k\ge 0$, take $\mathbb{P}_k(T;\mathbb{T})$ as the space of shape functions. The degrees of freedom (DoFs) are given by

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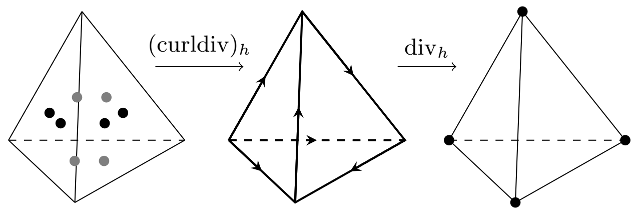

FIG. 1. *The simplest elements* $\Sigma_{0,h}^{\mathrm{tn}} - \mathring{\mathbb{V}}_{(1,0),h}^{\mathrm{curl}} - \mathring{\mathbb{V}}_{1,h}^{\mathrm{grad}}$ *: the first is a piecewise constant traceless matrix with tangential-normal continuity, the second is the lowest order edge element, and the third is the linear Lagrange element.*

$$\int_F \boldsymbol{t}_i^{\intercal} \boldsymbol{\tau} \boldsymbol{n}\, q \,\mathrm{d}S, \quad q \in \mathbb{P}_k(F), \quad i = 1, 2, \quad F \in \mathcal{F}(T), \tag{1.2a}$$

$$\int_T \boldsymbol{\tau} : \boldsymbol{q} \,\mathrm{d}x, \quad \boldsymbol{q} \in \mathbb{P}_{k-1}(T; \mathbb{T}), \tag{1.2b}$$

where $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ denote two mutually perpendicular unit tangential vectors of face F and are used to determine the tangential component of the vector $\boldsymbol{\tau n}$. The global finite element space $\Sigma_{k,h}^{\mathrm{tn}}$ by requiring single valued (1.2a) is tangential-normally continuous.

We use Nédélec elements $\mathring{\mathbb{V}}_h^{\mathrm{curl}} \subset V_0^{\mathrm{curl}}$ for the tangential continuous vector space, and use the Riesz representation of the L^2-inner product to bring the abstract dual to a concrete function. Define $(\mathrm{curl\ div})_h : \Sigma_h^{\mathrm{tn}} \to \mathring{\mathbb{V}}_h^{\mathrm{curl}}$ such that

$$((\mathrm{curl\ div})_h \boldsymbol{\tau}_h, \boldsymbol{v}_h) = \langle (\mathrm{curl\ div})_w \boldsymbol{\tau}_h, \boldsymbol{v}_h \rangle, \quad \forall\, \boldsymbol{v}_h \in \mathring{\mathbb{V}}_h^{\mathrm{curl}}, \tag{1.3}$$

and its L^2-adjoint operator $(\mathrm{grad\,curl})_h : \mathring{\mathbb{V}}_h^{\mathrm{curl}} \to \Sigma_h^{\mathrm{tn}}$.

By including the tensor version of the Nédélec elements $\mathbb{V}_{k,h}^{\mathrm{curl}}(\mathbb{M})$, and the Lagrange elements $\mathbb{V}_{k+1,h}^{\mathrm{grad}}(\mathbb{R}^3)$, we are able to construct the distributional finite element curl div complex:

$$\begin{aligned} \mathbb{R}^3 \times \{0\} \to \mathbb{V}_{k+1,h}^{\mathrm{grad}}(\mathbb{R}^3) \times \mathbb{R} \xrightarrow{(\mathrm{grad},\, \mathrm{mskw}\, \boldsymbol{x})} \mathbb{V}_{k,h}^{\mathrm{curl}}(\mathbb{M}) \xrightarrow{\mathrm{dev\,curl}} \\ \Sigma_{k-1,h}^{\mathrm{tn}} \xrightarrow{(\mathrm{curl\ div})_h} \mathring{\mathbb{V}}_{(k,\ell),h}^{\mathrm{curl}} \xrightarrow{\mathrm{div}_h} \mathring{\mathbb{V}}_{\ell+1,h}^{\mathrm{grad}} \to 0, \end{aligned} \tag{1.4}$$

where $(\mathrm{grad}\,, \mathrm{mskw}\, \boldsymbol{x})\binom{\boldsymbol{v}}{c} = \mathrm{grad}\, \boldsymbol{v} + c\, \mathrm{mskw}\, \boldsymbol{x}$, and $\ell = k-1$ or $\ell = k$ is introduced to distinguish the first and second kind of Nédélec element. The lowest order, i.e., $k = 1, \ell = 0$, of the last three elements are illustrated in Figure 1.

The finite element complex (1.4) is a discretization of the distributional curl div complex:

$$\begin{aligned} \mathbb{R}^3 \times \{0\} \to H^1(\Omega; \mathbb{R}^3) \times \mathbb{R} \xrightarrow{(\mathrm{grad}\,, \mathrm{mskw}\, \boldsymbol{x})} H(\mathrm{curl}, \Omega; \mathbb{M}) \xrightarrow{\mathrm{dev\ curl}} \\ H^{-1}(\mathrm{curl\ div}, \Omega; \mathbb{T}) \xrightarrow{\mathrm{curl\ div}} H^{-1}(\mathrm{div}, \Omega) \xrightarrow{\mathrm{div}} H^{-1}(\Omega) \to 0, \end{aligned}$$

where $H^{-1}(\Omega) := (H_0^1(\Omega))'$, and Sobolev spaces of negative order are

$$\begin{aligned} H^{-1}(\mathrm{div}, \Omega) &:= \left\{ \boldsymbol{u} \in H^{-1}(\Omega; \mathbb{R}^3) : \mathrm{div}\, \boldsymbol{u} \in H^{-1}(\Omega) \right\}, \\ H^{-1}(\mathrm{curl\ div}, \Omega; \mathbb{T}) &:= \left\{ \boldsymbol{\tau} \in L^2(\Omega; \mathbb{T}) : \mathrm{curl\ div}\, \boldsymbol{\tau} \in H^{-1}(\mathrm{div}, \Omega) \right\}. \end{aligned}$$

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As an application, we consider the fourth-order curl problem $-\operatorname{curl}\Delta\operatorname{curl}\boldsymbol{u}=\boldsymbol{f}, \operatorname{div}\boldsymbol{u}=0$ with boundary conditions $\boldsymbol{u}\times\boldsymbol{n}=\operatorname{curl}\boldsymbol{u}=0$ on $\partial\Omega$. The distributional mixed finite element method is to find $\boldsymbol{\sigma}_h\in\Sigma_{k-1,h}^{\mathrm{tn}}$, $\boldsymbol{u}_h\in\mathring{\mathbb{V}}_{(k,\ell),h}^{\mathrm{curl}}$, and $\phi_h\in\mathring{\mathbb{V}}_{\ell+1,h}^{\mathrm{grad}}$ such that

$$(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h)+b_h(\boldsymbol{\tau}_h,\psi_h;\boldsymbol{u}_h)=0, \qquad \forall\boldsymbol{\tau}_h\in\Sigma_{k-1,h}^{\mathrm{tn}},\psi_h\in\mathring{\mathbb{V}}_{\ell+1,h}^{\mathrm{grad}}, \tag{1.5a}$$

$$b_h(\boldsymbol{\sigma}_h,\phi_h;\boldsymbol{v}_h)=-(\boldsymbol{f},\boldsymbol{v}_h), \qquad \forall\boldsymbol{v}_h\in\mathring{\mathbb{V}}_{(k,\ell),h}^{\mathrm{curl}}, \tag{1.5b}$$

where $b_h(\boldsymbol{\tau},\psi;\boldsymbol{v}):=((\operatorname{curl}\operatorname{div})_h\boldsymbol{\tau},\boldsymbol{v})+(\operatorname{grad}\psi,\boldsymbol{v})$ and $(\operatorname{curl}\operatorname{div})_h$ is a discretization of distributional curl div operator; cf. (1.3).

We prove two discrete inf-sup conditions and thus obtain the well-posedness of (1.5a)–(1.5b) and optimal order convergence

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,h}+\|I_h^{\mathrm{curl}}\boldsymbol{u}-\boldsymbol{u}_h\|_{H((\operatorname{grad}\operatorname{curl})_h)}\lesssim h^k(|\boldsymbol{\sigma}|_k+|\boldsymbol{u}|_k),$$
$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{H(\operatorname{curl})}+h\|\boldsymbol{u}-\boldsymbol{u}_h\|_{H((\operatorname{grad}\operatorname{curl})_h)}\lesssim h^k(|\boldsymbol{\sigma}|_k+|\boldsymbol{u}|_k+|\operatorname{curl}\boldsymbol{u}|_k).$$

By the duality argument, the order of $\|\operatorname{curl}(I_h^{\mathrm{curl}}\boldsymbol{u}-\boldsymbol{u}_h)\|$ can be improved to h^{k+1} on convex domains. Both $\|I_h^{\mathrm{curl}}\boldsymbol{u}-\boldsymbol{u}_h\|_{H((\operatorname{grad}\operatorname{curl})_h)}$ and $\|\operatorname{curl}(I_h^{\mathrm{curl}}\boldsymbol{u}-\boldsymbol{u}_h)\|$ are superconvergent. Postprocessing can be applied to improve the approximation to $\boldsymbol{u}$.

Furthermore, we apply hybridization techniques to (1.5a)–(1.5b), leading to a stabilization-free weak Galerkin method and extending to the $H(\operatorname{grad}\operatorname{curl})$ nonconforming finite elements introduced in [29, 45] for solving the quad-curl problem. Equivalently, we identify the complex that accommodates these nonconforming finite elements and generalize them to arbitrary orders.

For other discretization of the quad-curl problem, we refer to the macro finite element method in [28], nonconforming finite element methods in [30, 39, 41], mixed finite element methods in [38, 10], decoupled finite element methods in [8, 44, 7], and references cited therein.

The rest of this paper is organized as follows. Section 2 focuses on the distributional curl div complex. A distributional finite element curl div complex is constructed in section 3, and applied to solve the quad-curl problem in section 4. The hybridization of the distributional mixed finite element method and the equivalence to other methods are presented in section 5.

2. Distributional curl div complex.

In this section, we present the distributional curl div complex and introduce the weak differential operator $(\operatorname{curl}\operatorname{div})_w$ which can be defined on the tangential-normal continuous matrix functions.

2.1. Notation.

Let $K\subset\mathbb{R}^3$ be a nondegenerated three-dimensional polyhedron. Denote by $\mathcal{F}(K)$ the set of all two-dimensional faces of K. For $F\in\mathcal{F}(K)$, denote by $\mathcal{E}(F)$ the set of all edges of F. For $F\in\mathcal{F}(K)$, choose a normal vector $\boldsymbol{n}_F$ and two mutually perpendicular unit tangential vectors $\boldsymbol{t}_{F,1}$ and $\boldsymbol{t}_{F,2}$, which will be abbreviated as $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ for simplicity. Let $\boldsymbol{n}_K$ be the unit outward normal vector to ∂K, which will be abbreviated as $\boldsymbol{n}$. For $F\in\mathcal{F}(K)$ and $e\in\mathcal{E}(F)$, denote by $\boldsymbol{n}_{F,e}$ the unit vector being parallel to F and outward normal to ∂F. Set $\boldsymbol{t}_{F,e}:=\boldsymbol{n}_K\times\boldsymbol{n}_{F,e}$.

Given a face $F\in\mathcal{F}(K)$, and a vector $\boldsymbol{v}\in\mathbb{R}^3$, define

$$\Pi_F\boldsymbol{v}=(\boldsymbol{n}\times\boldsymbol{v})\times\boldsymbol{n}=(\boldsymbol{I}-\boldsymbol{n}\boldsymbol{n}^{\intercal})\boldsymbol{v}$$

as the projection of $\boldsymbol{v}$ onto the face F which is called the tangential component of $\boldsymbol{v}$. The vector $\boldsymbol{n}\times\boldsymbol{v}=(\boldsymbol{n}\times\Pi_F)\boldsymbol{v}$ is called the tangential trace of $\boldsymbol{v}$, which is a rotation of $\Pi_F\boldsymbol{v}$ on F (90° counter-clockwise with respect to $\boldsymbol{n}$).

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Define the surface gradient operator as $\nabla_F := \Pi_F \nabla$. For a scalar function v, define the surface curl:

$$\operatorname{curl}_F v = \boldsymbol{n} \times \nabla v = \boldsymbol{n} \times \nabla_F v.$$

For a vector function $\boldsymbol{v}$, the surface rot operator is defined as

$$\operatorname{rot}_F \boldsymbol{v} := (\boldsymbol{n} \times \nabla) \cdot \boldsymbol{v} = (\boldsymbol{n} \times \nabla_F) \cdot \Pi_F \boldsymbol{v} = \boldsymbol{n} \cdot (\operatorname{curl} \boldsymbol{v}),$$

which represents the normal component of $\operatorname{curl} \boldsymbol{v}$.

Denote the space of all 3×3 matrices by $\mathbb{M}$, and all trace-free/traceless 3×3 matrices by $\mathbb{T}$. Define the deviation $\operatorname{dev} \boldsymbol{\tau} = \boldsymbol{\tau} - \frac{1}{3}(\operatorname{tr} \boldsymbol{\tau})\boldsymbol{I} \in \mathbb{T}$. Obviously, for a scalar function u, $\operatorname{dev}(u\boldsymbol{I}) = 0$. For a vector $\boldsymbol{w} = (w_1, w_2, w_3)^\intercal \in \mathbb{R}^3$, let

$$\operatorname{mskw} \boldsymbol{w} := \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}.$$

For a tensor-valued function $\boldsymbol{\tau}$, $\operatorname{div} \boldsymbol{\tau}$ and $\operatorname{curl} \boldsymbol{\tau}$ mean operators div and curl are applied row-wisely to $\boldsymbol{\tau}$. By direct calculation, we have the identities

$$\operatorname{div} \operatorname{mskw} \boldsymbol{v} = -\operatorname{curl} \boldsymbol{v}, \quad (\operatorname{mskw} \boldsymbol{v})\boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{n}. \tag{2.1}$$

We use $\{\mathcal{T}_h\}_{h>0}$ to denote a shape regular family of simplicial meshes of Ω with mesh size $h = \max_{T \in \mathcal{T}_h} h_T$ and h_T being the diameter of T. Let $\mathcal{F}_h$, $\mathring{\mathcal{F}}_h$, $\mathcal{E}_h$, $\mathring{\mathcal{E}}_h$, $\mathcal{V}_h$, and $\mathring{\mathcal{V}}_h$ be the set of all faces, interior faces, edges, interior edges, vertices, and interior vertices of $\mathcal{T}_h$, respectively. Let $T \in \mathcal{T}_h$ be a tetrahedron with four vertices $\texttt{v}_0, \ldots, \texttt{v}_3$. Denote by λ_i the ith barycentric coordinate with respect to the simplex T for $i = 0, \ldots, 3$. Set $\boldsymbol{t}_{ij} := \texttt{v}_j - \texttt{v}_i$ as the edge vector from $\texttt{v}_i$ to $\texttt{v}_j$.

Given a nonnegative integer k, let $\mathbb{P}_k(T)$ stand for the set of all polynomials in T with the total degree no more than k, and let $\mathbb{P}_k(T; \mathbb{X})$ denote the tensor or vector version with $\mathbb{X} = \mathbb{R}^3$, $\mathbb{M}$, and $\mathbb{T}$. When $k < 0$, set $\mathbb{P}_k(T) := \{0\}$.

Given a bounded domain $D \subset \mathbb{R}^3$ and a real number s, let $H^s(D)$ be the usual Sobolev space of functions over D, whose norm and seminorm are denoted by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$, respectively. Let $(\cdot,\cdot)_D$ be the standard inner product on $L^2(D)$. If D is Ω, we abbreviate $\|\cdot\|_{s,D}$, $|\cdot|_{s,D}$, and $(\cdot,\cdot)_D$ by $\|\cdot\|_s$, $|\cdot|_s$ and $(\cdot,\cdot)$, respectively. We also abbreviate $\|\cdot\|_{0,D}$ and $\|\cdot\|_0$ by $\|\cdot\|_D$ and $\|\cdot\|$, respectively. The duality pair will be denoted by $\langle \cdot, \cdot \rangle$.

Introduce the following Sobolev spaces:

$$\begin{aligned} H(\operatorname{curl}, D) &:= \{\boldsymbol{u} \in L^2(D;\mathbb{R}^3) : \operatorname{curl} \boldsymbol{u} \in L^2(D;\mathbb{R}^3)\}, \\ H(\operatorname{div}, D) &:= \{\boldsymbol{u} \in L^2(D;\mathbb{R}^3) : \operatorname{div} \boldsymbol{u} \in L^2(D)\}, \\ H(\operatorname{grad} \operatorname{curl}, D) &:= \{\boldsymbol{u} \in L^2(D;\mathbb{R}^3) : \operatorname{curl} \boldsymbol{u} \in H^1(D;\mathbb{R}^3)\}, \\ H(\operatorname{curl} \operatorname{div}, D; \mathbb{T}) &:= \left\{\boldsymbol{\tau} \in L^2(D;\mathbb{T}) : \operatorname{curl} \operatorname{div} \boldsymbol{\tau} \in L^2(D;\mathbb{R}^3)\right\}, \end{aligned}$$

where $H^s(D;\mathbb{X}) := H^s(D) \otimes \mathbb{X}$. Define piecewise smooth function space, for $s > 0$,

$$H^s(\mathcal{T}_h) := \{v \in L^2(\Omega) : v|_T \in H^s(T) \text{ for all } T \in \mathcal{T}_h\},$$

and $H^s(\mathcal{T}_h; \mathbb{X})$ its tensor or vector version with $\mathbb{X} = \mathbb{R}^3$, $\mathbb{M}$, and $\mathbb{T}$. Let $\operatorname{grad}_{\mathcal{T}_h}$ and $\operatorname{curl}_{\mathcal{T}_h}$ be the elementwise version of grad and curl associated with $\mathcal{T}_h$, respectively.

