SCALAR AND MATRIX TAIL BOUND

LONG CHEN

ABSTRACT. We give an introduction on the tail bound of sum of random matrices.

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1. TAIL BOUND OF ONE RANDOM VARIABLE

We collect several inequalities on the tail bound of random variables. Taking expectation of the inequality
\[ \chi(\{X \geq a\}) \leq X/a, \]
we obtain the Markov’s inequality.

Markov’s inequality. Let \( X \) be a non-negative random variable, i.e., \( X \geq 0 \). Then, for any value \( a > 0 \),
\[ \Pr\{X \geq a\} \leq \frac{\E[X]}{a}. \]

Apply Markov’s inequality to the non-negative RV: \( (X - \E[X])^2 \) to get the Chebyshev’s inequality.

Chebyshev’s inequality. If \( X \) is a random variable with finite mean and variance, then, for any value \( a > 0 \),
\[ \Pr\{|X - \E[X]| \geq a\} \leq \frac{\Var(X)}{a^2}. \]

If we know more moment of \( X \), we can obtain more effective bounds. For example, if \( \E[X^r] \) is finite for a positive integer \( r \), then for any \( a > 0 \)

\[ (1) \quad \Pr\{X \geq a\} = \Pr\{X^r \geq a^r\} \leq \frac{\E[X^r]}{a^r}, \]

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a bound that falls off as \(1/a^r\). The larger the \(r\), the greater the rate is, given a bound on \(\mathbb{E}[X^r]\) is available. If we write the probability \(\bar{F}(a) := \Pr\{X > a\} = 1 - \Pr\{X \leq a\} = 1 - F(a)\), then the bound (1) tells how fast the function \(\bar{F}\) decays. The moments \(\mathbb{E}[X^r]\) is finite implies the p.d.f. \(f\) decays faster than \(1/x^{r+1}\) and \(\bar{F}\) decays at most with rate \(1/x^r\).

We can improve the bound to be strictly less than one.

**Chebyshev-Cantell inequality.** If \(\mathbb{E}[X] = \mu\) and \(\text{Var}(X) = \sigma^2\), then, for any \(a > 0\),

\[
\Pr\{X \geq \mu + a\} \leq \frac{\sigma^2}{\sigma^2 + a^2},
\]

\[
\Pr\{X \leq \mu - a\} \leq \frac{\sigma^2}{\sigma^2 + a^2}.
\]

**Proof.** Let \(b > 0\) and note that \(X \geq a\) is equivalent to \(X + b \geq a + b\). Hence

\[
\Pr\{X \geq a\} = \Pr\{X + b \geq a + b\} \leq \Pr\{(X + b)^2 \geq (a + b)^2\}.
\]

Upon applying Markov’s inequality, the preceding yields that

\[
\Pr\{X \geq a\} \leq \frac{\mathbb{E}[(X + b)^2]}{(a + b)^2} = \frac{\sigma^2 + b^2}{(a + b)^2}.
\]

Letting \(b = \sigma^2/a\), which minimizes the upper bound, gives the desired result. \(\square\)

When the moment generating function \(M_X(t) = \mathbb{E}[e^{tX}]\) is available (all moments are finite), we have the Chernoff bound which usually implies exponential decay of the tail.

**Chernoff bounds.**

\[
\Pr\{X \geq a\} \leq \inf_{t>0} e^{-ta} M_X(t),
\]

\[
\Pr\{X \leq a\} \leq \inf_{t<0} e^{-ta} M_X(t).
\]

A proof of the first inequality is as follows: for all \(t > 0\)

\[
\Pr\{X \geq a\} = \Pr\{e^{tX} \geq e^{ta}\} \leq e^{-ta} M_X(t).
\]

Taking the inf over all \(t > 0\), we get the Chernoff bounds. Note that the moment generating function \(M_X(t)\) might exist only for a bounded interval \(I\) containing \(0\). Then in the inf of (2), the \(t \in I^* = \{x \in I, x > 0\}\).

**Example 1.1** (Chernoff bounds for the standard normal distribution). Let \(X \sim N(0, 1)\) be the standard normal distribution. Then \(M(t) = e^{t^2/2}\). So the Chernoff bound is given by

\[
\Pr\{X \geq a\} \leq e^{-a^2/2} \quad \text{for all } t > 0.
\]

The minimum is achieved at \(t = a\) which gives the exponential decay tail bound

\[
\Pr\{X \geq a\} \leq e^{-a^2/2} \quad \text{for all } a > 0.
\]

For a Gaussian with variance \(\sigma^2\), i.e., \(Y \sim N(0, \sigma)\), we have \(Y = \sigma X\) with \(X \sim N(0, 1)\). Then a simple change of variable leads to the M-bound and T-bound

\[
M_Y(t) = e^{\sigma^2 t^2/2}, \quad \Pr\{Y \geq a\} \leq e^{-a^2/(2\sigma^2)}.
\]

In general, if \(M_X(t) \leq e^{Ct^2/2}\), which is called sub-Gaussian, for some constant \(C\) and for all \(t > 0\), then \(X\) has a sub-Gaussian upper tail \(\Pr\{X \geq a\} \leq e^{-a^2/(2C)}\) with the same constant \(C\). If the M-bound only holds for \(t \in (0, t_0)\), then the T-bound holds for \(a \in (0, Ct_0)\).
Example 1.2. Let $X$ be the random variable taking values $\pm 1$ with probability $1/2$. Then
\[ \mathbb{E} [e^{tX}] = \frac{1}{2} (e^t + e^{-t}) \leq e^{t^2/2}. \]

Lemma 1.3. Let $X$ be a random variable with $\mathbb{E}[X] = 0$ and $a \leq X \leq b$. Then $\text{Var}(X) \leq (b-a)^2/4$.

Proof. Using $\mathbb{E}[X] = 0$, we can prove the inequality
\[ \mathbb{E}[X^2] \leq \mathbb{E}[(x - (b + a)/2)^2]. \]
Then the bound of variance is obtained by $(x - (b + a)/2)^2 \leq (b-a)^2/4$ for $x \in [a, b]$.

Exercise 1.4. Prove the equality holds if and only if the random variable is Bernoulli with $p_a = p_b = 1/2$.

For a bounded random variable, we have the Hoeffding’s inequality.

Proposition 1.5 (Hoeffding’s inequality). Let $X$ be a random variable with $\mathbb{E}[X] = 0$ and $a \leq X \leq b$. Then for $t > 0$,
\[ \mathbb{E}[e^{tx}] \leq e^{t^2(b-a)^2/8}. \]

Proof. We use the convexity of the exponential function to get the inequality
\[ e^{tx} \leq w_1(x)e^{ta} + w_2(x)e^{tb} \]
with $w_1(x) = (x-a)/(b-a)$ and $w_2 = (b-x)/(b-a)$. Apply the expectation operator and notice that $\mathbb{E}[X] = 0$ to get
\[ \mathbb{E}[e^{tx}] \leq w_2(0)e^{tb} - w_1(0)e^{ta} = (1 - p + pe^{t(b-a)} - e^{-pt(b-a)}) = e^{\Phi(u)}, \]
where $p = -a/(b-a), u = t(b-a)$, and $\Phi(u) = -pu + \log(1 - p + pe^u)$. Now it is a calculus problem to show $\Phi(u) \leq t^2(b-a)^2/8$, e.g. by Taylor series. $\star$\ *

2. Tail bound of sum of random variables

Consider $n$-independent random variables. We want to pass properties of individual random variable to the sum. A sequence of random variables $X_1, X_2, \ldots, X_n$ have a uniform sub-Gaussian tail if all of them have sub-Gaussian tails with the same constant.

The tail bound can be easily derived from the M-bound. When passing the properties of each random variable to the summation, working on the $M$-bound is much easier.

Lemma 2.1. Let $X_1, \ldots, X_n$ be independent random variables satisfying $\mathbb{E}[X_i] = 0$, $\text{Var}(X_i) = 1$, and having a uniform sub-Gaussian $M$-bound. Let $\alpha_1, \ldots, \alpha_n$ be real coefficients satisfying $\sum_{i=1}^n \alpha_i^2 = 1$. Then the sum $Y = \sum_{i=1}^n \alpha_i X_i$ has $\mathbb{E}[Y] = 0$, $\text{Var}(Y) = 1$, and a sub-Gaussian $M$-bound.

Proof. By the linearity of expectation, we get $\mathbb{E}[Y] = 0$. For independent random variables, the variance is additive and thus $\text{Var}(Y) = \sum_{i=1}^n \alpha_i^2 \text{Var}(X_i) = \sum_{i=1}^n \alpha_i^2 = 1$.

With assumption, $M_{X_i}(t) \leq e^{Ct^2/2}$ for all $t > 0$ and all $i = 1, \ldots, n$, we have
\[ M_Y(t) = \mathbb{E} \left[ e^{tY} \right] = \mathbb{E} \left[ \prod_{i=1}^n e^{t\alpha_i X_i} \right] = \prod_{i=1}^n \mathbb{E} \left[ e^{t\alpha_i X_i} \right] \leq e^{\frac{1}{2}Ct^2 \sum_{i=1}^n \alpha_i^2} = e^{Ct^2/2}. \]

$\square$
Example 2.2 (The 2-stability of Gaussian distribution). Let $X_1, \ldots, X_n$ be i.i.d $N(0, 1)$. Let $\alpha_1, \ldots, \alpha_n$ be real coefficients satisfying $\sum_{i=1}^{n} \alpha_i^2 = 1$. Then $Y = \sum_{i=1}^{n} \alpha_i X_i$ is still the standard normal distribution, i.e., $Y \sim N(0, 1)$ since the moment generation function uniquely determines the random variable.