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2.2. The curl div complexes. The curl div complex in three dimensions reads as [3, eq. (47)]

$$\begin{aligned}\mathbb{R}^3\times\{0\} \to & H^1(\Omega;\mathbb{R}^3)\times\mathbb{R} \xrightarrow{(\mathrm{grad},\,\mathrm{mskw}\,\boldsymbol{x})} H(\mathrm{curl},\Omega;\mathbb{M}) \xrightarrow{\mathrm{dev\ curl}} \\ & H(\mathrm{curl\ div},\Omega;\mathbb{T}) \xrightarrow{\mathrm{curl\ div}} H(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega)\to 0.\end{aligned} \tag{2.2}$$

When Ω is topologically trivial, i.e., all co-homology group of Ω is trivial, then (2.2) is exact. The smoothness of the potential can be further improved to be in H^1.

It is difficult to construct $H(\mathrm{curl\,div},\Omega;\mathbb{T})$-conforming finite element with lower order degree of polynomials. To relax the smoothness, we are going to present a distributional curl div complex with negative Sobolev spaces involved.

Define

$$H^{-1}(\mathrm{div},\Omega)=\{\boldsymbol{v}\in H^{-1}(\Omega;\mathbb{R}^3):\mathrm{div}\,\boldsymbol{v}\in H^{-1}(\Omega)\}.$$

In [12], we have shown that $(H_0(\mathrm{curl},\Omega))'=H^{-1}(\mathrm{div},\Omega)$. Define

$$H^{-1}(\mathrm{curl\,div},\Omega;\mathbb{T}):=\left\{\boldsymbol{\tau}\in L^2(\Omega;\mathbb{T}):\mathrm{curl\,div}\,\boldsymbol{\tau}\in H^{-1}(\mathrm{div},\Omega)\right\}$$

with squared norm

$$\|\boldsymbol{\tau}\|^2_{H(\mathrm{curl\,div})}:=\|\boldsymbol{\tau}\|^2+\|\mathrm{curl\ div}\,\boldsymbol{\tau}\|^2_{H^{-1}(\mathrm{div})}=\|\boldsymbol{\tau}\|^2+\|\mathrm{curl\ div}\,\boldsymbol{\tau}\|^2_{-1}.$$

LEMMA 2.1. *The distributional* curl div *complex in three dimensions is*

$$\begin{aligned}\mathbb{R}^3\times\{0\} \to & H^1(\Omega;\mathbb{R}^3)\times\mathbb{R} \xrightarrow{(\mathrm{grad}\,,\mathrm{mskw}\,\boldsymbol{x})} H(\mathrm{curl},\Omega;\mathbb{M}) \xrightarrow{\mathrm{dev\ curl}} \\ & H^{-1}(\mathrm{curl\ div},\Omega;\mathbb{T}) \xrightarrow{\mathrm{curl\ div}} H^{-1}(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} H^{-1}(\Omega)\to 0.\end{aligned} \tag{2.3}$$

When $\Omega\subset\mathbb{R}^3$ is a bounded and topologically trivial Lipschitz domain, (2.3) *is exact.*

Proof. Apparently (2.3) is a complex. The surjection $\mathrm{div}\,H^{-1}(\mathrm{div},\Omega)=H^{-1}(\Omega)$ follows from $\mathrm{div}\,L^2(\Omega;\mathbb{R}^3)=H^{-1}(\Omega)$ and $\mathrm{div}\,L^2(\Omega;\mathbb{R}^3)\subseteq \mathrm{div}\,H^{-1}(\mathrm{div},\Omega)$. We then verify its exactness.

$\boxed{1}$ $\mathrm{curl\ div}\,H^{-1}(\mathrm{curl\ div},\Omega;\mathbb{T})=H^{-1}(\mathrm{div},\Omega)\cap\ker(\mathrm{div})$.

For $\boldsymbol{v}\in H^{-1}(\mathrm{div},\Omega)\cap\ker(\mathrm{div})$, by the exactness of the de Rham complex [19], there exists $\boldsymbol{\tau}\in H^1(\Omega;\mathbb{M})$ such that $\boldsymbol{v}=\mathrm{curl\,div}\,\boldsymbol{\tau}$. Notice that $\mathrm{curl\,div}(p\boldsymbol{I})=\mathrm{curl\,grad}\,p=0$. Then $\boldsymbol{v}=\mathrm{curl\ div}(\mathrm{dev}\,\boldsymbol{\tau})\in\mathrm{curl\ div}\,H^{-1}(\mathrm{curl\ div},\Omega;\mathbb{T})$.

$\boxed{2}$ $\mathrm{dev\,curl}\,H(\mathrm{curl},\Omega;\mathbb{M})=H^{-1}(\mathrm{curl\ div},\Omega;\mathbb{T})\cap\ker(\mathrm{curl\ div})$.

For $\boldsymbol{\tau}\in H^{-1}(\mathrm{curl\ div},\Omega;\mathbb{T})\cap\ker(\mathrm{curl\ div})$, by the de Rham complex, there exists a function $u\in L^2(\Omega)$ s.t. $\mathrm{div}\,\boldsymbol{\tau}=\mathrm{grad}\,u$. Then $\mathrm{div}(\boldsymbol{\tau}-u\boldsymbol{I})=0$, which means $\boldsymbol{\tau}=u\boldsymbol{I}+\mathrm{curl}\,\boldsymbol{\sigma}$ with $\boldsymbol{\sigma}\in H^1(\Omega;\mathbb{M})$. By the traceless of $\boldsymbol{\tau}$, we get $\boldsymbol{\tau}=\mathrm{dev\,curl}\,\boldsymbol{\sigma}\in\mathrm{dev\,curl}\,H^1(\Omega;\mathbb{M})\subseteq\mathrm{dev\,curl}\,H(\mathrm{curl},\Omega;\mathbb{M})$.

$\boxed{3}$ $\mathrm{grad}\,H^1(\Omega;\mathbb{R}^3)\oplus\mathrm{span}\{\mathrm{mskw}\,\boldsymbol{x}\}=H(\mathrm{curl},\Omega;\mathbb{M})\cap\ker(\mathrm{dev\,curl})$.

Since $\mathrm{curl}\,(\mathrm{mskw}\,\boldsymbol{x})=2\boldsymbol{I}$, we have $\mathrm{grad}\,H^1(\Omega;\mathbb{R}^3)\cap\mathrm{span}\{\mathrm{mskw}\,\boldsymbol{x}\}=\{0\}$. For $\boldsymbol{\tau}\in H(\mathrm{curl},\Omega;\mathbb{M})\cap\ker(\mathrm{dev\,curl})$, we have $\mathrm{curl}\,\boldsymbol{\tau}=\frac{1}{3}\mathrm{tr}(\mathrm{curl}\,\boldsymbol{\tau})\boldsymbol{I}$. Apply div on both sides to get $\mathrm{grad}\,(\mathrm{tr}(\mathrm{curl}\,\boldsymbol{\tau}))=0$. Then $\mathrm{tr}(\mathrm{curl}\,\boldsymbol{\tau})$ is constant, and $\mathrm{curl}\,\boldsymbol{\tau}=2c\boldsymbol{I}$ with $c\in\mathbb{R}$. This implies $\mathrm{curl}\,(\boldsymbol{\tau}-c\,\mathrm{mskw}\,\boldsymbol{x})=0$. Therefore, $\boldsymbol{\tau}\in\mathrm{grad}\,H^1(\Omega;\mathbb{R}^3)\oplus\mathrm{span}\{\mathrm{mskw}\,\boldsymbol{x}\}$. ∎

Next, we use the framework developed in [12] to present a Helmholtz decomposition of $H^{-1}(\mathrm{curl\ div},\Omega;\mathbb{T})$. Denote by

$$K^c=H_0(\mathrm{curl},\Omega)\cap\ker(\mathrm{div})=\{\boldsymbol{v}\in H_0(\mathrm{curl},\Omega):\boldsymbol{v}\perp\mathrm{grad}\,H_0^1(\Omega)\}.$$

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Then $\operatorname{curl}\operatorname{curl}: K^c \to H^{-1}(\operatorname{div},\Omega)\cap\ker(\operatorname{div})$ is isomorphic. Indeed, by the Helmholtz decomposition $L^2(\Omega;\mathbb{R}^3) = \operatorname{curl} K^c + \nabla H^1(\Omega)$ [2], we have

$$\operatorname{curl}\operatorname{curl} K^c = \operatorname{curl} L^2(\Omega;\mathbb{R}^3) = H^{-1}(\operatorname{div},\Omega)\cap\ker(\operatorname{div}).$$

LEMMA 2.2. *It holds the Helmholtz decomposition*

$$H^{-1}(\operatorname{curl}\operatorname{div},\Omega;\mathbb{T}) = \operatorname{dev}\operatorname{curl} H(\operatorname{curl},\Omega;\mathbb{M}) \oplus \operatorname{mskw} K^c.$$

Proof. With complex (2.3) and identity (2.1), we build up the commutative diagram

$$\begin{array}{ccccc} H(\operatorname{curl},\Omega;\mathbb{M}) & \xrightarrow{\operatorname{dev}\operatorname{curl}} & H^{-1}(\operatorname{curl}\operatorname{div},\Omega;\mathbb{T}) & \xrightarrow{\operatorname{curl}\operatorname{div}} & H^{-1}(\operatorname{div},\Omega)\cap\ker(\operatorname{div}) \to 0 \\ & & \nwarrow{\scriptstyle -\operatorname{mskw}} & & \uparrow{\scriptstyle \operatorname{curl}\operatorname{curl}} \\ & & & & K^c. \end{array}$$

Apply the framework in [12] to get the required Helmholtz decomposition. ∎

We introduce the tensor space with tangential-normal continuity

$$\Sigma^{\mathrm{tn}} := \{\boldsymbol{\tau}\in H^1(\mathcal{T}_h;\mathbb{T}) : [\![\boldsymbol{n}\times\boldsymbol{\tau}\boldsymbol{n}]\!]|_F = 0 \text{ for each } F\in\mathring{\mathcal{F}}_h\},$$

where $[\![\boldsymbol{n}\times\boldsymbol{\tau}\boldsymbol{n}]\!]$ is the jump of $\boldsymbol{n}\times\boldsymbol{\tau}\boldsymbol{n}$ across F. Let space $V_0^{\operatorname{curl}} := H_0(\operatorname{curl},\Omega)\cap H^1(\operatorname{curl},\mathcal{T}_h)$, where $H^1(\operatorname{curl},\mathcal{T}_h) := \{\boldsymbol{v}\in H^1(\mathcal{T}_h;\mathbb{R}^3) : \operatorname{curl}\boldsymbol{v}\in H^1(\mathcal{T}_h;\mathbb{R}^3)\}$. We define a weak operator $(\operatorname{curl}\operatorname{div})_w : \Sigma^{\mathrm{tn}} \to (V_0^{\operatorname{curl}})' \subset (C_0^\infty(\Omega;\mathbb{R}^3))'$ by

$$\langle(\operatorname{curl}\operatorname{div})_w\boldsymbol{\tau},\boldsymbol{v}\rangle := \sum_{T\in\mathcal{T}_h}(\operatorname{div}\boldsymbol{\tau},\operatorname{curl}\boldsymbol{v})_T - \sum_{F\in\mathring{\mathcal{F}}_h}([\![\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n}]\!],\boldsymbol{n}_F\cdot\operatorname{curl}\boldsymbol{v})_F. \tag{2.4}$$

Notice that only interior faces are included in the second term as $\boldsymbol{n}_F\cdot\operatorname{curl}\boldsymbol{v} = \operatorname{rot}_F\boldsymbol{v} = 0$ for $\boldsymbol{v}\in H_0(\operatorname{curl},\Omega)$.

LEMMA 2.3. *For $\boldsymbol{\tau}\in\Sigma^{\mathrm{tn}}$, the following identity holds in the distribution sense:*

$$(\operatorname{curl}\operatorname{div})_w\boldsymbol{\tau} = \operatorname{curl}\operatorname{div}\boldsymbol{\tau}.$$

Proof. By the definition of the distributional derivative and employing the integration by parts elementwise we get for $\boldsymbol{v}\in C_0^\infty(\Omega;\mathbb{R}^3)$ that

$$\begin{aligned}\langle\operatorname{curl}\operatorname{div}\boldsymbol{\tau},\boldsymbol{v}\rangle := -(\boldsymbol{\tau},\operatorname{grad}\operatorname{curl}\boldsymbol{v}) &= \sum_{T\in\mathcal{T}_h}(\operatorname{div}\boldsymbol{\tau},\operatorname{curl}\boldsymbol{v})_T - \sum_{T\in\mathcal{T}_h}(\boldsymbol{\tau}\boldsymbol{n},\operatorname{curl}\boldsymbol{v})_{\partial T} \\ &= \sum_{T\in\mathcal{T}_h}(\operatorname{div}\boldsymbol{\tau},\operatorname{curl}\boldsymbol{v})_T - \sum_{T\in\mathcal{T}_h}(\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n},\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{v})_{\partial T} \\ &\quad - \sum_{T\in\mathcal{T}_h}(\boldsymbol{n}\times\boldsymbol{\tau}\boldsymbol{n},\boldsymbol{n}\times\operatorname{curl}\boldsymbol{v})_{\partial T}.\end{aligned}$$

As $\boldsymbol{n}\times\boldsymbol{\tau}\boldsymbol{n}$ is continuous and $\boldsymbol{v}\in C_0^\infty(\Omega;\mathbb{R}^3)$, the last term is canceled. Then rearrange the second term facewisely to derive

$$\langle\operatorname{curl}\operatorname{div}\boldsymbol{\tau},\boldsymbol{v}\rangle = \langle(\operatorname{curl}\operatorname{div})_w\boldsymbol{\tau},\boldsymbol{v}\rangle \quad \forall\,\boldsymbol{v}\in C_0^\infty(\Omega;\mathbb{R}^3).$$

Thus, $(\operatorname{curl}\operatorname{div})_w\boldsymbol{\tau} = \operatorname{curl}\operatorname{div}\boldsymbol{\tau}$ in the distribution sense. ∎

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Similarly, we can define the weak operator $(\operatorname{grad}\operatorname{curl})_w : V_0^{\text{curl}} \to (\Sigma^{\text{tn}})'$ as

$$\langle (\operatorname{grad}\operatorname{curl})_w \boldsymbol{v}, \boldsymbol{\tau}\rangle := \sum_{T\in\mathcal{T}_h} (\boldsymbol{\tau}, \operatorname{grad}\operatorname{curl}\boldsymbol{v})_T - \sum_{F\in\mathcal{F}_h} (\boldsymbol{n}\times\boldsymbol{\tau}\boldsymbol{n}, [\![\boldsymbol{n}\times\operatorname{curl}\boldsymbol{v}]\!])_F.$$

By definition, we have the duality

$$\langle (\operatorname{curl}\operatorname{div})_w \boldsymbol{\tau}, \boldsymbol{v}\rangle = -\langle \boldsymbol{\tau}, (\operatorname{grad}\operatorname{curl})_w \boldsymbol{v}\rangle, \quad \boldsymbol{\tau}\in\Sigma^{\text{tn}}, \boldsymbol{v}\in V_0^{\text{curl}}. \tag{2.5}$$

When $\boldsymbol{\tau} \in H(\operatorname{curl}\operatorname{div},\Omega;\mathbb{T})\cap\Sigma^{\text{tn}}$, $\langle (\operatorname{curl}\operatorname{div})_w\boldsymbol{\tau},\boldsymbol{v}\rangle = (\operatorname{curl}\operatorname{div}\boldsymbol{\tau},\boldsymbol{v})$ and when $\boldsymbol{v}\in H(\operatorname{grad}\operatorname{curl},\Omega)\cap V_0^{\text{curl}}$, $\langle\boldsymbol{\tau},(\operatorname{grad}\operatorname{curl})_w\boldsymbol{v}\rangle = (\boldsymbol{\tau},\operatorname{grad}\operatorname{curl}\boldsymbol{v})$. The duality (2.5) strikes a balance of the smoothness of $\boldsymbol{\tau}$ and $\boldsymbol{v}$ so that the second order differential operators can be defined for less smooth functions.

3. Distributional finite element curl div complex.

We shall construct a finite element counterpart of the distributional curl div complex (2.3).

3.1. Finite element spaces.

We first recall the tangential-normal continuous finite element for traceless tensors in [21]. Take $\mathbb{P}_k(T;\mathbb{T})$ as the space of shape functions with $k\geq 0$. The DoFs are given by

$$\int_F \boldsymbol{t}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}\, q\,\mathrm{d}S, \quad q\in\mathbb{P}_k(F), \quad i=1,2, \quad F\in\mathcal{F}(T), \tag{3.1a}$$

$$\int_T \boldsymbol{\tau}:\boldsymbol{q}\,\mathrm{d}x, \quad \boldsymbol{q}\in\mathbb{P}_{k-1}(T;\mathbb{T}). \tag{3.1b}$$

In order to give a geometric decomposition of space $\mathbb{P}_k(T;\mathbb{T})$, we present two intrinsic bases of $\mathbb{T}$ which are variants of a basis constructed in [24].