Theorem 2.3 (Hoeffding’s inequality). Let $X_1, \ldots, X_n$ be independent random variables with zero mean $\mathbb{E}[X_i] = 0$ and the bound $a_i \leq X_i \leq b_i$ for $i = 1, \ldots, n$. Let $\sigma_i^2 = \frac{1}{n} \sum_{i=1}^{n} (b_i - a_i)^2$. Then by the Hoeffding’s inequality for one random variables, we have, for any $\epsilon > 0$,

$$\Pr\left\{ \frac{1}{n} \sum_{i=1}^{n} X_i \geq \epsilon \right\} \leq \exp\left(-\frac{2n\epsilon^2}{\sigma^2}\right).$$

We can rewrite $\frac{1}{n} \sum_{i=1}^{n} X_i \geq \epsilon$ as $Y \geq \sqrt{n}\epsilon$ with $Y = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$. The variance

$$\operatorname{Var}(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i) = \frac{1}{n} \sum_{i=1}^{n} \operatorname{Var}(X_i) \leq \frac{1}{4n} \sum_{i=1}^{n} (b_i - a_i)^2 = \sigma^2 / 4.$$

Then the bound is consistent with the T-bound for Gaussian $\exp\left(-\frac{(\sqrt{n}\epsilon)^2}{2 \operatorname{Var}(Y)}\right)$.

If the variance is finite, we could improve to the Bernstein’s inequality. We refer to [3] for a proof; see also Theorem 5.5 in Section 5.

Theorem 2.4 (Bernstein’s inequality). Let $X_1, \ldots, X_n$ be independent random variables with zero mean $\mathbb{E}[X_i] = 0$ and uniform bound $|X_i| \leq M$ for $i = 1, \ldots, n$. Let $\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} \operatorname{Var}(X_i)$. Then for any $\epsilon > 0$,

$$\Pr\left\{ \frac{1}{n} \sum_{i=1}^{n} X_i \geq \epsilon \right\} \leq \exp\left(-\frac{n\epsilon^2}{2\sigma^2 + 2M\epsilon / 3}\right).$$

When $X_i \sim N(0, \sigma_i)$, the power in the bound is $-n\epsilon^2 / (2\sigma^2)$. For general independent random variables, by the central limit theorem, the bound holds with power $-n\epsilon^2 / (2\sigma^2)$, if $n$ is large enough. The Bernstein’s inequality is qualitatively right up to a term $2M\epsilon / 3$ which is small if $\epsilon \ll 1$. On the other hand, if $M\epsilon > 3\sigma^2$, then the bound behaves like $e^{-n\epsilon}$ which is an improvement of $e^{-n\epsilon^2}$. It is an improvement since now to get the probability less than $\delta \in (0, 1)$, we can chose smaller $n = C\epsilon^{-1} \ln(1/\delta)$ instead of $n = C\epsilon^{-2} \ln(1/\delta)$.

We also point out this improvement is only for bounded random variables which rules out the most popular Gaussian.

The independence of $X_i$ can relaxed. Consider a martingale.

Definition of martingale and A-H inequality.

Theorem 2.5 (Azuma-Hoeffding’s inequality). Suppose $S_n, n = 0, 1, 2 \ldots$ is a martingale and $|S_n - S_{n-1}| < c_n$ almost surely. Then for all positive integers $N$ and all $t > 0$

$$\Pr\{X_N - X_0 \geq t\} \leq \exp\left(-\frac{t^2}{2 \sum_{i=1}^{N} c_k^2}\right).$$

3. Trace Inequalities and Functions of Matrices

When generalizing properties of functions of single random variable to functions of random matrices, we face some difficulties due to the more complex structures of spaces of matrices:

1. no ordering: $A \leq B$ is meaningless;
2. non-commutative: $AB \neq BA$. 

We fix the ordering issue by restricting to a nice class of matrices. More precisely, denoted by:

- \( \mathcal{M}_n \) the space of \( n \times n \) matrices;
- \( \mathcal{H}_n \) the space of Hermitian \( n \times n \) matrices;
- \( \mathcal{H}_n^+ \) the set of positive semi-definite Hermitian matrices;
- \( \mathcal{H}_n^{++} \) the set of positive definite Hermitian matrices;
- \( S_n \) the set of density matrices i.e. \( \text{tr}(\rho) = 1 \) for \( \rho \in \mathcal{M}_n \).