LEMMA 3.1. *Let $(i\,j\,\ell\,m)$ be a cyclic permutation of $(0\,1\,2\,3)$. Then the set*

$$\{\operatorname{dev}(\nabla\lambda_i\otimes\boldsymbol{t}_{i\ell}), \operatorname{dev}(\nabla\lambda_j\otimes\boldsymbol{t}_{j\ell}), \ell=0,\ldots,3\} \tag{3.2}$$

is dual to

$$\{\boldsymbol{t}_{mi}\otimes\nabla\lambda_\ell, \boldsymbol{t}_{mj}\otimes\nabla\lambda_\ell\}_{\ell=0}^3. \tag{3.3}$$

Consequently, both are bases of $\mathbb{T}$.

Proof. The duality follows from the identity $\boldsymbol{t}_{ij}\cdot\nabla\lambda_\ell = \delta_{j\ell}-\delta_{i\ell}$, where $\delta_{i\ell}$ is the Kronecker delta function, and $(\boldsymbol{t}\otimes\nabla\lambda_\ell):\boldsymbol{I} = \boldsymbol{t}\cdot\nabla\lambda_\ell = 0$ for vector $\boldsymbol{t}$ tangent to F_ℓ. ☐

As $\nabla\lambda_i \parallel \boldsymbol{n}_{F_i}$, the basis (3.3) is facewise and each face contributes two while the basis (3.2) is vertexwise. They are illustrated in Figure 2.

Define

$$\operatorname{tr}_F^{\text{tn}}\boldsymbol{\tau} = \boldsymbol{n}_F\times\boldsymbol{\tau}\boldsymbol{n}_F,$$

and $\operatorname{tr}^{\text{tn}} : C(T;\mathbb{T})\to L^2(\partial T;\mathbb{R}^2)$ as $\operatorname{tr}^{\text{tn}}|_F = \operatorname{tr}_F^{\text{tn}}$. Let the bubble polynomial space of degree k be

$$\mathbb{B}_k^{\text{tn}}(T;\mathbb{T}) := \{\boldsymbol{\tau}\in\mathbb{P}_k(T;\mathbb{T}) : \operatorname{tr}^{\text{tn}}\boldsymbol{\tau}=0\}.$$

Notice that for the identity matrix $\boldsymbol{I}$, $\operatorname{tr}^{\text{tn}}\boldsymbol{I}=0$ but $\boldsymbol{I}\notin\mathbb{B}_k^{\text{tn}}(T;\mathbb{T})$ as $\operatorname{trace}(\boldsymbol{I})\neq 0$.

LEMMA 3.2. *For $\ell, i = 0,\ldots,3, i\neq\ell$, we have*

$$\lambda_\ell\operatorname{dev}(\nabla\lambda_i\otimes\boldsymbol{t}_{i\ell})\in\mathbb{B}_k^{\text{tn}}(T;\mathbb{T}).$$

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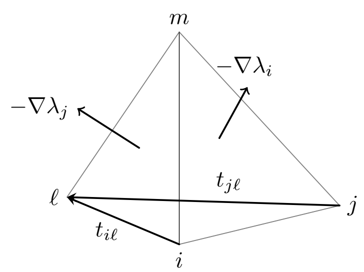
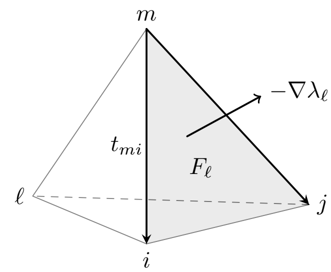


(a) Basis $\{\mathrm{dev}(\nabla\lambda_i \otimes \boldsymbol{t}_{i\ell}), \mathrm{dev}(\nabla\lambda_j \otimes \boldsymbol{t}_{j\ell})\}_{\ell=0}^3$.



(b) Basis $\{\boldsymbol{t}_{mi} \otimes \nabla\lambda_\ell, \boldsymbol{t}_{mj} \otimes \nabla\lambda_\ell\}_{\ell=0}^3$.

FIG. 2. *Two intrinsic bases of traceless matrix* $\mathbb{T}$.

Proof. Let $(i\,j\,\ell\,m)$ be a cyclic permutation of $(0\,1\,2\,3)$. The edge $\mathtt{v}_i\mathtt{v}_\ell$ is contained in faces F_m and F_j and thus $\boldsymbol{t}_{i\ell} \cdot \boldsymbol{n}_F = 0$ for $F = F_m, F_j$. The vector $\nabla\lambda_i \parallel \boldsymbol{n}_{F_i}$ and thus $\boldsymbol{n}_{F_i} \times \nabla\lambda_i = 0$ on F_i. On the face F_ℓ, $\lambda_\ell|_{F_\ell} = 0$. Notice that the identity matrix $\boldsymbol{I}$ satisfies $\mathrm{tr}^{\mathrm{tn}}\boldsymbol{I} = 0$.

So we have verified $\mathrm{tr}^{\mathrm{tn}}(\lambda_\ell\,\mathrm{dev}(\nabla\lambda_i \otimes \boldsymbol{t}_{i\ell})) = 0$. ∎

By changing $\nabla\lambda_i$ to the parallel vector $\boldsymbol{n}_i$, we present the following geometric decomposition of $\mathbb{P}_k(T;\mathbb{T})$.

LEMMA 3.3. *We have the geometric decomposition*

$$\mathbb{P}_k(T;\mathbb{T}) = \oplus_{\ell=0}^3\big(\mathbb{P}_k(F_\ell) \otimes \mathrm{span}\{\mathrm{dev}(\boldsymbol{n}_i \otimes \boldsymbol{t}_{i\ell}), \mathrm{dev}(\boldsymbol{n}_j \otimes \boldsymbol{t}_{j\ell})\}\big) \oplus \mathbb{B}_k^{\mathrm{tn}}(T;\mathbb{T}),$$

where we give a characterization of the bubble space

$$\mathbb{B}_k^{\mathrm{tn}}(T;\mathbb{T}) = \mathbb{P}_{k-1}(T) \otimes \mathrm{span}\{\lambda_\ell\,\mathrm{dev}(\boldsymbol{n}_i \otimes \boldsymbol{t}_{i\ell}), \lambda_\ell\,\mathrm{dev}(\boldsymbol{n}_j \otimes \boldsymbol{t}_{j\ell}), \ell = 0, \ldots, 3\}.$$

Define the global finite element space

$$\begin{aligned}\Sigma_{k,h}^{\mathrm{tn}} := \{&\boldsymbol{\tau}_h \in L^2(\Omega;\mathbb{T}) : \boldsymbol{\tau}_h|_T \in \mathbb{P}_k(T;\mathbb{T}) \text{ for each } T \in \mathcal{T}_h,\\ &\text{and all the DoFs (3.1a)–(3.1b) are single-valued}\}.\end{aligned}$$

Remark 3.4. The construction can be readily extended to arbitrary dimension $\mathbb{R}^d$ for $d \geq 2$. There are $d+1$ faces for a d-simplex. At each face, we have $d-1$ linearly independent traceless matrices $\{\boldsymbol{t}_{F,i} \otimes \boldsymbol{n}_F\}_{i=1}^{d-1}$ and, in total, $(d+1)(d-1)$ such matrices form a basis of $\mathbb{T}$. The basis of bubble functions is constructed verticeswisely $\{\lambda_\ell\,\mathrm{dev}(\nabla\lambda_i \otimes \boldsymbol{t}_{i\ell})\}$ for $\ell = 0, \ldots, d$ and $d-1$ different i for each ℓ.

Remark 3.5. Since the basis functions are expressed in terms of intrinsic geometric quantities—tangential and normal vectors along with barycentric coordinates—the defined finite element spaces are affine invariant, meaning the space remains unchanged under affine transformations. Thus, the traditional method of using a reference tetrahedron and transforming to the physical element can be applied, resulting in the same finite element space. In particular, curved elements can be handled using the Piola transform, as described in equations (3.76)–(3.77) of [33]. Extending the two-dimensional work of [4] to a distributional mixed finite element method specifically tailored for the quad-curl problem on domains with curved boundaries is an interesting topic for future research.

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As $H^{-1}(\mathrm{div},\Omega)=(H_0(\mathrm{curl},\Omega))'$, we can use $H(\mathrm{curl})$-conforming finite elements, i.e., Nédélec elements [34, 35] for the pair space. Take

$$\mathcal{N}^c_{k,\ell}(T):=\boldsymbol{x}\times\mathbb{P}_{k-1}(T;\mathbb{R}^3)+\mathrm{grad}\,\mathbb{P}_{\ell+1}(T),\quad \ell=k \text{ or } k-1$$

as the space of shape functions. The element $\mathcal{N}^c_{k,k-1}$ is the first kind and $\mathcal{N}^c_{k,k}$ is the second kind Nédélec elements [34, 35]. The DoFs $\mathcal{N}^c_{k,\ell}(T)$ are given by

$$(\boldsymbol{v}\cdot\boldsymbol{t},q)_e,\quad q\in\mathbb{P}_\ell(e), e\in\mathcal{E}(T), \tag{3.4a}$$

$$(\boldsymbol{n}\times\boldsymbol{v}\times\boldsymbol{n},\boldsymbol{q})_F,\quad \boldsymbol{q}\in\mathrm{curl}_F\mathbb{P}_{k-1}(F)\oplus\boldsymbol{x}\mathbb{P}_{\ell-2}(F), F\in\mathcal{F}(T), \tag{3.4b}$$

$$(\boldsymbol{v},\boldsymbol{q})_T,\quad \boldsymbol{q}\in\mathrm{curl}\,\mathbb{P}_{k-2}(T;\mathbb{R}^3)\oplus\boldsymbol{x}\mathbb{P}_{\ell-3}(T). \tag{3.4c}$$

Define global finite element spaces

$$\begin{aligned}\mathbb{V}^{\mathrm{curl}}_{(k,\ell),h}:=\{&\boldsymbol{v}_h\in H(\mathrm{curl},\Omega):\boldsymbol{v}_h|_T\in\mathcal{N}^c_{k,\ell}(T) \text{ for } T\in\mathcal{T}_h\\ &\text{and all the DoFs (3.4a)–(3.4c) are single-valued}\}.\end{aligned}$$

Let $\mathring{\mathbb{V}}^{\mathrm{curl}}_{(k,\ell),h}:=\mathbb{V}^{\mathrm{curl}}_{(k,\ell),h}\cap H_0(\mathrm{curl},\Omega)$ and $\mathbb{V}^{\mathrm{curl}}_{k,h}:=\mathbb{V}^{\mathrm{curl}}_{(k,k),h}$.

We use the standard Lagrange element for $H^1(\Omega)$,

$$\mathbb{V}^{\mathrm{grad}}_{\ell+1,h}:=\{\psi_h\in H^1(\Omega):\psi_h|_T\in\mathbb{P}_{\ell+1}(T) \text{ for } T\in\mathcal{T}_h\},$$

and let $\mathring{\mathbb{V}}^{\mathrm{grad}}_{\ell+1,h}:=\mathbb{V}^{\mathrm{grad}}_{\ell+1,h}\cap H^1_0(\Omega)$. Let

$$\mathbb{V}^{\mathrm{grad}}_{k+1,h}(\mathbb{R}^3):=\mathbb{V}^{\mathrm{grad}}_{k+1,h}\otimes\mathbb{R}^3,\quad\text{and}\quad \mathbb{V}^{\mathrm{curl}}_{k,h}(\mathbb{M}):=\mathbb{R}^3\otimes\mathbb{V}^{\mathrm{curl}}_{k,h}.$$

The degree of polynomial may be skipped in the notation of finite element spaces when it is clear from the context.

3.2. Distributional finite element complex. By treating the right-hand side of (2.4) as a bilinear form defined on $\boldsymbol{\Sigma}^{\mathrm{tn}}\times V_0^{\mathrm{curl}}$, the weak operators can be naturally extended to the discrete spaces by restricting the bilinear form to subspaces. Define

$$\begin{aligned}(\mathrm{curl\ div})_h&=J_h(\mathrm{curl\ div})_w:\Sigma^{\mathrm{tn}}_h\to(\mathring{\mathbb{V}}^{\mathrm{curl}}_h)'\cong\mathring{\mathbb{V}}^{\mathrm{curl}}_h,\\ (\mathrm{grad\,curl})_h&=J_h(\mathrm{grad\,curl})_w:\mathring{\mathbb{V}}^{\mathrm{curl}}_h\to(\Sigma^{\mathrm{tn}}_h)'\cong\Sigma^{\mathrm{tn}}_h,\end{aligned}$$

where the isomorphism J_h is the Reisz representation of the L^2-inner product and realized by the inverse of the mass matrix of the corresponding finite element spaces. More precisely, for $\boldsymbol{\sigma}_h\in\Sigma^{\mathrm{tn}}_h$, $(\mathrm{curl\ div})_h\boldsymbol{\sigma}_h\in\mathring{\mathbb{V}}^{\mathrm{curl}}_h$ such that

$$\begin{aligned}((\mathrm{curl\ div})_h\boldsymbol{\sigma}_h,\boldsymbol{v}_h)&=\langle(\mathrm{curl\ div})_w\boldsymbol{\sigma}_h,\boldsymbol{v}_h\rangle\\ &=\sum_{T\in\mathcal{T}_h}(\mathrm{div}\,\boldsymbol{\sigma}_h,\mathrm{curl}\,\boldsymbol{v}_h)_T-\sum_{F\in\mathring{\mathcal{F}}_h}([\![\boldsymbol{n}^\intercal\boldsymbol{\sigma}_h\boldsymbol{n}]\!],\boldsymbol{n}_F\cdot\mathrm{curl}\,\boldsymbol{v}_h)_F\\ &=-\sum_{T\in\mathcal{T}_h}(\boldsymbol{\sigma}_h,\mathrm{grad\,curl}\,\boldsymbol{v}_h)_T+\sum_{F\in\mathcal{F}_h}(\boldsymbol{n}\times\boldsymbol{\sigma}_h\boldsymbol{n},[\![\boldsymbol{n}\times\mathrm{curl}\,\boldsymbol{v}_h]\!])_F\\ &=-(\boldsymbol{\sigma}_h,(\mathrm{grad\,curl})_h\boldsymbol{v}_h).\end{aligned} \tag{3.5}$$

For a fixed triangulation $\mathcal{T}_h$, $(\mathrm{curl\,div})_h$ is well defined which can be obtained by inverting the mass matrix of $\mathring{\mathbb{V}}^{\mathrm{curl}}_h$. However, $\{(\mathrm{curl\,div})_h\}$ is not uniformly bounded when $h\to 0$ as (3.5) is not a bounded linear functional of $L^2(\Omega)$.

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Similarly, define discrete div operator $\operatorname{div}_h : H_0(\operatorname{curl},\Omega) \to \mathring{\mathbb{V}}_h^{\rm grad}$ by

$$(\operatorname{div}_h \boldsymbol{v}, \psi_h) = -(\boldsymbol{v}, \operatorname{grad} \psi_h), \quad \forall\, \psi_h \in \mathring{\mathbb{V}}_h^{\rm grad}.$$

Notice that div_h is the L^2-adjoint of $-\operatorname{grad}$ restricted to $\mathring{\mathbb{V}}_h^{\rm grad}$. Again for a fixed triangulation $\mathcal{T}_h$, div_h is well defined which can be obtained by inverting the mass matrix of $\mathring{\mathbb{V}}_h^{\rm grad}$. However, $\{\operatorname{div}_h\}$ is not uniformly bounded as $h \to 0$ as $(\boldsymbol{v}, \operatorname{grad} \psi)$ is, in general, not a bounded linear functional of L^2 unless $\boldsymbol{v} \in H(\operatorname{div},\Omega)$.

THEOREM 3.6. *The distributional finite element* curl div *complex is*

$$\begin{aligned}(3.6)\qquad &\mathbb{R}^3 \times \{0\} \to \mathbb{V}_{k+1,h}^{\rm grad}(\mathbb{R}^3) \times \mathbb{R} \xrightarrow{(\operatorname{grad}, \operatorname{mskw} \boldsymbol{x})} \mathbb{V}_{k,h}^{\rm curl}(\mathbb{M}) \xrightarrow{\operatorname{dev}\operatorname{curl}} \\ &\qquad \Sigma_{k-1,h}^{\rm tn} \xrightarrow{(\operatorname{curl}\operatorname{div})_h} \mathring{\mathbb{V}}_{(k,\ell),h}^{\rm curl} \xrightarrow{\operatorname{div}_h} \mathring{\mathbb{V}}_{\ell+1,h}^{\rm grad} \to 0.\end{aligned}$$

When $\Omega \subset \mathbb{R}^3$ is a bounded and topologically trivial Lipschitz domain, (3.6) *is exact.*

Proof. As curl grad $=0$, it is straightforward to verify $\operatorname{div}_h(\operatorname{curl}\operatorname{div})_h = 0$. Take $\boldsymbol{\tau}_h = \operatorname{dev}\operatorname{curl}\boldsymbol{\sigma}_h$ with $\boldsymbol{\sigma}_h \in \mathbb{V}_h^{\rm curl}(\mathbb{M})$. Since $\boldsymbol{n} \times \boldsymbol{\tau}_h \boldsymbol{n} = \boldsymbol{n} \times (\operatorname{curl}\boldsymbol{\sigma}_h)\boldsymbol{n}$ is single-value across each face $F \in \mathring{\mathcal{F}}_h$, we get $\boldsymbol{\tau}_h \in \Sigma_h^{\rm tn}$. In the distribution sense, curl div dev curl $=$ 0 and so is $(\operatorname{curl}\operatorname{div})_h \operatorname{dev}\operatorname{curl}$. In summary, we have verified (3.6) is a complex.

Verification of $\ker(\operatorname{grad}, \operatorname{mskw}\boldsymbol{x}) = \mathbb{R}^3 \times \{0\}$ is trivial. For $\boldsymbol{\tau}_h \in \mathbb{V}_{k,h}^{\rm curl}(\mathbb{M})$ and $\operatorname{dev}\operatorname{curl}(\boldsymbol{\tau}_h) = 0$, by the exactness of (2.3), we can find $(\boldsymbol{v}, c) \in H^1(\Omega;\mathbb{R}^3) \times \mathbb{R}$ s.t. $\operatorname{grad}\boldsymbol{v} + c\operatorname{mskw}\boldsymbol{x} = \boldsymbol{\tau}_h$. As $\boldsymbol{\tau}_h$ is polynomial of degree k, we conclude that $\boldsymbol{v} \in \mathbb{V}_{k+1,h}^{\rm grad}(\mathbb{R}^3)$.

From the finite element de Rham complex $\operatorname{grad}\mathring{\mathbb{V}}_{\ell+1,h}^{\rm grad} = \mathring{\mathbb{V}}_{(k,\ell),h}^{\rm curl} \cap \ker(\operatorname{curl})$, we have

$$\operatorname{div}_h \mathring{\mathbb{V}}_{(k,\ell),h}^{\rm curl} = \mathring{\mathbb{V}}_{\ell+1,h}^{\rm grad}. \tag{3.7}$$

It remains to prove

$$(\operatorname{curl}\operatorname{div})_h \Sigma_{k-1,h}^{\rm tn} = \mathring{\mathbb{V}}_{(k,\ell),h}^{\rm curl} \cap \ker(\operatorname{div}_h), \tag{3.8}$$

$$\operatorname{dev}\operatorname{curl}\mathbb{V}_{k,h}^{\rm curl}(\mathbb{M}) = \Sigma_{k-1,h}^{\rm tn} \cap \ker((\operatorname{curl}\operatorname{div})_h). \tag{3.9}$$

We will prove (3.8) in Corollary 3.12 and (3.9) in Corollary 3.13. ◻

3.3. Characterization of null spaces. Define $K_h^c = \mathring{\mathbb{V}}_{(k,\ell),h}^{\rm curl} \cap \ker(\operatorname{div}_h)$ and $(\operatorname{curl}\operatorname{curl})_h : K_h^c \to K_h^c$ so that

$$((\operatorname{curl}\operatorname{curl})_h \boldsymbol{u}_h, \boldsymbol{v}_h) = (\operatorname{curl}\boldsymbol{u}_h, \operatorname{curl}\boldsymbol{v}_h), \quad \boldsymbol{v}_h \in K_h^c. \tag{3.10}$$

LEMMA 3.7. *We have the discrete Poincaré inequalities*

$$\|\boldsymbol{v}_h\| \lesssim \|\operatorname{curl}\boldsymbol{v}_h\|, \qquad \boldsymbol{v}_h \in K_h^c, \tag{3.11}$$

$$\|\boldsymbol{v}_h\|_{H(\operatorname{curl})} \lesssim \|(\operatorname{curl}\operatorname{curl})_h \boldsymbol{v}_h\|, \quad \boldsymbol{v}_h \in K_h^c. \tag{3.12}$$

Proof. The first Poincaré inequality (3.11) can be found in [20, Lemma 3.4 and Theorem 3.6] and [33, Lemma 7.20]. Consequently, $(\operatorname{curl}\cdot, \operatorname{curl}\cdot)$ is an inner product on K_h^c and the operator $(\operatorname{curl}\operatorname{curl})_h$ is isomorphic.

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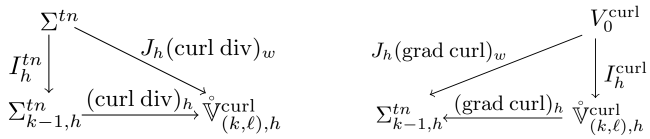

FIG. 3. *Identities connecting the weak differential operators and interpolation operators.*

Taking $\boldsymbol{u}_h = \boldsymbol{v}_h$ in (3.10) and applying (3.11), we get

$$\|\mathrm{curl}\,\boldsymbol{v}_h\|^2 \le \|(\mathrm{curl}\,\mathrm{curl})_h\boldsymbol{v}_h\|\|\boldsymbol{v}_h\| \lesssim \|(\mathrm{curl}\,\mathrm{curl})_h\boldsymbol{v}_h\|\|\mathrm{curl}\,\boldsymbol{v}_h\|.$$

Hence $\|\mathrm{curl}\,\boldsymbol{v}_h\| \lesssim \|(\mathrm{curl}\,\mathrm{curl})_h\boldsymbol{v}_h\|$. The proof is finished by applying (3.11) again to bound $\|\boldsymbol{v}_h\|$. ∎

We will introduce interpolation operators satisfying the commutative diagrams in Figure 3.

Let $I_h^{\mathrm{tn}} : \Sigma^{\mathrm{tn}} \to \Sigma_{k-1,h}^{\mathrm{tn}}$ be the interpolation operator using DoF (3.1) and denote by $\boldsymbol{\sigma}_I = I_h^{\mathrm{tn}}\boldsymbol{\sigma}$.

LEMMA 3.8. *For $\boldsymbol{\tau} \in \Sigma^{\mathrm{tn}}$ and $\boldsymbol{v}_h \in \mathring{\mathbb{V}}_{(k,\ell),h}^{\mathrm{curl}}$, it holds that*

$$((\mathrm{curl\ div})_h(I_h^{\mathrm{tn}}\boldsymbol{\tau}), \boldsymbol{v}_h) = \langle(\mathrm{curl\ div})_w\boldsymbol{\tau}, \boldsymbol{v}_h\rangle. \tag{3.13}$$

Proof. By definition,

$$\sum_{T\in\mathcal{T}_h} -(\boldsymbol{\tau} - I_h^{\mathrm{tn}}\boldsymbol{\tau}, \mathrm{grad}\,\mathrm{curl}\,\boldsymbol{v}_h)_T + (\boldsymbol{n}\times(\boldsymbol{\tau} - I_h^{\mathrm{tn}}\boldsymbol{\tau})\boldsymbol{n}, \boldsymbol{n}\times\mathrm{curl}\,\boldsymbol{v}_h)_{\partial T} = 0,$$

which is true due to DoFs (3.1a)–(3.1b) and the fact that $\mathrm{grad}\,\mathrm{curl}\,\boldsymbol{v}_h|_T \in \mathbb{P}_{k-2}(T;\mathbb{R}^3)$ and $\boldsymbol{n}\times\mathrm{curl}\,\boldsymbol{v}_h|_F \in \mathbb{P}_{k-1}(F;\mathbb{R}^2)$. ∎

We introduce the interpolation operator $I_h^{\mathrm{curl}} : V_0^{\mathrm{curl}} \to \mathring{\mathbb{V}}_{(k,\ell),h}^{\mathrm{curl}}$ defined by DoF (3.4) and denote by $\boldsymbol{v}_I = I_h^{\mathrm{curl}}\boldsymbol{v}$.

LEMMA 3.9. *For $\boldsymbol{v} \in V_0^{\mathrm{curl}}$ and $\boldsymbol{\tau}_h \in \Sigma_{k-1,h}^{\mathrm{tn}}$, it holds that*

$$\langle(\mathrm{curl\ div})_w\boldsymbol{\tau}_h, \boldsymbol{v}\rangle = ((\mathrm{curl\ div})_h\boldsymbol{\tau}_h, I_h^{\mathrm{curl}}\boldsymbol{v}). \tag{3.14}$$

Proof. By integration by parts and $\boldsymbol{\tau}_h\,|_T \in \mathbb{P}_{k-1}(T;\mathbb{T})$, we have

$$\begin{aligned}
&\langle(\mathrm{curl\ div})_w\boldsymbol{\tau}_h, \boldsymbol{v}\rangle \\
&\quad= \sum_{T\in\mathcal{T}_h}(\mathrm{curl\ div}\,\boldsymbol{\tau}_h, \boldsymbol{v})_T - \sum_{F\in\mathring{\mathcal{F}}_h}([\boldsymbol{n}\times\mathrm{div}\,\boldsymbol{\tau}_h + \mathrm{curl}_F(\boldsymbol{n}^\intercal\boldsymbol{\tau}_h\boldsymbol{n})], \boldsymbol{v})_F \\
&\qquad- \sum_{e\in\mathring{\mathcal{E}}_h}([\boldsymbol{n}^\intercal\boldsymbol{\tau}_h\boldsymbol{n}]_e, \boldsymbol{v}\cdot\boldsymbol{t}_e)_e \\
&\quad= \sum_{T\in\mathcal{T}_h}(\mathrm{curl\ div}\,\boldsymbol{\tau}_h, \boldsymbol{v}_I)_T - \sum_{F\in\mathring{\mathcal{F}}_h}([\boldsymbol{n}\times\mathrm{div}\,\boldsymbol{\tau}_h + \mathrm{curl}_F(\boldsymbol{n}^\intercal\boldsymbol{\tau}_h\boldsymbol{n})], \boldsymbol{v}_I)_F \\
&\qquad- \sum_{e\in\mathring{\mathcal{E}}_h}([\boldsymbol{n}^\intercal\boldsymbol{\tau}_h\boldsymbol{n}]_e, \boldsymbol{v}_I\cdot\boldsymbol{t}_e)_e = ((\mathrm{curl\ div})_h\boldsymbol{\tau}_h, \boldsymbol{v}_I),
\end{aligned}$$

where $[\boldsymbol{n}^\intercal\boldsymbol{\tau}_h\boldsymbol{n}]_e = \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}(T),e\subset\partial F}(\boldsymbol{n}^\intercal\boldsymbol{\tau}_h|_T\boldsymbol{n})|_e(\boldsymbol{t}_e\cdot\boldsymbol{t}_{F,e})$. ∎

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COROLLARY 3.10. *Let* $\boldsymbol{\tau}_h \in \Sigma^{\rm tn}_{k-1,h}$ *satisfy* $(\operatorname{curl} \operatorname{div})_h \boldsymbol{\tau}_h = 0$. *Then we have* $\operatorname{curl} \operatorname{div} \boldsymbol{\tau}_h = 0$ *and* $\boldsymbol{\tau}_h \in H(\operatorname{curl} \operatorname{div}, \Omega; \mathbb{T})$.

Proof. Let $\boldsymbol{\tau}_h \in \Sigma^{\rm tn}_h$ satisfy $(\operatorname{curl} \operatorname{div})_h \boldsymbol{\tau}_h = 0$. By Lemmas 3.9 and 2.3, curl div $\boldsymbol{\tau}_h = 0$ in the distribution sense. ∎

At first glance, the null space $\Sigma^{\rm tn}_{k-1,h} \cap \ker((\operatorname{curl} \operatorname{div})_h)$ is larger than the null space $\Sigma^{\rm tn}_{k-1,h} \cap \ker(\operatorname{curl} \operatorname{div})$ as the test function space is $\mathring{\mathbb{V}}^{\rm curl}_{(k,\ell),h}$ not C_0^∞. However, Lemma 3.9 implies that they are the same due to the design of finite element spaces and weak differential operators. This is in the same spirit of the Hellan–Herrmann–Johnson (HHJ) element [22, 23, 31, 11] in two dimensions.

LEMMA 3.11. *It holds that*

$$(\operatorname{curl} \operatorname{div})_h (I^{\rm tn}_h (\operatorname{mskw} \boldsymbol{u}_h)) = -(\operatorname{curl} \operatorname{curl})_h \boldsymbol{u}_h, \quad \boldsymbol{u}_h \in K^c_h. \tag{3.15}$$

Proof. Since $(\operatorname{mskw} \boldsymbol{u}_h)\boldsymbol{n} = \boldsymbol{u}_h \times \boldsymbol{n}$, it follows that $\operatorname{mskw} \boldsymbol{u}_h \in \Sigma^{\rm tn}$. By the fact that $\operatorname{div} \operatorname{mskw} \boldsymbol{u}_h = -\operatorname{curl} \boldsymbol{u}_h$, we have

$$\langle (\operatorname{curl} \operatorname{div})_w (\operatorname{mskw} \boldsymbol{u}_h), \boldsymbol{v_h} \rangle = (\operatorname{div} \operatorname{mskw} \boldsymbol{u}_h, \operatorname{curl} \boldsymbol{v_h}) = -(\operatorname{curl} \boldsymbol{u}_h, \operatorname{curl} \boldsymbol{v_h}).$$

Then the result holds from (3.13). ∎

With complex (3.6) and identity (3.15), we have the commutative diagram

$$\begin{array}{ccccccc} \mathbb{V}^{\rm curl}_{k,h}(\mathbb{M}) & \xrightarrow{\operatorname{dev} \operatorname{curl}} & \Sigma^{\rm tn}_{k-1,h} & \xrightarrow{(\operatorname{curl} \operatorname{div})_h} & K^c_h & \longrightarrow & 0 \\ & & & \nwarrow^{-I^{\rm tn}_h \operatorname{mskw}} & \uparrow {\scriptstyle (\operatorname{curl} \operatorname{curl})_h} & & \\ & & & & K^c_h . & & \end{array} \tag{3.16}$$

COROLLARY 3.12. *It holds that*

$$(\operatorname{curl} \operatorname{div})_h \Sigma^{\rm tn}_{k-1,h} = K^c_h = \mathring{\mathbb{V}}^{\rm curl}_{(k,\ell),h} \cap \ker(\operatorname{div}_h). \tag{3.17}$$

Proof. It is straightforward to verify that $(\operatorname{curl} \operatorname{div})_h \Sigma^{\rm tn}_h \subseteq K^c_h$. On the other side, take $\boldsymbol{w}_h \in K^c_h$. Let $\boldsymbol{u}_h = (\operatorname{curl} \operatorname{curl})^{-1}_h \boldsymbol{w}_h \in K^c_h$ and set $\boldsymbol{\tau}_h = -I^{\rm tn}_h (\operatorname{mskw} \boldsymbol{u}_h) \in \Sigma^{\rm tn}_h$. By (3.15),

$$(\operatorname{curl} \operatorname{div})_h \boldsymbol{\tau}_h = (\operatorname{curl} \operatorname{curl})_h \boldsymbol{u}_h = \boldsymbol{w}_h,$$

which ends the proof. ∎

COROLLARY 3.13. *We have*

$$\operatorname{dev} \operatorname{curl} \mathbb{V}^{\rm curl}_{k,h}(\mathbb{M}) = \Sigma^{\rm tn}_{k-1,h} \cap \ker((\operatorname{curl} \operatorname{div})_h). \tag{3.18}$$

Proof. By complex (3.6), $\operatorname{dev} \operatorname{curl} \mathbb{V}^{\rm curl}_{k,h}(\mathbb{M}) \subseteq \Sigma^{\rm tn}_{k-1,h} \cap \ker((\operatorname{curl} \operatorname{div})_h)$. Then we prove (3.18) by dimension count. By (3.7) and (3.17),

$$\begin{aligned} \dim \Sigma^{\rm tn}_{k-1,h} \cap \ker((\operatorname{curl} \operatorname{div})_h) &= \dim \Sigma^{\rm tn}_{k-1,h} - \dim \mathring{\mathbb{V}}^{\rm curl}_{(k,k-1),h} + \dim \mathring{\mathbb{V}}^{\rm grad}_{k,h} \\ &= |\mathring{\mathcal{V}}_h| - |\mathring{\mathcal{E}}_h| + 2\binom{k+1}{2}|\mathcal{F}_h| - \binom{k+1}{2}|\mathring{\mathcal{F}}_h| + |\mathring{\mathcal{F}}_h| \\ &\quad + |\mathcal{T}_h|\left(8\binom{k+1}{3} - 3\binom{k}{3} + \binom{k-1}{3}\right). \end{aligned}$$

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Similarly,

$$\begin{aligned}\dim \operatorname{dev}\operatorname{curl} \mathbb{V}_{k,h}^{\text{curl}}(\mathbb{M}) &= \dim \mathbb{V}_{k,h}^{\text{curl}}(\mathbb{M}) - \dim \mathbb{V}_{k+1,h}^{\text{grad}}(\mathbb{R}^3) + 2\\ &= -3|\mathcal{V}_h| + 3|\mathcal{E}_h| + 3\binom{k+1}{2}|\mathcal{F}_h| - 3|\mathcal{F}_h|\\ &\quad + |\mathcal{T}_h|\left(9\binom{k}{3} - 3\binom{k-1}{3}\right) + 2.\end{aligned}$$

Combine the last two identities to get

$$\begin{aligned}&\dim \Sigma_{k-1,h}^{\text{tn}} \cap \ker((\operatorname{curl}\operatorname{div})_h) - \dim \operatorname{dev}\operatorname{curl} \mathbb{V}_{k,h}^{\text{curl}}(\mathbb{M})\\ &\quad = |\mathring{\mathcal{V}}_h| + 3|\mathcal{V}_h| - |\mathring{\mathcal{E}}_h| - 3|\mathcal{E}_h| + |\mathring{\mathcal{F}}_h| + 3|\mathcal{F}_h| - 4|\mathcal{T}_h| - 2\\ &\qquad + \binom{k+1}{2}(4|\mathcal{T}_h| - |\mathcal{F}_h| - |\mathring{\mathcal{F}}_h|).\end{aligned}$$

Finally, we conclude the result from $4|\mathcal{T}_h| = |\mathcal{F}_h| + |\mathring{\mathcal{F}}_h|$, and the Euler's formulas $|\mathcal{V}_h| - |\mathcal{E}_h| + |\mathcal{F}_h| - |\mathcal{T}_h| = 1$ and $|\mathring{\mathcal{V}}_h| - |\mathring{\mathcal{E}}_h| + |\mathring{\mathcal{F}}_h| - |\mathcal{T}_h| = -1$. ∎

Remark 3.14. Here is another proof of (3.18). Take $\boldsymbol{\tau}_h \in \Sigma_{k-1,h}^{\text{tn}} \cap \ker((\operatorname{curl}\operatorname{div})_h)$. By Corollary 3.10, we have $\operatorname{curl}\operatorname{div}\boldsymbol{\tau}_h = 0$. Therefore, $\boldsymbol{\tau}_h = \operatorname{dev}\operatorname{curl}\boldsymbol{\sigma}$ with $\boldsymbol{\sigma} \in H^1(\Omega;\mathbb{M})$. Let $\boldsymbol{\sigma}_I \in \mathbb{V}_h^{\text{curl}}(\mathbb{M})$ be the interpolation of $\boldsymbol{\sigma}$ based on DoFs (3.4a)–(3.4c). The edge moment is not well defined for H^1 function but may be fixed by the fact that $\operatorname{dev}\operatorname{curl}\boldsymbol{\sigma} = \boldsymbol{\tau}_h$ has extra smoothness. It is easy to verify that all the DoFs (3.1a)–(3.1b) of $\boldsymbol{\tau}_h - \operatorname{dev}\operatorname{curl}\boldsymbol{\sigma}_I$ vanish. Therefore, $\boldsymbol{\tau}_h = \operatorname{dev}\operatorname{curl}\boldsymbol{\sigma}_I \in \operatorname{dev}\operatorname{curl} \mathbb{V}_h^{\text{curl}}(\mathbb{M})$.