We can introduce a partial ordering of \( \mathcal{H}_n \): for \( A, M \in \mathcal{H}_n \),

\[
A \leq M \iff x^T Ax \leq x^T M x, \forall x \in \mathbb{R}^n.
\]

Let \( A \) be a \( n \times n \) matrix. The trace of \( A \) is defined as:

\[
\text{tr}(A) = \sum_{i=1}^{n} a_{ii}.
\]

If \( A \) is an \( m \times n \) matrix and \( B \) is an \( n \times m \) matrix, then

\[
\text{tr}(AB) = \text{tr}(BA),
\]

which can be easily verified by direct computation. By taking trace, we gain the commutative property.

In summary, the matrix should be restricted to the space of Hermitian matrices and most generalization works for trace of matrix functions.

3.1. Trace. We follow Carlen [2] to provide some background on the trace inequalities involving functions of matrices.

**Lemma 3.1.** For any orthonormal basis \( \{u_1, u_2, \ldots, u_n\} \)

\[
\text{tr}(A) = \sum_{i=1}^{n} (u_j, Au_j).
\]

**Proof.** Let \( U = (u_1, u_2, \ldots, u_n) \) be the orthonormal matrix. Then by (5),

\[
\text{tr}(U^*AU) = \text{tr}(AUU^*) = \text{tr}(A).
\]

We can also prove it by direct computation

\[
\text{tr}(U^*AU) = \sum_{i=1}^{n} (U^*AU)_{ii} = \sum_{i=1}^{n} (u_i, Au_i).
\]

If we chose \( U \) formed by eigen-vectors of \( A \), then we obtain the intrinsic definition of the trace

\[
\text{tr}(A) = \sum_{j=1}^{n} \lambda_j,
\]

and thus is invariant with respect to a change of basis: for a non-singular matrix \( P \)

\[
\text{tr}(P^{-1}AP) = \text{tr}(APP^{-1}) = \text{tr}(A).
\]

The trace of a projection matrix is the dimension of the target space since the Jordan form will be a truncated identity.

The trace of a product can be rewritten as the sum of entry-wise products of elements:

\[
\text{tr}(X^T Y) = X : Y = \sum_{ij} X_{ij} Y_{ij} = \sum_{ij} (X \ast Y)_{ij} = \text{vec}(X)^T \text{vec}(Y).
\]
This means that the trace of a product of matrices functions similarly to a dot product of vectors. For this reason, generalizations of vector operations to matrices often involve a trace of matrix products. The norm induced by the above inner product is called the Frobenius norm.

Unlike the determinant, the trace of the product is not the product of traces. It is true, however, when apply to the tensor product:

\[
\text{tr}(X \otimes Y) = \text{tr}(X) \text{tr}(Y).
\]

### 3.2. Function of matrices

We now present definitions of function of matrices. Consider a scalar function \( f : \mathbb{R} \rightarrow \mathbb{R} \). For a diagonal matrix \( D = \text{diag}(d_1, \cdots, d_n) \), define \( f(D) := \text{diag}(f(d_1), \cdots, f(d_n)) \). For \( A \in \mathcal{H}_n \), let \( A = Q\Lambda Q^* \) be the eigen-decomposition of \( A \). We define

\[
f(A) := Qf(\Lambda)Q^*.
\]

If we write

\[
Q\Lambda Q^* = \sum_{i=1}^{n} \lambda_i Q(:, i)Q^(:, i) := \sum_{i=1}^{n} \lambda_i P_i
\]

with the rank-1 projection \( P_i = Q(:, i)Q^(:, i) \), an equivalent definition of matrix function is

\[
f(A) = \sum_{i=1}^{n} f(\lambda_i)P_i.
\]

Taking trace and using \( \text{tr}(P_i) = 1 \), we obtain

\[
\text{tr } f(A) = \sum_{i=1}^{n} f(\lambda_i).
\]

Two important matrix functions are exponential and logarithm functions of matrices. We have the power series expansion:

\[
\exp(A) = I + \sum_{k=1}^{\infty} \frac{A^k}{k!}.
\]

Inequalities of scalar functions can be generalized to the matrix version by the following transfer rule:

**Lemma 3.2.** If \( f(x) \leq g(x) \) for \( x \in I \), then for all \( A \in \mathcal{H}_n \) and \( \sigma(A) \subseteq I \), \( f(A) \leq g(A) \).

As a consequence, some inequalities of exponential functions can be generalized to exponential of matrices.