3.4. Helmholtz decompositions. The right half of the distributional finite element curl div complex (3.6) is listed below

$$\mathbb{V}_{k,h}^{\text{curl}}(\mathbb{M}) \xrightarrow{\operatorname{dev}\operatorname{curl}} \Sigma_{k-1,h}^{\text{tn}} \xrightarrow{(\operatorname{curl}\operatorname{div})_h} \mathring{\mathbb{V}}_{(k,\ell),h}^{\text{curl}} \xrightarrow{\operatorname{div}_h} \mathring{\mathbb{V}}_{\ell+1,h}^{\text{grad}} \to 0, \quad \ell = k \text{ or } k-1.$$

By taking the dual, we have the short exact sequence

$$0 \leftarrow (\operatorname{grad}\operatorname{curl})_h K_h^c \xleftarrow{(\operatorname{grad}\operatorname{curl})_h} \mathring{\mathbb{V}}_{(k,\ell),h}^{\text{curl}} = K_h^c \oplus \operatorname{grad}(\mathring{\mathbb{V}}_{\ell+1,h}^{\text{grad}}) \xleftarrow{\operatorname{grad}} \mathring{\mathbb{V}}_{\ell+1,h}^{\text{grad}} \leftarrow 0.$$

By the framework in [12], we get the following Helmholtz decompositions from the last two complexes, commutative diagram (3.16) and (3.15).

COROLLARY 3.15. *We have the discrete Helmholtz decompositions*

$$\begin{aligned}\Sigma_{k-1,h}^{\text{tn}} &= \operatorname{dev}\operatorname{curl} \mathbb{V}_{k,h}^{\text{curl}}(\mathbb{M}) \oplus^{L^2} (\operatorname{grad}\operatorname{curl})_h K_h^c,\\ \Sigma_{k-1,h}^{\text{tn}} &= \operatorname{dev}\operatorname{curl} \mathbb{V}_{k,h}^{\text{curl}}(\mathbb{M}) \oplus I_h^{\text{tn}}(\operatorname{mskw} K_h^c).\end{aligned}$$

COROLLARY 3.16. *We have the L^2-orthogonal Helmholtz decomposition of space $\mathring{\mathbb{V}}_{(k,\ell),h}^{\text{curl}}$,*

$$\begin{aligned}\mathring{\mathbb{V}}_{(k,\ell),h}^{\text{curl}} &= K_h^c \oplus^{L^2} \operatorname{grad} \mathring{\mathbb{V}}_{\ell+1,h}^{\text{grad}} = (\operatorname{curl}\operatorname{curl})_h K_h^c \oplus^{L^2} \operatorname{grad} \mathring{\mathbb{V}}_{\ell+1,h}^{\text{grad}}\\ &= (\operatorname{curl}\operatorname{div})_h \Sigma_{k-1,h}^{\text{tn}} \oplus^{L^2} \operatorname{grad} \mathring{\mathbb{V}}_{\ell+1,h}^{\text{grad}}.\end{aligned} \tag{3.19}$$

LEMMA 3.17. *We have the discrete Poincaré inequality*

$$\|\boldsymbol{v}_h\|_{H(\operatorname{curl})} \lesssim \|(\operatorname{grad}\operatorname{curl})_h \boldsymbol{v}_h\|, \quad \boldsymbol{v}_h \in K_h^c. \tag{3.20}$$

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Proof. Set $\boldsymbol{\tau}_h = I_h^{\rm tn}(\operatorname{mskw} \boldsymbol{v}_h) \in \Sigma_{k-1,h}^{\rm tn}$. Then $(\operatorname{curl}\operatorname{div})_h \boldsymbol{\tau}_h = -(\operatorname{curl}\operatorname{curl})_h \boldsymbol{v}_h$ follows from (3.15). By the scaling argument, the inverse inequality, and the Poincaré inequality (3.11),

$$\|\boldsymbol{\tau}_h\| \lesssim \|\boldsymbol{v}_h\| \lesssim \|\operatorname{curl} \boldsymbol{v}_h\|. \tag{3.21}$$

It follows that

$$\|\operatorname{curl} \boldsymbol{v}_h\|^2 = ((\operatorname{curl}\operatorname{curl})_h \boldsymbol{v}_h, \boldsymbol{v}_h) = -((\operatorname{curl}\operatorname{div})_h \boldsymbol{\tau}_h, \boldsymbol{v}_h) = (\boldsymbol{\tau}_h, (\operatorname{grad}\operatorname{curl})_h \boldsymbol{v}_h).$$

Applying Cauchy–Schwarz inequality and (3.21), we obtain

$$\|\operatorname{curl} \boldsymbol{v}_h\| \lesssim \|(\operatorname{grad}\operatorname{curl})_h \boldsymbol{v}_h\|,$$

which implies (3.20). ∎

4. Mixed finite element method of the quad-curl problem. Let $\Omega \subset \mathbb{R}^3$ be a bounded polygonal domain. Consider the fourth order problem

$$\begin{cases} -\operatorname{curl} \Delta \operatorname{curl} \boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \operatorname{curl} \boldsymbol{u} \times \boldsymbol{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\boldsymbol{f} \in H^{-1}(\operatorname{div}, \Omega) \cap \ker(\operatorname{div})$ is known. Such a problem arises from multiphysics simulation such as modeling a magnetized plasma in magnetohydrodynamics [9].

4.1. Distributional mixed formulation. Introducing $\boldsymbol{\sigma} := \operatorname{grad}\operatorname{curl} \boldsymbol{u}$, we have $\operatorname{tr} \boldsymbol{\sigma} = \operatorname{tr} \operatorname{grad}\operatorname{curl} \boldsymbol{u} = \operatorname{div}\operatorname{curl} \boldsymbol{u} = 0$. Then rewrite problem (4.1) as the second-order system

$$\begin{cases} \boldsymbol{\sigma} - \operatorname{grad}\operatorname{curl} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \operatorname{curl}\operatorname{div} \boldsymbol{\sigma} = -\boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \operatorname{curl} \boldsymbol{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

A mixed formulation of the system (4.2) is to find $\boldsymbol{\sigma} \in H^{-1}(\operatorname{curl}\operatorname{div}, \Omega; \mathbb{T})$, $\boldsymbol{u} \in H_0(\operatorname{curl}, \Omega)$, and $\phi \in H_0^1(\Omega)$ such that

$$(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \psi; \boldsymbol{u}) = 0, \qquad \forall \boldsymbol{\tau} \in H^{-1}(\operatorname{curl}\operatorname{div}, \Omega; \mathbb{T}), \psi \in H_0^1(\Omega), \tag{4.3a}$$

$$b(\boldsymbol{\sigma}, \phi; \boldsymbol{v}) = -\langle \boldsymbol{f}, \boldsymbol{v} \rangle, \qquad \forall \boldsymbol{v} \in H_0(\operatorname{curl}, \Omega), \tag{4.3b}$$

where the bilinear form $b(\cdot, \cdot; \cdot) : (H^{-1}(\operatorname{curl}\operatorname{div}, \Omega; \mathbb{T}) \times H_0^1(\Omega)) \times H_0(\operatorname{curl}, \Omega)$ is defined by

$$b(\boldsymbol{\tau}, \psi; \boldsymbol{v}) := \langle \operatorname{curl}\operatorname{div} \boldsymbol{\tau}, \boldsymbol{v} \rangle + (\operatorname{grad} \psi, \boldsymbol{v}).$$

The term $(\operatorname{grad} \psi, \boldsymbol{u})$ is introduced to impose the divergence free condition $\operatorname{div} \boldsymbol{u} = 0$.

LEMMA 4.1. *For $\boldsymbol{v} \in H_0(\operatorname{curl}, \Omega)$, it holds that*

$$\|\boldsymbol{v}\|_{H(\operatorname{curl})} \lesssim \sup_{\boldsymbol{\tau} \in H^{-1}(\operatorname{curl}\operatorname{div}, \Omega; \mathbb{T}), \psi \in H_0^1(\Omega)} \frac{b(\boldsymbol{\tau}, \psi; \boldsymbol{v})}{\|\boldsymbol{\tau}\|_{H^{-1}(\operatorname{curl}\operatorname{div})} + |\psi|_1}. \tag{4.4}$$

Proof of this lemma is similar to, indeed simpler than, that of the discrete inf-sup condition (cf. Lemma 4.4), we thus skip the details here.

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LEMMA 4.2. *For* $\boldsymbol{\tau} \in H^{-1}(\operatorname{curl}\operatorname{div}, \Omega; \mathbb{T})$ *and* $\psi \in H_0^1(\Omega)$ *satisfying*

$$b(\boldsymbol{\tau}, \psi; \boldsymbol{v}) = 0 \quad \forall\ \boldsymbol{v} \in H_0(\operatorname{curl}, \Omega), \tag{4.5}$$

it holds that

$$\|\boldsymbol{\tau}\|_{H^{-1}(\operatorname{curl}\operatorname{div})}^2 + |\psi|_1^2 = \|\boldsymbol{\tau}\|^2. \tag{4.6}$$

Proof. By taking $\boldsymbol{v} = \operatorname{grad}\psi$ in (4.5), we get $\psi = 0$. Then (4.5) becomes

$$\langle \operatorname{curl}\operatorname{div}\boldsymbol{\tau}, \boldsymbol{v}\rangle = 0 \quad \forall\ \boldsymbol{v} \in H_0(\operatorname{curl}, \Omega).$$

Hence $\operatorname{curl}\operatorname{div}\boldsymbol{\tau} = 0$, and (4.6) follows. ∎

Combining (4.4), (4.6) and the Babuška–Brezzi theory [5] yields the well-posedness of the mixed formulation (4.3a)–(4.3b).

THEOREM 4.3. *The mixed formulation* (4.3a)–(4.3b) *is well-posed. Namely, for any* $\boldsymbol{f} \in H^{-1}(\operatorname{div}, \Omega) \cap \ker(\operatorname{div})$*, there exists a unique solution* $(\boldsymbol{\sigma}, \boldsymbol{u}, \phi)$ *to* (4.3a)–(4.3b)*. Furthermore, we have* $\phi = 0$*, and the stability*

$$\|\boldsymbol{\sigma}\|_{H^{-1}(\operatorname{curl}\operatorname{div})} + \|\boldsymbol{u}\|_{H(\operatorname{curl})} \lesssim \|\boldsymbol{f}\|_{H^{-1}(\operatorname{div})}. \tag{4.7}$$

Proof. Combine the inf-sup condition (4.4) and the coercivity (4.6) to get (4.7) and the well-posedness of the mixed formulation (4.3a)–(4.3b). By choosing $\boldsymbol{v} = \operatorname{grad}\phi$ in (4.3b), we get $\phi = 0$ from $\operatorname{div}\boldsymbol{f} = 0$. ∎

4.2. Distributional mixed finite element method. For $(\boldsymbol{\tau}, \psi) \in \Sigma^{\mathrm{tn}} \times H_0^1(\Omega)$ and $\boldsymbol{v} \in V_0^{\mathrm{curl}}$, introduce the bilinear form

$$b_h(\boldsymbol{\tau}, \psi; \boldsymbol{v}) := \langle (\operatorname{curl}\operatorname{div})_w \boldsymbol{\tau}, \boldsymbol{v}\rangle + (\operatorname{grad}\psi, \boldsymbol{v}).$$

By (3.5), we have for $(\boldsymbol{\tau}, \psi) \in \Sigma_{k-1,h}^{\mathrm{tn}} \times \mathring{\mathbb{V}}_{\ell+1,h}^{\mathrm{grad}}$ and $\boldsymbol{v} \in \mathring{\mathbb{V}}_{(k,\ell),h}^{\mathrm{curl}}$ that

$$b_h(\boldsymbol{\tau}, \psi; \boldsymbol{v}) = ((\operatorname{curl}\operatorname{div})_h \boldsymbol{\tau}, \boldsymbol{v}) + (\operatorname{grad}\psi, \boldsymbol{v}).$$

Then the distributional mixed finite element method finds $\boldsymbol{\sigma}_h \in \Sigma_h^{\mathrm{tn}}$, $\phi_h \in \mathring{\mathbb{V}}_h^{\mathrm{grad}}$, and $\boldsymbol{u}_h \in \mathring{\mathbb{V}}_h^{\mathrm{curl}}$ such that

$$(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h, \psi_h; \boldsymbol{u}_h) = 0, \qquad \forall\, \boldsymbol{\tau}_h \in \Sigma_h^{\mathrm{tn}}, \psi_h \in \mathring{\mathbb{V}}_h^{\mathrm{grad}}, \tag{4.8a}$$

$$b_h(\boldsymbol{\sigma}_h, \phi_h; \boldsymbol{v}_h) = -\langle \boldsymbol{f}, \boldsymbol{v}_h\rangle, \qquad \forall\, \boldsymbol{v}_h \in \mathring{\mathbb{V}}_h^{\mathrm{curl}}. \tag{4.8b}$$

We will derive two discrete inf-sup conditions for the linear form $b_h(\cdot, \cdot; \cdot)$. To this end, introduce some mesh dependent norms. For $\boldsymbol{\tau} \in \Sigma_h^{\mathrm{tn}}$, equip squared norm

$$\|\boldsymbol{\tau}\|_{H^{-1}((\operatorname{curl}\operatorname{div})_h)}^2 := \|\boldsymbol{\tau}\|^2 + \|(\operatorname{curl}\operatorname{div})_h \boldsymbol{\tau}\|_{H_h^{-1}(\operatorname{div})}^2,$$

where $\|\boldsymbol{v}\|_{H_h^{-1}(\operatorname{div})} := \sup_{\boldsymbol{w}_h \in \mathring{\mathbb{V}}_h^{\mathrm{curl}}} \frac{(\boldsymbol{v}, \boldsymbol{w}_h)}{\|\boldsymbol{w}_h\|_{H(\operatorname{curl})}}$. The continuity of the bilinear form

$$b_h(\boldsymbol{\tau}, \psi; \boldsymbol{v}) \le (\|\boldsymbol{\tau}\|_{H^{-1}((\operatorname{curl}\operatorname{div})_h)} + |\psi|_1)\|\boldsymbol{v}\|_{H(\operatorname{curl})},$$

for all $\boldsymbol{\tau} \in \Sigma^{\mathrm{tn}}, \psi \in \mathring{\mathbb{V}}_h^{\mathrm{grad}}, \boldsymbol{v} \in \mathring{\mathbb{V}}_h^{\mathrm{curl}}$, is straightforward by the definition of these norms.

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LEMMA 4.4. *For* $\boldsymbol{v}_h \in \mathring{\mathbb{V}}_h^{\mathrm{curl}}$, *it holds that*

$$\|\boldsymbol{v}_h\|_{H(\mathrm{curl})} \lesssim \sup_{\boldsymbol{\tau}_h \in \Sigma_h^{\mathrm{tn}}, \psi_h \in \mathring{\mathbb{V}}_h^{\mathrm{grad}}} \frac{b_h(\boldsymbol{\tau}_h, \psi_h; \boldsymbol{v}_h)}{\|\boldsymbol{\tau}_h\|_{H^{-1}((\mathrm{curl\,div})_h)} + |\psi_h|_1}. \tag{4.9}$$

Proof. By Helmholtz decomposition (3.19), given a $\boldsymbol{v}_h \in \mathring{\mathbb{V}}_h^{\mathrm{curl}}$, there exists $\boldsymbol{u}_h \in K_h^c$, $\tilde{\boldsymbol{v}}_h = (\mathrm{curl\,curl})_h \boldsymbol{u}_h$, and $\psi_h \in \mathring{\mathbb{V}}_{\ell+1,h}^{\mathrm{grad}}$ s.t.

$$\boldsymbol{v}_h = (\mathrm{curl\,curl})_h \boldsymbol{u}_h \oplus^{L_2} \mathrm{grad}\, \psi_h = \tilde{\boldsymbol{v}}_h \oplus^{L_2} \mathrm{grad}\, \psi_h. \tag{4.10}$$

Then

$$\mathrm{curl}\, \boldsymbol{v}_h = \mathrm{curl}\, \tilde{\boldsymbol{v}}_h, \quad \|\boldsymbol{v}_h\|^2 = \|(\mathrm{curl\,curl})_h \boldsymbol{u}_h\|^2 + |\psi_h|_1^2.$$

Set $\boldsymbol{\tau}_h = -I_h^{\mathrm{tn}} \mathrm{mskw}(\boldsymbol{u}_h + \tilde{\boldsymbol{v}}_h) \in \Sigma_{k-1,h}^{\mathrm{tn}}$. By (3.15),

$$((\mathrm{curl\,div})_h \boldsymbol{\tau}_h, \boldsymbol{v}_h) = ((\mathrm{curl\,curl})_h (\boldsymbol{u}_h + \tilde{\boldsymbol{v}}_h), \boldsymbol{v}_h) = \|(\mathrm{curl\,curl})_h \boldsymbol{u}_h\|^2 + \|\mathrm{curl}\, \boldsymbol{v}_h\|^2.$$

Consequently, $b_h(\boldsymbol{\tau}_h, \psi_h; \boldsymbol{v}_h) = \|\boldsymbol{v}_h\|^2 + \|\mathrm{curl}\, \boldsymbol{v}_h\|^2$.

It remains to control the norms. As the decomposition (4.10) is L^2-orthogonal, $|\psi_h|_1 \le \|\boldsymbol{v}_h\|$ and $\|(\mathrm{curl\,curl})_h \boldsymbol{u}_h\| \le \|\boldsymbol{v}_h\|$. We control the negative norm by

$$\begin{aligned}\|(\mathrm{curl\,div})_h \boldsymbol{\tau}_h\|_{H_h^{-1}(\mathrm{div})} &= \sup_{\boldsymbol{w}_h \in \mathring{\mathbb{V}}_h^{\mathrm{curl}}} \frac{(\mathrm{curl}\, \boldsymbol{u}_h + \mathrm{curl}\, \tilde{\boldsymbol{v}}_h, \mathrm{curl}\, \boldsymbol{w}_h)}{\|\boldsymbol{w}_h\|_{H(\mathrm{curl})}} \\ &\le \|\mathrm{curl}\, \boldsymbol{u}_h\| + \|\mathrm{curl}\, \boldsymbol{v}_h\| \\ &\lesssim \|(\mathrm{curl\,curl})_h \boldsymbol{u}_h\| + \|\mathrm{curl}\, \boldsymbol{v}_h\| \le \|\boldsymbol{v}_h\|_{H(\mathrm{curl})},\end{aligned}$$

where we have used the discrete Poincaré inequality (3.12).

By the scaling argument, the inverse inequality, and the Poincaré inequality (3.12),

$$\|\boldsymbol{\tau}_h\| \lesssim \|\boldsymbol{u}_h\| + \|\tilde{\boldsymbol{v}}_h\| \lesssim \|(\mathrm{curl\,curl})_h \boldsymbol{u}_h\| + \|\mathrm{curl}\, \boldsymbol{v}_h\| \le 2\|\boldsymbol{v}_h\|_{H(\mathrm{curl})},$$

as required. ∎

Introduce

$$\|\boldsymbol{v}_h\|_{H((\mathrm{grad\,curl})_h)}^2 := \|\boldsymbol{v}_h\|^2 + \|(\mathrm{grad\,curl})_h \boldsymbol{v}_h\|^2.$$

Again the continuity of the bilinear form in these norms

$$b_h(\boldsymbol{\tau}, \psi; \boldsymbol{v}) \lesssim (\|\boldsymbol{\tau}\| + |\psi|_1)\|\boldsymbol{v}\|_{H((\mathrm{grad\,curl})_h)}, \boldsymbol{\tau} \in \Sigma_h^{\mathrm{tn}}, \psi \in \mathring{\mathbb{V}}_h^{\mathrm{grad}}, \boldsymbol{v} \in \mathring{\mathbb{V}}_h^{\mathrm{curl}}$$

is straightforward by the definition of these norms.

LEMMA 4.5. *For* $\boldsymbol{v}_h \in \mathring{\mathbb{V}}_h^{\mathrm{curl}}$, *it holds that*

$$\|\boldsymbol{v}_h\|_{H((\mathrm{grad\,curl})_h)} \lesssim \sup_{\boldsymbol{\tau}_h \in \Sigma_h^{\mathrm{tn}}, \psi_h \in \mathring{\mathbb{V}}_h^{\mathrm{grad}}} \frac{b_h(\boldsymbol{\tau}_h, \psi_h; \boldsymbol{v}_h)}{\|\boldsymbol{\tau}_h\| + |\psi_h|_1}. \tag{4.11}$$

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Proof. We still use the Helmholtz decomposition (4.10) but choose

$$\boldsymbol{\tau}_h = -(\operatorname{grad}\operatorname{curl})_h \boldsymbol{v}_h = -(\operatorname{grad}\operatorname{curl})_h \tilde{\boldsymbol{v}}_h.$$

Then

$$b_h(\boldsymbol{\tau}_h, \psi_h; \boldsymbol{v}_h) = \|(\operatorname{grad}\operatorname{curl})_h \boldsymbol{v}_h\|^2 + |\psi_h|_1^2.$$

We end the proof by the estimates $\|\boldsymbol{\tau}_h\| + |\psi_h|_1 \lesssim \|\boldsymbol{v}_h\|_{H((\operatorname{grad}\operatorname{curl})_h)}$ and $\|\boldsymbol{v}_h\| \lesssim \|(\operatorname{grad}\operatorname{curl})_h \boldsymbol{v}_h\| + |\psi_h|_1$, in which we use the Poincaré inequality (3.20) for $\tilde{\boldsymbol{v}}_h$. ☐

There are other variants of mesh-dependent norms. For $\boldsymbol{\tau} \in \Sigma^{\mathrm{tn}}$, equip a mesh-dependent squared norm

$$\|\boldsymbol{\tau}\|_{0,h}^2 := \|\boldsymbol{\tau}\|^2 + \sum_{F \in \mathcal{F}_h} h_F \|\boldsymbol{n} \times \boldsymbol{\tau}\boldsymbol{n}\|_F^2.$$

By the inverse trace inequality, clearly we have $\|\boldsymbol{\tau}_h\|_{0,h} \eqsim \|\boldsymbol{\tau}_h\|$ for $\boldsymbol{\tau}_h \in \Sigma_h^{\mathrm{tn}}$. For piecewise smooth vector-valued function $\boldsymbol{v}$, equip a mesh-dependent squared norm

$$|\boldsymbol{v}|_{1,h}^2 := \sum_{T \in \mathcal{T}_h} \|\operatorname{grad} \boldsymbol{v}\|_T^2 + \sum_{F \in \mathcal{F}_h} h_F^{-1} \|[\![\boldsymbol{v}]\!]\|_F^2.$$

Then

$$b_h(\boldsymbol{\tau}, \psi; \boldsymbol{v}) \lesssim \|\boldsymbol{\tau}\|_{0,h} |\operatorname{curl} \boldsymbol{v}|_{1,h} + |\psi|_1 \|\boldsymbol{v}\|, \quad \boldsymbol{\tau} \in \Sigma^{\mathrm{tn}}, \psi \in H_0^1(\Omega), \boldsymbol{v} \in V_0^{\mathrm{curl}}.$$

One can prove the norm equivalence

$$\|(\operatorname{grad}\operatorname{curl})_h \boldsymbol{v}_h\| \eqsim |\operatorname{curl} \boldsymbol{v}_h|_{1,h}, \quad \boldsymbol{v}_h \in \mathring{\mathbb{V}}_h^{\mathrm{curl}} \tag{4.12}$$

and thus obtain the discrete inf-sup condition from (4.11)

$$\|\operatorname{curl} \boldsymbol{v}_h\|_{1,h} \lesssim \sup_{\boldsymbol{\tau}_h \in \Sigma_h^{\mathrm{tn}}, \psi \in \mathring{\mathbb{V}}_h^{\mathrm{grad}}} \frac{b_h(\boldsymbol{\tau}_h, \psi_h; \boldsymbol{v}_h)}{\|\boldsymbol{\tau}_h\|_{0,h} + |\psi_h|_1},$$

where $\|\operatorname{curl} \boldsymbol{v}_h\|_{1,h}^2 := \|\operatorname{curl} \boldsymbol{v}_h\|^2 + |\operatorname{curl} \boldsymbol{v}_h|_{1,h}^2$.

The discrete coercivity on the null space is similar to Lemma 4.2.

Lemma 4.6. *For* $\boldsymbol{\tau}_h \in \Sigma_h^{\mathrm{tn}}$ *and* $\psi_h \in \mathring{\mathbb{V}}_h^{\mathrm{grad}}$ *satisfying*

$$b_h(\boldsymbol{\tau}_h, \psi_h; \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \mathring{\mathbb{V}}_h^{\mathrm{curl}},$$

it holds that

$$\|\boldsymbol{\tau}_h\|_{H^{-1}((\operatorname{curl}\operatorname{div})_h)}^2 + |\psi_h|_1^2 = \|\boldsymbol{\tau}_h\|^2. \tag{4.13}$$

Applying the Babuška–Brezzi theory [5], from the discrete inf-sup conditions (4.9) and (4.11), and the discrete coercivity (4.13), we achieve the well-posedness of the mixed finite element method (4.8a)–(4.8b).

Theorem 4.7. *The distributional mixed finite element method* (4.8a)–(4.8b) *for the quad-curl problem is well-posed. We have the discrete stability results*

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$$\|\boldsymbol{\sigma}_h\|_{H^{-1}((\operatorname{curl}\operatorname{div})_h)} + |\phi_h|_1 + \|\boldsymbol{u}_h\|_{H(\operatorname{curl})} \lesssim \sup_{\boldsymbol{\tau}_h\in\Sigma_h^{\mathrm{tn}},\psi_h\in\mathring{\mathbb{V}}_h^{\mathrm{grad}},\boldsymbol{v}_h\in\mathring{\mathbb{V}}_h^{\mathrm{curl}}} \frac{A_h(\boldsymbol{\sigma}_h,\phi_h,\boldsymbol{u}_h;\boldsymbol{\tau}_h,\psi_h,\boldsymbol{v}_h)}{\|\boldsymbol{\tau}_h\|_{H^{-1}((\operatorname{curl}\operatorname{div})_h)} + |\psi_h|_1 + \|\boldsymbol{v}_h\|_{H(\operatorname{curl})}}, \tag{4.14}$$

$$\|\boldsymbol{\sigma}_h\| + |\phi_h|_1 + \|\boldsymbol{u}_h\|_{H((\operatorname{grad}\operatorname{curl})_h)} \lesssim \sup_{\boldsymbol{\tau}_h\in\Sigma_h^{\mathrm{tn}},\psi_h\in\mathring{\mathbb{V}}_h^{\mathrm{grad}},\boldsymbol{v}_h\in\mathring{\mathbb{V}}_h^{\mathrm{curl}}} \frac{A_h(\boldsymbol{\sigma}_h,\phi_h,\boldsymbol{u}_h;\boldsymbol{\tau}_h,\psi_h,\boldsymbol{v}_h)}{\|\boldsymbol{\tau}_h\| + |\psi_h|_1 + \|\boldsymbol{v}_h\|_{H((\operatorname{grad}\operatorname{curl})_h)}}, \tag{4.15}$$

for any $\boldsymbol{\sigma}_h\in\Sigma_h^{\mathrm{tn}}$, $\phi_h\in\mathring{\mathbb{V}}_h^{\mathrm{grad}}$, *and* $\boldsymbol{u}_h\in\mathring{\mathbb{V}}_h^{\mathrm{curl}}$, *where*

$$A_h(\boldsymbol{\sigma}_h,\phi_h,\boldsymbol{u}_h;\boldsymbol{\tau}_h,\psi_h,\boldsymbol{v}_h) := (\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h,\psi_h;\boldsymbol{u}_h) + b_h(\boldsymbol{\sigma}_h,\phi_h;\boldsymbol{v}_h).$$

By choosing $\boldsymbol{v}_h = \operatorname{grad}\phi_h$ in (4.8b), we get $\phi_h = 0$ from $\operatorname{div}\boldsymbol{f} = 0$.

4.3. Error analysis.

LEMMA 4.8. *Let* $(\boldsymbol{\sigma},0,\boldsymbol{u})$ *and* $(\boldsymbol{\sigma}_h,0,\boldsymbol{u}_h)$ *be the solution of the mixed formulation* (4.3a)–(4.3b) *and the mixed finite element method* (4.8a)–(4.8b)*, respectively. Assume* $\boldsymbol{\sigma}\in\Sigma^{\mathrm{tn}}$, *and* $\boldsymbol{u},\operatorname{curl}\boldsymbol{u}\in H^1(\Omega;\mathbb{R}^3)$. *Then*

$$A_h(I_h^{\mathrm{tn}}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,0,I_h^{\mathrm{curl}}\boldsymbol{u}-\boldsymbol{u}_h;\boldsymbol{\tau}_h,\psi_h,\boldsymbol{v}_h) = (I_h^{\mathrm{tn}}\boldsymbol{\sigma}-\boldsymbol{\sigma},\boldsymbol{\tau}_h) + (I_h^{\mathrm{curl}}\boldsymbol{u}-\boldsymbol{u},\operatorname{grad}\psi_h) \tag{4.16}$$

holds for any $\boldsymbol{\tau}_h\in\Sigma_h^{\mathrm{tn}}$, $\psi_h\in\mathring{\mathbb{V}}_h^{\mathrm{grad}}$, *and* $\boldsymbol{v}_h\in\mathring{\mathbb{V}}_h^{\mathrm{curl}}$.

Proof. Subtract (4.8a)–(4.8b) from (4.3a)–(4.3b) and use (3.13) and (3.14) to get error equations

$$(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h,\psi_h;I_h^{\mathrm{curl}}\boldsymbol{u}-\boldsymbol{u}_h) = (I_h^{\mathrm{curl}}\boldsymbol{u}-\boldsymbol{u},\operatorname{grad}\psi_h),$$
$$b_h(I_h^{\mathrm{tn}}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,0;\boldsymbol{v}_h) = 0.$$

Then subtract $(\boldsymbol{\sigma}-I_h^{\mathrm{tn}}\boldsymbol{\sigma},\boldsymbol{\tau}_h)$ to get (4.16). ∎

THEOREM 4.9. *Let* $(\boldsymbol{\sigma},0,\boldsymbol{u})$ *and* $(\boldsymbol{\sigma}_h,0,\boldsymbol{u}_h)$ *be the solution of the mixed formulation* (4.3a)–(4.3b) *and the mixed finite element method* (4.8a)–(4.8b)*, respectively. Assume* $\boldsymbol{\sigma}\in H^k(\Omega;\mathbb{T})$ *and* $\boldsymbol{u},\operatorname{curl}\boldsymbol{u}\in H^k(\Omega;\mathbb{R}^3)$. *Then*

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\| + \|I_h^{\mathrm{curl}}\boldsymbol{u}-\boldsymbol{u}_h\|_{H((\operatorname{grad}\operatorname{curl})_h)} \lesssim h^k(|\boldsymbol{\sigma}|_k + |\boldsymbol{u}|_k), \tag{4.17}$$

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{H(\operatorname{curl})} + h|\operatorname{curl}(\boldsymbol{u}-\boldsymbol{u}_h)|_{1,h} \lesssim h^k(|\boldsymbol{\sigma}|_k + |\boldsymbol{u}|_k + |\operatorname{curl}\boldsymbol{u}|_k). \tag{4.18}$$

Proof. It follows from the stability results (4.14)–(4.15) and (4.16) that

$$\|I_h^{\mathrm{tn}}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\| + \|I_h^{\mathrm{curl}}\boldsymbol{u}-\boldsymbol{u}_h\|_{H(\operatorname{curl})} + \|I_h^{\mathrm{curl}}\boldsymbol{u}-\boldsymbol{u}_h\|_{H((\operatorname{grad}\operatorname{curl})_h)} \lesssim \|\boldsymbol{\sigma}-I_h^{\mathrm{tn}}\boldsymbol{\sigma}\| + \|\boldsymbol{u}-I_h^{\mathrm{curl}}\boldsymbol{u}\|.$$

Hence (4.17)–(4.18) follow from the triangle inequality, the norm equivalence (4.12), and interpolation error estimates. ∎

We perform numerical experiments to support the theoretical results of the distributional mixed method (4.8a)–(4.8b). Let $\Omega = (0,1)^3$. Choose the function $\boldsymbol{f}$ in (4.1) such that the exact solution of (4.1) is

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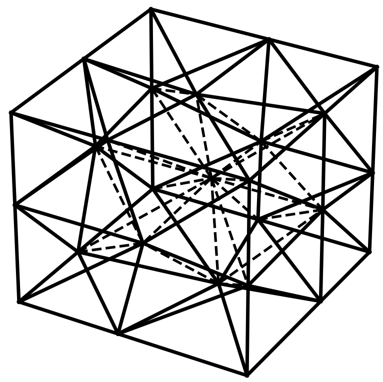

FIG. 4. *An initial perturbed mesh of the uniform mesh with $h = 1/2$.*

TABLE 1
Errors for the distributional mixed finite element method (4.8a)–(4.8b) *with $k = 1$.*

h	2^{-2}	2^{-3}	2^{-4}	2^{-5}
# DoFs for $\ell = 0$	2,332	17,240	132,400	1,037,408
# DoFs for $\ell = 1$	2,936	21,424	163,424	1,276,096
$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|$	1.51E + 02	8.62E + 01	4.48E + 01	2.26E + 01
order	–	0.81	0.95	0.99
$\|\mathrm{curl}\,(\boldsymbol{u} - \boldsymbol{u}_h)\|$	1.48E + 01	5.60E + 00	2.39E + 00	1.13E + 00
order	–	1.40	1.23	1.08
$\|\mathrm{grad}_{\mathcal{T}_h}\mathrm{curl}\,(\boldsymbol{u} - \boldsymbol{u}_h)\|$	1.46E + 02	1.46E + 02	1.46E + 02	1.46E + 02
order	–	0	0	0
$\|\boldsymbol{u} - \boldsymbol{u}_h\|$ for $\ell = 0$	1.49E + 00	5.23E-01	2.19E-01	1.03E-01
order	–	1.51	1.25	1.09
$\|\boldsymbol{u} - \boldsymbol{u}_h\|$ for $\ell = 1$	1.04E + 00	2.82E-01	7.32E-02	1.85E-02
order	–	1.88	1.95	1.98

$$\boldsymbol{u} = \mathrm{curl}\begin{pmatrix}\sin^3(\pi x)\sin^3(\pi y)\sin^3(\pi z)\\ \sin^3(\pi x)\sin^3(\pi y)\sin^3(\pi z)\\ 0\end{pmatrix},$$

and let $\boldsymbol{\sigma} := \mathrm{grad}\,\mathrm{curl}\,\boldsymbol{u}$. To break the symmetry, the initial unstructured mesh of Ω is shown in Figure 4, which is a perturbation of the uniform mesh with $h = 1/2$. Then we take uniform refinement of this initial unstructured triangulation.