**Exercise 3.3.** Prove Lemma 3.2 and consequently, for \( A \in \mathcal{H}_n \), prove that

1. \( e^A \geq 0 \).
2. \( I + A \leq e^A \).
3. \( \cosh(A) \leq \exp(A^2/2) \).

\[ \square \]

The von Neuman entropy of \( \rho \in \mathcal{S}_n \), \( S(\rho) \) is defined by

\[
S(\rho) = -\text{tr}(\rho \log \rho).
\]

**Exercise 3.4.** Prove that, for \( \rho \in \mathcal{S}_n \),

\[
0 \leq S(\rho) \leq \log n.
\]

And give conditions when the equality holds.
A related function is
\[ A \to \log \text{tr}(e^A). \]
It can be shown that the function \( \log \text{tr}(e^A) \) is the Legendre transforms of \( S(A) \) if we extend the definition of \( S(A) = -\infty \) for \( A \not\in S_n \); see [2] Theorem 2.13.

We want to extend some properties of scalar functions to matrix functions. In the following we restrict matrices in \( \mathcal{H}_n \) with spectrum restricted to the domain of \( f \). The properties we concern are:

**Operator monotone.** A function \( f \) is operator monotone if the following holds
\[ A \geq B \implies f(A) \geq f(B). \]
Obviously if \( f \) is operator monotone, then \( f(x) \) is monotone. But not all monotone functions are operator monotone due to the non-commutative algebra structure of \( \mathcal{M}_n \).

**Operator convex.** A function \( f : I \to \mathbb{R} \) is operator convex if for all \( A, B \in \mathcal{H}_n \) and \( \sigma(A), \sigma(B) \subset I \) and for \( \theta \in (0, 1) \), the following holds
\[ f(\theta A + (1 - \theta)B) \leq \theta f(A) + (1 - \theta) f(B). \]
A function \( f \) is operator concave if \(-f\) is operator convex.

**Example 3.5** (\( A^2 \) is not operator monotone). The function \( f(x) = x^2 \) is monotone but for \( A, B \in \mathcal{H}_n^+ \),
\[ (A + B)^2 = A^2 + (AB + BA) + B^2. \]
Due to the non-commutative property of matrix product, the term \( AB + BA \) could have negative eigenvalue and thus \( (A + tB)^2 \geq A^2 \) could fail for sufficiently small \( t \). An example is \( A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \).

**Example 3.6** (\( A^2 \) is operator convex). With the parallelogram law
\[ \left( \frac{A + B}{2} \right)^2 + \left( \frac{A - B}{2} \right)^2 = \frac{1}{2} A^2 + \frac{1}{2} B^2, \]
we get the convexity for \( \theta = 1/2 \), which is known as midpoint convexity. From that, we can prove the convexity for all \( \theta \in (0, 1) \) by the continuity argument.

**Exercise 3.7** (\( A^{1/2} \) is operator monotone). Prove that if \( A, B \in \mathcal{H}_n^+ \) and \( A^2 \leq B^2 \), then \( A \leq B \).

**Example 3.8** (\( -A^{-1} \) is operator monotone and concave). The monotonicity can be proved by the following identity: for \( A, B \in \mathcal{H}_n^+ \)
\[ A^{-1} - (A + B)^{-1} = A^{-1/2} \left[ I - (I + C)^{-1} \right] A^{-1/2}, \]
with \( C = A^{-1/2} BA^{-1/2} \in \mathcal{H}_n^+ \).

The midpoint convexity can be proved by the identity:
\[ \frac{1}{2} A^{-1} + \frac{1}{2} B^{-1} - \left( \frac{A + B}{2} \right)^{-1} = A^{-1/2} \left[ \frac{1}{2} I + \frac{1}{2} C^{-1} - \left( \frac{I + C}{2} \right)^{-1} \right] A^{-1/2}, \]
and then the convexity of the function \( f(x) = 1/x \).

**Theorem 3.9** (Löwner-Heinz). On the function \( f(x) = x^p \), we have
- For \(-1 \leq p \leq 0\), the function \( f(A) = -A^p \) is operator concave and operator monotone.
• For $0 \leq p \leq 1$, the function $f(A) = A^p$ is operator concave and operator monotone.

• For $1 < p \leq 2$, the function $f(A) = A^p$ is operator convex but not operator monotone when $p$ is near 2.

In short for the power $p \in [-1, 1]$, the monotonicity and convexity/concavity is preserved for function $x^p$. When $p \in [1, 2]$, the convexity is still preserved. When $p > 2$, even the convexity could be missing.