We will implement the hybridized version of (4.8a)–(4.8b); see (5.1a)–(5.1b). Therefore, the number of DoFs is $\dim \mathring{\mathbb{V}}^{\mathrm{curl}}_{(k,\ell),h} + \dim \Lambda_{k-1,h}$, where $\Lambda_{k-1,h}$ is the space of Lagrange multiplier. Numerical results of $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|$, $\|\boldsymbol{u} - \boldsymbol{u}_h\|$, $\|\mathrm{curl}\,(\boldsymbol{u} - \boldsymbol{u}_h)\|$, and $\|\mathrm{grad}_{\mathcal{T}_h}\mathrm{curl}\,(\boldsymbol{u} - \boldsymbol{u}_h)\|$ of the distributional mixed method (4.8a)–(4.8b) with $k = 1$ and $\ell = 0, 1$ are shown in Table 1. It is observed that $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| = \mathcal{O}(h)$, and $\|\mathrm{curl}\,(\boldsymbol{u} - \boldsymbol{u}_h)\| = \mathcal{O}(h)$ numerically, which coincide with the theoretical error estimates in (4.17) and (4.18). The error $\|\boldsymbol{u} - \boldsymbol{u}_h\| = \mathcal{O}(h^{\ell+1})$ is also observed but not included in (4.17) and (4.18).

4.4. Postprocessing. We can additionally derive the superconvergence result of $\|\mathrm{curl}\,(I_h^{\mathrm{curl}}\boldsymbol{u} - \boldsymbol{u}_h)\| \lesssim h^{k+1}$ by using the duality argument [17, section 4.3]. Postprocessing can be also applied to improve the approximation. It follows from the standard procedure of the stable mixed methods and will be briefly reviewed below.

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TABLE 2
Errors for the postprocessing for $k = 1$.

h	2^{-2}	2^{-3}	2^{-4}	2^{-5}
# DoFs for $\ell = 0$	2,332	17,240	132,400	1,037,408
# DoFs for $\ell = 1$	2,936	21,424	163,424	1,276,096
$\lVert \operatorname{curl}_{\mathcal{T}_h}(\boldsymbol{u} - \boldsymbol{u}_h^*) \rVert$	9.25E + 00	2.71E + 00	7.18E-01	1.83E-01
order	–	1.77	1.92	1.98
$\lVert \operatorname{grad}_{\mathcal{T}_h} \operatorname{curl}_{\mathcal{T}_h}(\boldsymbol{u} - \boldsymbol{u}_h^*) \rVert$	1.52E + 02	8.62E + 01	4.48E + 01	2.26E + 01
order	–	0.81	0.95	0.99

We will construct a new superconvergent approximation to $\boldsymbol{u}$ in virtue of the optimal result of $\boldsymbol{\sigma}$ in (4.17). Introduce discrete space

$$\mathbb{V}_h^* := \{\boldsymbol{v}_h \in L^2(\Omega;\mathbb{R}^3) : \boldsymbol{v}_h|_T \in \mathbb{P}_k(T;\mathbb{R}^3) + \boldsymbol{x} \times \mathbb{P}_k(T;\mathbb{R}^3) \text{ for } T \in \mathcal{T}_h\}.$$

Let $\mathring{\mathbb{P}}_{k+1}(T)$ be the subspace of $\mathbb{P}_{k+1}(T)$ with vanishing funcition values at vertices of T. For each $T \in \mathcal{T}_h$, define the new approximation $\boldsymbol{u}_h^* \in \mathbb{V}_h^*$ to $\boldsymbol{u}$ piecewisely as a solution of the following problem:

$$(\boldsymbol{u}_h^* \cdot \boldsymbol{t}, q)_e = (\boldsymbol{u}_h \cdot \boldsymbol{t}, q)_e \quad \forall\, q \in \mathbb{P}_0(e), e \in \mathcal{E}(T), \tag{4.19a}$$

$$(\operatorname{grad}\operatorname{curl}\boldsymbol{u}_h^*, \boldsymbol{q})_T = (\boldsymbol{\sigma}_h, \boldsymbol{q})_T \quad \forall\, \boldsymbol{q} \in \operatorname{grad}\operatorname{curl}\mathbb{P}_{k+1}(T;\mathbb{R}^3), \tag{4.19b}$$

$$(\boldsymbol{u}_h^*, \boldsymbol{q})_T = (\boldsymbol{u}_h, \boldsymbol{q})_T \quad \forall\, \boldsymbol{q} \in \operatorname{grad}\mathring{\mathbb{P}}_{k+1}(T). \tag{4.19c}$$

It is easy to verify that the local problem (4.19a)–(4.19c) is well-posed. We can prove

$$\lVert \operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}_{\mathcal{T}_h}(\boldsymbol{u} - \boldsymbol{u}_h^*)\rVert \lesssim h^k(|\boldsymbol{\sigma}|_k + |\boldsymbol{u}|_k + |\operatorname{curl}\boldsymbol{u}|_{k+1}),$$

and from $\lVert \operatorname{curl}(I_h^{\mathrm{curl}}\boldsymbol{u} - \boldsymbol{u}_h)\rVert \lesssim h^{k+1}$ that

$$\lVert \operatorname{curl}_{\mathcal{T}_h}(\boldsymbol{u} - \boldsymbol{u}_h^*)\rVert \lesssim h^{k+1}(|\boldsymbol{\sigma}|_k + |\boldsymbol{u}|_k + |\operatorname{curl}\boldsymbol{u}|_{k+1} + \delta_{k1}\lVert \boldsymbol{f}\rVert).$$

Numerical results of the postprocessing $\boldsymbol{u}_h^*$ for $k = 1$ and $\ell = 0, 1$ are listed in Table 2. We can observe that $\lVert \operatorname{curl}_{\mathcal{T}_h}(\boldsymbol{u} - \boldsymbol{u}_h^*)\rVert = \mathcal{O}(h^2)$ and $\lVert \operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}_{\mathcal{T}_h}(\boldsymbol{u} - \boldsymbol{u}_h^*)\rVert = \mathcal{O}(h)$ numerically, which are one order higher than $\lVert \operatorname{curl}(\boldsymbol{u} - \boldsymbol{u}_h)\rVert$ and $\lVert \operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}(\boldsymbol{u} - \boldsymbol{u}_h)\rVert$, respectively. Indeed, $\lVert \operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}(\boldsymbol{u} - \boldsymbol{u}_h)\rVert = \lVert \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\rVert$.

5. Hybridization of distributional mixed finite element method. In this section we will hybridize the distributional mixed finite element method (4.8a)–(4.8b) following the framework in [1]. We introduce a Lagrange multiplier for the tangential-normal continuity of $\boldsymbol{\sigma}$ which can be treated as an approximation of $(\boldsymbol{n}_F \times \operatorname{curl}\boldsymbol{u})|_{\mathcal{F}_h}$. As $\boldsymbol{\sigma}$ is discontinuous, it can be eliminated elementwise and the size of the resulting linear system is reduced, which is easier to solve than the saddle point system obtained by the mixed method (4.8a)–(4.8b). With the hybridized method, we can also establish the equivalence of the mixed method (4.8a)–(4.8b) to a weak Galerkin method without stabilization and nonconforming finite element methods in [29, 45].

To this end, introduce two finite element spaces

$$\Sigma_{k-1,h}^{-1} := \{\boldsymbol{\tau}_h \in L^2(\Omega;\mathbb{T}) : \boldsymbol{\tau}_h|_T \in \mathbb{P}_{k-1}(T;\mathbb{T}) \text{ for each } T \in \mathcal{T}_h\},$$
$$\Lambda_{k-1,h} := \{\boldsymbol{\mu}_h \in L^2(\mathcal{F}_h;\mathbb{R}^3) : \boldsymbol{\mu}_h|_F \in \mathbb{P}_{k-1}(F;\mathbb{R}^3) \text{ and } \boldsymbol{\mu}_h \cdot \boldsymbol{n}|_F = 0 \text{ for each } F \in \mathring{\mathcal{F}}_h, \text{ and } \boldsymbol{\mu}_h = 0 \text{ on } \mathcal{F}_h \backslash \mathring{\mathcal{F}}_h\}.$$

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The space Λ_h is introduced as the Lagrange multiplier to impose the tangential-normal continuity and is equipped with squared norm

$$\|\boldsymbol{\mu}_h\|_{\alpha,h}^2 := \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}(T)} h_F^{-2\alpha}\|\boldsymbol{\mu}_h\|_F^2, \quad \alpha=\pm 1/2.$$

5.1. Hybridization. The hybridization of the mixed finite element method (4.8a)–(4.8b) is to find $(\boldsymbol{\sigma}_h,\boldsymbol{u}_h,\phi_h,\boldsymbol{\lambda}_h)\in\Sigma_{k-1,h}^{-1}\times\mathring{\mathbb{V}}_{(k,\ell),h}^{\text{curl}}\times\mathring{\mathbb{V}}_{\ell+1,h}^{\text{grad}}\times\Lambda_{k-1,h}$ such that

(5.1a)
$$(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h)+b_h(\boldsymbol{\tau}_h,\psi_h;\boldsymbol{u}_h)+c_h(\boldsymbol{\tau}_h,\boldsymbol{\lambda}_h)=0, \qquad \forall\,\boldsymbol{\tau}_h\in\Sigma_{k-1,h}^{-1},\psi_h\in\mathring{\mathbb{V}}_{\ell+1,h}^{\text{grad}},$$

(5.1b)
$$b_h(\boldsymbol{\sigma}_h,\phi_h;\boldsymbol{v}_h)+c_h(\boldsymbol{\sigma}_h,\boldsymbol{\mu}_h)=-\langle\boldsymbol{f},\boldsymbol{v}_h\rangle,\forall\,\boldsymbol{v}_h\in\mathring{\mathbb{V}}_{(k,\ell),h}^{\text{curl}},\boldsymbol{\mu}_h\in\Lambda_{k-1,h},$$

where the bilinear form $c_h(\boldsymbol{\tau}_h,\boldsymbol{\lambda}_h):=-\sum_{T\in\mathcal{T}_h}(\boldsymbol{n}\times\boldsymbol{\tau}_h\boldsymbol{n}_F,\boldsymbol{\lambda}_h)_{\partial T}$ is introduced to impose the tangential-normal continuity.

LEMMA 5.1. *There holds the following inf-sup condition:*

(5.2)
$$\begin{aligned}&\|\boldsymbol{v}_h\|_{H((\operatorname{grad}\operatorname{curl})_h)}+\|\boldsymbol{n}_F\times\operatorname{curl}\boldsymbol{v}_h-\boldsymbol{\mu}_h\|_{1/2,h}\\&\quad\lesssim\sup_{\boldsymbol{\tau}_h\in\Sigma_{k-1,h}^{-1},\psi_h\in\mathring{\mathbb{V}}_{\ell+1,h}^{\text{grad}}}\frac{b_h(\boldsymbol{\tau}_h,\psi_h;\boldsymbol{v}_h)+c_h(\boldsymbol{\tau}_h,\boldsymbol{\mu}_h)}{\|\boldsymbol{\tau}_h\|+|\psi_h|_1},\ \forall\,\boldsymbol{v}_h\in\mathring{\mathbb{V}}_{(k,\ell),h}^{\text{curl}},\boldsymbol{\mu}_h\in\Lambda_{k-1,h}.\end{aligned}$$

Proof. Let $\boldsymbol{\tau}_h\in\Sigma_{k-1,h}^{-1}$ be determined as follows: for $T\in\mathcal{T}_h$,

$$\begin{aligned}(\boldsymbol{n}\times\boldsymbol{\tau}_h\boldsymbol{n}_F)|_F&=\frac{1}{h_F}(\boldsymbol{n}_F\times\operatorname{curl}\boldsymbol{v}_h-\boldsymbol{\mu}_h), && F\in\mathcal{F}(T),\\(\boldsymbol{\tau}_h,\boldsymbol{q})_T&=-(\operatorname{grad}\operatorname{curl}\boldsymbol{v}_h,\boldsymbol{q})_T, && \boldsymbol{q}\in\mathbb{P}_{k-2}(T;\mathbb{T}).\end{aligned}$$

Clearly, we have

$$\begin{aligned}&\|\boldsymbol{\tau}_h\|\lesssim\|\operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}\boldsymbol{v}_h\|+\|\boldsymbol{n}_F\times\operatorname{curl}\boldsymbol{v}_h-\boldsymbol{\mu}_h\|_{1/2,h},\\&b_h(\boldsymbol{\tau}_h,0;\boldsymbol{v}_h)+c_h(\boldsymbol{\tau}_h,\boldsymbol{\mu}_h)=\|\operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}\boldsymbol{v}_h\|^2+\|\boldsymbol{n}_F\times\operatorname{curl}\boldsymbol{v}_h-\boldsymbol{\mu}_h\|_{1/2,h}^2.\end{aligned}$$

Then

(5.3)
$$\begin{aligned}&\|\operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}\boldsymbol{v}_h\|+\|\boldsymbol{n}_F\times\operatorname{curl}\boldsymbol{v}_h-\boldsymbol{\mu}_h\|_{1/2,h}\\&\quad\lesssim\sup_{\boldsymbol{\tau}_h\in\Sigma_{k-1,h}^{-1}}\frac{b_h(\boldsymbol{\tau}_h,0;\boldsymbol{v}_h)+c_h(\boldsymbol{\tau}_h,\boldsymbol{\mu}_h)}{\|\boldsymbol{\tau}_h\|},\end{aligned}$$

which together with (4.12) indicates

$$\begin{aligned}&\|(\operatorname{grad}\operatorname{curl})_h\boldsymbol{v}_h\|+\|\boldsymbol{n}_F\times\operatorname{curl}\boldsymbol{v}_h-\boldsymbol{\mu}_h\|_{1/2,h}\\&\quad\lesssim\sup_{\boldsymbol{\tau}_h\in\Sigma_h^{-1},\psi_h\in\mathring{\mathbb{V}}_h^{\text{grad}}}\frac{b_h(\boldsymbol{\tau}_h,\psi_h;\boldsymbol{v}_h)+c_h(\boldsymbol{\tau}_h,\boldsymbol{\mu}_h)}{\|\boldsymbol{\tau}_h\|+|\psi_h|_1}.\end{aligned}$$

Therefore, the inf-sup condition (5.2) holds from (4.11). ∎

THEOREM 5.2. *The hybridized mixed finite element method* (5.1a)–(5.1b) *is well-posed, and the solution* $(\boldsymbol{\sigma}_h,\boldsymbol{u}_h,\phi_h)\in\Sigma_h^{\text{tn}}\times\mathring{\mathbb{V}}_{(k,\ell),h}^{\text{curl}}\times\mathring{\mathbb{V}}_{\ell+1,h}^{\text{grad}}$ *coincides with the mixed finite element method* (4.8a)–(4.8b).

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Proof. For $\boldsymbol{\tau}_h \in \Sigma_h^{-1}$ and $\psi_h \in \mathring{\mathbb{V}}_h^{\mathrm{grad}}$ satisfying

$$b_h(\boldsymbol{\tau}_h, \psi_h; \boldsymbol{v}_h) + c_h(\boldsymbol{\tau}_h, \boldsymbol{\mu}_h) = 0, \quad \forall \boldsymbol{v}_h \in \mathring{\mathbb{V}}_h^{\mathrm{curl}}, \boldsymbol{\mu}_h \in \Lambda_h,$$

we have $\boldsymbol{\tau}_h \in \Sigma_h^{\mathrm{tn}}$, and $b_h(\boldsymbol{\tau}_h, \psi_h; \boldsymbol{v}_h) = 0$ for $\boldsymbol{v}_h \in \mathring{\mathbb{V}}_h^{\mathrm{curl}}$. By (4.13), we obtain the discrete coercivity $\|\boldsymbol{\tau}_h\|^2 + |\psi_h|_1^2 \lesssim \|\boldsymbol{\tau}_h\|^2$. Then we get the well-posedness of the hybridized mixed finite element method (5.1a)–(5.1b) by applying the Babuška–Brezzi theory [5] with the discrete inf-sup condition (5.2).