An elementary proof of Löwner-Heinz Theorem given by Carlen [2] is based on the integral form of $A^p$. For example, for $p \in (-1, 0)$,

$$A^p = \frac{\sin((p + 1)\pi)}{\pi} \int_0^\infty t^p (tI + A)^{-1} dt. \tag{9}$$

The integral is a weighted sum of monotone and convex function $A \to (tI + A)^{-1}$ and thus preserves the monotonicity and convexity. The case for $1 < p < 2$ is different since it is related to the sum of two operator convex functions but the difference of two operator monotone functions.

**Exercise 3.10.**

(1) Prove (9) using a contour integral for scalar, i.e., $A = a$.

(2) Write out integral form for $A^p$ when $p \in (0, 1)$ and $p \in (1, 2)$.

**Corollary 3.11.**

(1) $\log(A)$ is operator concave and operator monotone.

(2) $A \log(A)$ is operator convex but not monotone.

**Proof.** We have the limit

$$\log(A) = \lim_{p \to 0} \frac{A^p - I}{p}$$

and $A^p$ is nice for $p$ near 0. Thus the properties pass to the limit.

The other limit is

$$A \log(A) = \lim_{p \to 1} \frac{A^p - A}{p - 1}$$

and the convexity holds for $p$ near 1 but not the monotonicity.

We emphasize that the function $\exp(A)$ and $A^p$ for $p > 2$ are neither operator monotone nor operator convex.

3.3. **Trace inequalities.** A trivial inequality connecting trace and max eigenvalues are:

$$\lambda_{\max}(A) \leq \text{tr } A \leq d\lambda_{\max}(A).$$

for all $A \in \mathcal{H}_n$.

Monotone and convex operators are very rare. Taking trace of matrix functions, however, we can preserve the convexity and monotonicity property.

**Theorem 3.12.** The trace of a matrix function will preserve the convexity and monotonicity property. Namely

- if $f$ is monotone increasing, so is $\text{tr } f(A)$ on $\mathcal{H}_n$.
- if $f$ is convex (or concave), so is $\text{tr } f(A)$ on $\mathcal{H}_n$.

**Proof.** Let $A, B \in \mathcal{H}_n$ and $C = A - B > 0$. We first consider the differentiable monotone function $f$ with $f' > 0$. Then **2**
\[
\text{tr}(f(A)) - \text{tr}(f(B)) = \int_0^1 \frac{d}{dt} \text{tr}(f(B + t(A - B))) \, dt
= \int_0^1 \text{tr}(C^{1/2} f'(B + tC) C^{1/2}) \, dt \geq 0.
\]

Using the continuity argument, the differentiability can be relaxed to monotone only.

To prove the convexity, we need the following Peierls inequality: for any orthonormal basis \( \{ u_1, \ldots, u_n \} \) and for a convex function \( f \)
\[
\sum_{i=1}^n f((u_i, Au_i)) \leq \text{tr}(f(A)).
\]

And the equality holds when each \( u_i \) is an eigenvector of \( A \). Note that we can write the right hand side
\[
\text{tr}(f(A)) = \sum_{i=1}^n f(\lambda_i) = \sum_{i=1}^n f((v_i, Av_i))
\]
with \( (\lambda_i, v_i) \) being the eigenv-pair of \( A \). This verifies the equality.

To prove inequality (10), we write \( A = \sum_{k=1}^n \lambda_k P_k \) and thus
\[
(u_i, Au_i) = \sum_{k=1}^n \lambda_k (u_i, P_k u_i) = \sum_{k=1}^n \lambda_k \| P_k u_i \|^2.
\]
As the nonnegative weight \( w_k = \| P_k u_i \|^2 \) satisfies \( \sum_k w_k = \| u_i \|^2 = 1 \), we use the convexity of \( f \) and the definition of \( f(A) \) to conclude that
\[
f \left( \sum_{k=1}^n w_k \lambda_k \right) \leq \sum_{k=1}^n \| P_k u_i \|^2 f(\lambda_k) = \sum_{k=1}^n (u_i, f(\lambda_k) P_k u_i) = (u_i, f(A) u_i).
\]
Sum over index \( i \) and use the identity for \( \text{tr}(A) \), c.f. Lemma 3.1, we finish the proof. □

**Theorem 3.13** (Golden-Thompson inequality). For any \( A, B \in \mathcal{H}_n \),
\[
\text{tr}(e^{A+B}) \leq \text{tr}(e^A e^B).
\]

An elementary proof is given by Oliveira in [4]; see also Terrance Tao [?].

**Theorem 3.14** (Lieb). For a fixed Hermitian matrix \( L \in \mathcal{H}_n \), the function \( f(A) = \text{tr}(\exp(L + \log A)) \) is concave on \( \mathcal{H}_n^+ \).

3.4. **Functions of random matrices.** Let \( X \) be a random matrix in \( \mathcal{H}_n \). We define the moment generating function (mgf) and the cumulant-generating function (cgf):
\[
M_X(t) := \mathbb{E} \left[ e^{tX} \right], \quad \text{and} \quad \Xi(t) := \log \mathbb{E} \left[ e^{tX} \right],
\]
for \( t \) in an open interval containing zero. They admit a formal power series expansions:
\[
M_X(t) = I + \sum_{k=1}^\infty \frac{t^k}{k!} \mathbb{E}[X^k],
\]
\[
\Xi_X(t) = \sum_{k=1}^\infty \frac{t^k}{k!} \Psi_k.
\]
The coefficients \( \mathbb{E}[X^k] \) are called *matrix moment* and \( \Psi_k \) as a *matrix cumulant*. In particular, \( \Psi_1 = \mathbb{E}[X] \) is the mean and \( \Psi_2 = \mathbb{E}[X^2] - \mathbb{E}^2[X] \) is the variance.

The expectation operator \( \mathbb{E} \) is exchangeable to a linear operator especially
\[
\mathbb{E}[\text{tr}(X)] = \text{tr} \mathbb{E}[X].
\]
For an operator convex function $f$, as the expectation is a weighted average, we have the Jensen’s inequality

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)].$$

In particular, as $f(x) = x^2$ is operator convex,

$$\mathbb{E}^2[X] \leq \mathbb{E}[X^2].$$

And the expectation preserves the partial ordering in $\mathcal{H}_n$, i.e.

$$\mathbb{E}[A] \leq \mathbb{E}[B] \quad \text{if} \quad A \preceq B.$$

### 4. Tail Bound and M-Bound of One Random Matrix

We provide matrix version of various tail bounds.

**Lemma 4.1 (Matrix Markov inequality).** Let $X$ be a random matrix in $\mathcal{H}_n^+$. Then for all deterministic matrix $A \in \mathcal{H}_n^{++}$, we have

$$\Pr\{X \preceq A\} \leq \text{tr}(A^{-1}\mathbb{E}[X]).$$

**Proof.** We consider the matrix $A^{-1}X$ which is Hermitian w.r.t the $(\cdot, \cdot)_A$ inner product. $X \preceq A$ is equivalent to $A^{-1}X \preceq_A I$ which implies that $\|A^{-1}X\| = \lambda_{\max}(A^{-1}X) > 1$. Then we obtain the inequality

$$\chi\{X \preceq A\} \leq \lambda_{\max}(A^{-1}X) \leq \text{tr}(A^{-1}X).$$

Taking the expectation and using the fact $\mathbb{E}$ is linear, we obtain the desired inequality. $\square$

**Exercise 4.2.** Use the fact $f(x) = x^{1/2}$ is operator monotone to derive a matrix Chebyshev inequality.

We now present a matrix Chernoff bound established by Oliverira [4].

**Theorem 4.3 (Matrix Chernoff bound).** Let $X$ be a random matrix in $\mathcal{H}_n$. For all $a \in \mathbb{R}$,

$$\Pr\{\lambda_{\max}(X) \geq a\} \leq \inf_{t > 0} e^{-ta} \text{tr} \mathbb{E}[e^{tX}].$$

**Proof.** By the scalar Chernoff bound, we obtain

$$\Pr\{\lambda_{\max}(X) \geq a\} \leq \inf_{t > 0} e^{-ta} \mathbb{E}[e^{t\lambda_{\max}(X)}].$$

Use

$$e^{t\lambda_{\max}(X)} = \lambda_{\max}(e^{tX}) \leq \text{tr}(e^{tX})$$

and $\text{tr} \mathbb{E} = \mathbb{E} \text{tr}$ to get the desired inequality. $\square$

We then try to generalize the M-bound $\mathbb{E}[e^{tX}] \leq e^{C\sigma^2 t^2}$ for a random variable to a random matrix. One possible form is:

$$\mathbb{E}[e^{tX}] \leq e^{g(t)A}$$

with a deterministic matrix $A \in \mathcal{H}_n$.

The following bound of mgf is important for the Bernstein T-bound.