By (5.1b) with $\boldsymbol{v}_h = 0$, $\boldsymbol{\sigma}_h \in \Sigma_h^{\mathrm{tn}}$, thus $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \phi_h)$ satisfies (4.8a)–(4.8b). ∎

THEOREM 5.3. *Let $(\boldsymbol{\sigma}, 0, \boldsymbol{u})$ and $(\boldsymbol{\sigma}_h, 0, \boldsymbol{u}_h, \boldsymbol{\lambda}_h)$ be the solution of the mixed formulation* (4.3a)–(4.3b) *and the mixed finite element method* (5.1)*, respectively. Assume $\boldsymbol{\sigma} \in H^k(\Omega; \mathbb{T})$ and $\boldsymbol{u}, \mathrm{curl}\, \boldsymbol{u} \in H^k(\Omega; \mathbb{R}^3)$. Then*

$$\|\boldsymbol{n}_F \times \mathrm{curl}\, \boldsymbol{u}_h - \boldsymbol{\lambda}_h\|_{-1/2,h} \lesssim h^k (h|\boldsymbol{\sigma}|_k + h|\boldsymbol{u}|_k + |\mathrm{curl}\, \boldsymbol{u}|_k). \tag{5.4}$$

Proof. By the proof of the discrete inf-sup condition (5.2),

$$\begin{aligned}&\left\|\boldsymbol{n}_F \times \mathrm{curl}\,(I_h^{\mathrm{curl}} \boldsymbol{u} - \boldsymbol{u}_h) - (Q_{\mathcal{F}_h}^{k-1}(\boldsymbol{n}_F \times \mathrm{curl}\, \boldsymbol{u}) - \boldsymbol{\lambda}_h)\right\|_{1/2,h} \\ &\quad \lesssim \sup_{\boldsymbol{\tau}_h \in \Sigma_{k-1,h}^{-1}} \frac{b_h(\boldsymbol{\tau}_h, 0; I_h^{\mathrm{curl}} \boldsymbol{u} - \boldsymbol{u}_h) + c_h(\boldsymbol{\tau}_h, Q_{\mathcal{F}_h}^{k-1}(\boldsymbol{n}_F \times \mathrm{curl}\, \boldsymbol{u}) - \boldsymbol{\lambda}_h)}{\|\boldsymbol{\tau}_h\|}.\end{aligned}$$

Thanks to (3.14) and (5.1a), we have

$$b_h(\boldsymbol{\tau}_h, 0; I_h^{\mathrm{curl}} \boldsymbol{u} - \boldsymbol{u}_h) + c_h(\boldsymbol{\tau}_h, Q_{\mathcal{F}_h}^{k-1}(\boldsymbol{n}_F \times \mathrm{curl}\, \boldsymbol{u}) - \boldsymbol{\lambda}_h) = (\boldsymbol{\sigma}_h - \boldsymbol{\sigma}, \boldsymbol{\tau}_h).$$

Hence

$$\left\|\boldsymbol{n}_F \times \mathrm{curl}\,(I_h^{\mathrm{curl}} \boldsymbol{u} - \boldsymbol{u}_h) - (Q_{\mathcal{F}_h}^{k-1}(\boldsymbol{n}_F \times \mathrm{curl}\, \boldsymbol{u}) - \boldsymbol{\lambda}_h)\right\|_{1/2,h} \lesssim \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|.$$

Therefore, we can derive estimate (5.4) from (4.17) and the error estimate of I_h^{curl}. ∎

5.2. Connection to other methods. The pair $(\boldsymbol{v}_h, \boldsymbol{\mu}_h)$ can be understood as a weak function. Define $(\mathrm{grad}\,\mathrm{curl})_w : \mathring{\mathbb{V}}_h^{\mathrm{curl}} \times \Lambda_h \to \Sigma_h^{-1}$ by

$$((\mathrm{grad}\,\mathrm{curl})_w(\boldsymbol{v}_h, \boldsymbol{\mu}_h), \boldsymbol{\tau}_h) = -b_h(\boldsymbol{\tau}_h, 0; \boldsymbol{v}_h) - c_h(\boldsymbol{\tau}_h, \boldsymbol{\mu}_h), \quad \forall \boldsymbol{\tau}_h \in \Sigma_{k-1,h}^{-1}.$$

By (5.1a), we can eliminate $\boldsymbol{\sigma}_h$ elementwise and write $\boldsymbol{\sigma}_h = (\mathrm{grad}\,\mathrm{curl})_w(\boldsymbol{u}_h, \boldsymbol{\lambda}_h)$. Then the hybridized mixed finite element method (5.1a)–(5.1b) can be recast as follows: find $(\boldsymbol{u}_h, \lambda_h, \phi_h) \in \mathring{\mathbb{V}}_{(k,\ell),h}^{\mathrm{curl}} \times \Lambda_{k-1,h} \times \mathring{\mathbb{V}}_{\ell+1,h}^{\mathrm{grad}}$ such that

$$\begin{aligned}((\mathrm{grad}\,\mathrm{curl})_w(\boldsymbol{u}_h, \boldsymbol{\lambda}_h), (\mathrm{grad}\,\mathrm{curl})_w(\boldsymbol{v}_h, \boldsymbol{\mu}_h)) - (\boldsymbol{v}_h, \mathrm{grad}\, \phi_h) &= \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle, \\ (\boldsymbol{u}_h, \mathrm{grad}\, \psi_h) &= 0\end{aligned}$$

for all $(\boldsymbol{v}_h, \mu_h, \psi_h) \in \mathring{\mathbb{V}}_{(k,\ell),h}^{\mathrm{curl}} \times \Lambda_{k-1,h} \times \mathring{\mathbb{V}}_{\ell+1,h}^{\mathrm{grad}}$. That is, we obtain a stabilization free weak Galerkin method for solving the quad-curl problem. The stabilization free is due to the inf-sup condition (5.3). Indeed, the inf-sup condition (5.3) is equivalent to

$$\|\mathrm{grad}_{\mathcal{T}_h} \mathrm{curl}\, \boldsymbol{v}_h\| + \|\boldsymbol{n}_F \times \mathrm{curl}\, \boldsymbol{v}_h - \boldsymbol{\mu}_h\|_{1/2,h} \lesssim \|(\mathrm{grad}\,\mathrm{curl})_w(\boldsymbol{v}_h, \boldsymbol{\mu}_h)\|$$

for $\boldsymbol{v}_h \in \mathring{\mathbb{V}}_{(k,\ell),h}^{\mathrm{curl}}$ and $\boldsymbol{\mu}_h \in \Lambda_{k-1,h}$. This means $\|(\mathrm{grad}\,\mathrm{curl})_w(\boldsymbol{v}_h, \boldsymbol{\mu}_h)\|$ is a norm on space $K_h^c \times \Lambda_{k-1,h}$.

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For the hybridized mixed finite element method (5.1a)–(5.1b) of the lowest order $k=1$ and $\ell=0,1$, it is also related to a nonconforming finite element method.

We first recall the $H(\operatorname{grad}\operatorname{curl})$ nonconforming finite elements constructed in [29, 45]. The space of shape functions is $\operatorname{grad}\mathbb{P}_{\ell+1}(T)\oplus(\boldsymbol{x}-\boldsymbol{x}_T)\times\mathbb{P}_1\left(T;\mathbb{R}^3\right)$ for $\ell=0,1$. The degrees of freedom are given by

$$\int_e \boldsymbol{v}\cdot\boldsymbol{t}\,q\,\mathrm{d}s,\quad q\in\mathbb{P}_\ell(e)\text{ on each }e\in\mathcal{E}(T),\tag{5.5a}$$

$$\int_F(\operatorname{curl}\boldsymbol{v})\times\boldsymbol{n}\,\mathrm{d}S\quad\text{ on each }F\in\mathcal{F}(T).\tag{5.5b}$$

Define the global $H(\operatorname{grad}\operatorname{curl})$-nonconforming element space

$$\begin{aligned}W_h:=\{&\boldsymbol{v}_h\in L^2(\Omega;\mathbb{R}^3):\boldsymbol{v}_h|_T\in\operatorname{grad}\mathbb{P}_{\ell+1}(T)\oplus(\boldsymbol{x}-\boldsymbol{x}_T)\times\mathbb{P}_1\left(T;\mathbb{R}^3\right)\ \ \forall\,T\in\mathcal{T}_h,\\&\text{all the DoFs (5.5a)–(5.5b) are single-valued, and vanish on boundary}\}.\end{aligned}$$

The interpolation operator I_h^{curl} can be extended to W_h, which is well-defined. We have $I_h^{\mathrm{curl}}W_h=\mathring{\mathbb{V}}_h^{\mathrm{curl}}$, and (3.14) still holds for $\boldsymbol{v}\in W_h$.

An $H(\operatorname{grad}\operatorname{curl})$-nonconforming finite element method for quad-curl problem (4.1) is to find $\boldsymbol{w}_h\in W_h$ and $\phi_h\in\mathring{\mathbb{V}}_h^{\mathrm{grad}}$ such that

$$(\operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{w}_h,\operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{v}_h)-(I_h^{\mathrm{curl}}\boldsymbol{v}_h,\operatorname{grad}\phi_h)=\langle\boldsymbol{f},I_h^{\mathrm{curl}}\boldsymbol{v}_h\rangle,\tag{5.6a}$$

$$(I_h^{\mathrm{curl}}\boldsymbol{w}_h,\operatorname{grad}\psi_h)=0\tag{5.6b}$$

for all $\boldsymbol{v}_h\in W_h$ and $\psi_h\in\mathring{\mathbb{V}}_h^{\mathrm{grad}}$. Nonconforming finite element method (5.6a)–(5.6b) is a modification of those in [29, 45] by introducing interpolation operator I_h^{curl}. In other words, we identify the complex that accommodates the nonconforming finite elements constructed in [29, 45] and generalize these elements to arbitrary orders.

LEMMA 5.4. *For $\boldsymbol{v}_h\in W_h$ satisfying $(I_h^{\mathrm{curl}}\boldsymbol{v}_h,\operatorname{grad}q_h)=0$ for all $q_h\in\mathring{\mathbb{V}}_h^{\mathrm{grad}}$, we have the discrete Poincaré inequality*

$$\|\boldsymbol{v}_h\|+\|\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{v}_h\|\lesssim\|\operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{v}_h\|.\tag{5.7}$$

Proof. By (4.12) in [29], it follows that

$$\|\boldsymbol{v}_h-I_h^{\mathrm{curl}}\boldsymbol{v}_h\|\lesssim h\|\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{v}_h\|.$$

Thanks to the discrete Poincaré inequality for space $\mathring{\mathbb{V}}_h^{\mathrm{curl}}$, we have

$$\|I_h^{\mathrm{curl}}\boldsymbol{v}_h\|\lesssim\|\operatorname{curl}(I_h^{\mathrm{curl}}\boldsymbol{v}_h)\|.$$

Combining the last two inequalities gives

$$\|\boldsymbol{v}_h\|\le\|\boldsymbol{v}_h-I_h^{\mathrm{curl}}\boldsymbol{v}_h\|+\|I_h^{\mathrm{curl}}\boldsymbol{v}_h\|\lesssim h\|\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{v}_h\|+\|\operatorname{curl}(I_h^{\mathrm{curl}}\boldsymbol{v}_h)\|.$$

By the commutative property of I_h^{curl}, it holds that

$$\|\operatorname{curl}(I_h^{\mathrm{curl}}\boldsymbol{v}_h)\|\lesssim\|\operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{v}_h\|.$$

Hence

$$\|\boldsymbol{v}_h\|\lesssim\|\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{v}_h\|+\|\operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{v}_h\|.$$

Finally, apply the discrete Poincaré inequality for H^1-nonconforming linear element to end the proof. ∎

Using the discrete Poincaré inequality (5.7), we have the well-posedness.

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LEMMA 5.5. *Nonconforming finite element method* (5.6a)–(5.6b) *is well-posed.*

We will show the equivalence between the hybridized mixed finite element method (5.1a)–(5.1b) with $k=1$ and nonconforming finite element method (5.6a)–(5.6b).

THEOREM 5.6. *Let* $(\boldsymbol{w}_h, \phi_h) \in W_h \times \mathring{\mathbb{V}}_h^{\mathrm{grad}}$ *be the solution of the nonconforming finite element method* (5.6a)–(5.6b). *Then* $(\operatorname{grad}_{\mathcal{T}_h} \operatorname{curl}_{\mathcal{T}_h} \boldsymbol{w}_h, I_h^{\mathrm{curl}} \boldsymbol{w}_h, \phi_h, Q_{\mathcal{F}_h}(\boldsymbol{n}_F \times \operatorname{curl} \boldsymbol{w}_h)) \in \Sigma_{0,h}^{-1} \times \mathring{\mathbb{V}}_h^{\mathrm{curl}} \times \mathring{\mathbb{V}}_h^{\mathrm{grad}} \times \Lambda_h$ *is the solution of the hybridized mixed finite element method* (5.1a)–(5.1b) *with* $k=1$.

Proof. Choose $\boldsymbol{v}_h \in W_h$ such that DoF (5.5a) vanishes, then $I_h^{\mathrm{curl}} \boldsymbol{v}_h = 0$. Applying the integration by parts on the left-hand side of (5.6a), we get

$$\begin{aligned} 0 &= \sum_{T\in\mathcal{T}_h} ((\operatorname{grad}\operatorname{curl}\boldsymbol{w}_h)\boldsymbol{n}, \operatorname{curl}\boldsymbol{v}_h)_{\partial T} \\ &= \sum_{T\in\mathcal{T}_h} (\boldsymbol{n}\times(\operatorname{grad}\operatorname{curl}\boldsymbol{w}_h)\boldsymbol{n}, \boldsymbol{n}\times\operatorname{curl}\boldsymbol{v}_h)_{\partial T} \\ &= \sum_{F\in\mathring{\mathcal{F}}_h} ([\![\boldsymbol{n}\times(\operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{w}_h)\boldsymbol{n}]\!], \boldsymbol{n}_F\times\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{v}_h)_F. \end{aligned}$$

By the arbitrariness of the DoF (5.5b) for $\boldsymbol{v}_h$, we obtain $[\![\boldsymbol{n}\times(\operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{w}_h)\boldsymbol{n}]\!]_F = 0$ for all $F\in\mathring{\mathcal{F}}_h$, that is $\operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{w}_h \in \Sigma_h^{\mathrm{tn}}$. For all $\boldsymbol{v}_h\in W_h$ and $\boldsymbol{\mu}_h\in\Lambda_h$, we get from (3.14), the integration by parts, and (5.6a) that

$$\begin{aligned} & b_h(\operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{w}_h, \phi_h; I_h^{\mathrm{curl}}\boldsymbol{v}_h) + c_h(\operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{w}_h, \boldsymbol{\mu}_h) \\ &\quad = b_h(\operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{w}_h, \phi_h; I_h^{\mathrm{curl}}\boldsymbol{v}_h) \\ &\quad = b_h(\operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{w}_h, 0; \boldsymbol{v}_h) + (I_h^{\mathrm{curl}}\boldsymbol{v}_h, \operatorname{grad}\phi_h) \\ &\quad = -(\operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{w}_h, \operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{v}_h) + (I_h^{\mathrm{curl}}\boldsymbol{v}_h, \operatorname{grad}\phi_h) = -\langle \boldsymbol{f}, I_h^{\mathrm{curl}}\boldsymbol{v}_h\rangle. \end{aligned}$$

Notice that $I_h^{\mathrm{curl}}: W_h \to \mathring{\mathbb{V}}_h^{\mathrm{curl}}$ is onto, hence $(\operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{w}_h, \phi_h)$ satisfies (5.1b).

On the side hand, for all $\boldsymbol{\tau}_h\in\Sigma_{0,h}^{-1}$ and $\psi_h\in\mathring{\mathbb{V}}_h^{\mathrm{grad}}$, apply (3.14), (5.6b), and the integration by parts to get

$$\begin{aligned} & (\operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{w}_h, \boldsymbol{\tau}_h) + b_h\left(\boldsymbol{\tau}_h, \psi_h; I_h^{\mathrm{curl}}\boldsymbol{w}_h\right) + c_h(\boldsymbol{\tau}_h, Q_{\mathcal{F}_h}(\boldsymbol{n}_F\times\operatorname{curl}\boldsymbol{w}_h)) \\ &\quad = (\operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{w}_h, \boldsymbol{\tau}_h) + b_h\left(\boldsymbol{\tau}_h, 0; \boldsymbol{w}_h\right) + c_h(\boldsymbol{\tau}_h, \boldsymbol{n}_F\times\operatorname{curl}\boldsymbol{w}_h) \\ &\qquad + (I_h^{\mathrm{curl}}\boldsymbol{w}_h, \operatorname{grad}\psi_h) \\ &\quad = (\operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{w}_h, \boldsymbol{\tau}_h) + b_h\left(\boldsymbol{\tau}_h, 0; \boldsymbol{w}_h\right) - \sum_{T\in\mathcal{T}_h} (\boldsymbol{n}\times\boldsymbol{\tau}_h\boldsymbol{n}, \boldsymbol{n}\times\operatorname{curl}\boldsymbol{w}_h)_{\partial T} = 0. \end{aligned}$$

That is, $(\operatorname{grad}_{\mathcal{T}_h}\operatorname{curl}_{\mathcal{T}_h}\boldsymbol{w}_h, I_h^{\mathrm{curl}}\boldsymbol{w}_h, Q_{\mathcal{F}_h}(\boldsymbol{n}_F\times\operatorname{curl}\boldsymbol{w}_h))$ satisfies (5.1a). ∎

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