**Lemma 4.4.** Let $X$ be a random matrix in $\mathcal{H}_n$ and

$$\mathbb{E}[X] = 0 \quad \text{and} \quad \lambda_{\max}(X) \leq 1.$$

Then for all $t > 0$

$$\mathbb{E}[e^{tX}] \leq \exp\left((e^t - t - 1)\mathbb{E}[X^2]\right).$$
Proof. We can prove the scalar inequality
\[ e^{tx} \leq 1 + tx + (e^t - t - 1)x^2, \quad \forall x \leq 1, \]
by power series of exponential functions (consider \(|x| \leq 1\) and \(x < -1\) separately). We then transfer to the matrix inequality by Lemma 3.2
\[ e^{tX} \leq I + tX + (e^t - t - 1)X^2. \]
Taking expectation and using the inequality \(I + A \leq e^A\) to get the desired inequality. □

5. Tail Bound of Sum of Random Matrices

In this section, we follow Tropp [5] to present tail bounds of sums of random matrices. In the scalar case, we use the fact
\[ E[e^{t \sum_i X_i}] = \prod_i E[e^{tX_i}] \]
to get the additivity of independent sub-Gaussians. For matrices, first of all,
\[ e^{A+B} \neq e^A e^B \]
due to the non-commutative algebra structure. It can be shown that in (13) the equality holds if and only if \(A, B\) commutes, i.e. \(AB = BA\).

Taking trace, we could get a desired inequality (using Golden-Thompson inequality, c.f. Theorem ??) for two independent random matrices
\[ E[tr(e^{X_1+X_2})] \leq tr(e^{X_1} e^{X_2}) = tr(E[e^{X_1}] E[e^{X_2}]). \]
Unfortunately, the above inequality cannot be generalized to three or more matrices.

The route Ahlswede and Winter [1] take is to recursively apply inequality (14) and the inequality \(tr(AB) \leq tr(A)\lambda_{\text{max}}(B)\) for \(A, B \in \mathcal{H}_n^+\) and end up with
\[ E[tr(e^{\sum_k X_k})] \leq d \exp\left(\sum_k \lambda_{\text{max}}(\log E[e^{X_k}])\right). \]

We shall follow Tropp [5] to present a sharper result with upper bound involving a smaller quantity \(\lambda_{\text{max}}(\sum_k \log E[e^{X_k}])\). We shall use the subadditivity of Matrix cgf’s
\[ \Xi_{\sum X_k}(t) \leq \sum_k \Xi_{X_k}(t). \]
To do so, we use a random matrix version of Lieb’s concave result, c.f. Theorem 3.14.

Lemma 5.1. Let \(L \in \mathcal{H}_n\) be a fixed Hermitian matrix and let \(X\) be a random matrix in \(\mathcal{H}_n\). Then
\[ E[tr \exp(L + X)] \leq tr \exp(L + \log E[e^X]). \]
Proof. Let \(A = e^X\). By Lieb’s concave Theorem 3.14, \(A \rightarrow tr \exp(L + \log A)\) is concave. Then apply Jensen’s inequality to the expectation to get the desired result. □

Lemma 5.2 (Subadditivity of Matrix cgf’s). Let \(\{X_k\}\) be a sequence of independent random matrices in \(\mathcal{H}_d\). Then
\[ E\left[\operatorname{tr} \exp\left(\sum_k X_k\right)\right] \leq \operatorname{tr} \exp\left(\sum_k \log E[e^{X_k}]\right). \]
Proof. Tropp’s Lemma 3.4. □
Theorem 5.3 (Master Tail Bound). Let \( \{X_k\} \) be a sequence of independent random matrices in \( \mathcal{H}_d \). For all \( a \in \mathbb{R} \),
\[
\Pr\{\lambda_{\max}(X) \geq a\} \leq \inf_{t > 0} e^{-ta} \text{tr} \exp \left( \sum_{k=1}^{n} \log \mathbb{E} \left[ e^{tX_k} \right] \right).
\]

Proof. It is a combination of matrix Chernoff bound, c.f. Theorem 4.3 and the sub-additivity of matrix cgf. \(\square\)

Then we combine bounds of mgf/cgf to a Chernoff bound.

Corollary 5.4. Let \( \{X_k\} \) be a sequence of independent random matrices in \( \mathcal{H}_d \). Assume
\[
\mathbb{E} \left[ e^{tX_k} \right] \leq e^{g(t)A_k}
\]
with a function \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) and deterministic matrices \( A_k \in \mathcal{H}_d \) for \( k = 1, 2, \ldots, n \). Then, for all \( a \in \mathbb{R} \),
\[
\Pr \left\{ \lambda_{\max} \left( \sum_{k=1}^{n} X_k \right) \geq a \right\} \leq d \inf_{t > 0} e^{-ta + g(t)\rho_A},
\]
with parameter \( \rho_A = \lambda_{\max} \left( \sum_{k} A_k \right) \).

Proof. Use the inequality \( \text{tr}(M) \leq d\lambda_{\max}(M) \) for \( M \in \mathcal{H}_d \). \(\square\)

Theorem 5.5 (Matrix Bennett and Bernstein). Let \( \{X_k\} \) be a sequence of independent random matrices in \( \mathcal{H}_d \). Assume
\[
\mathbb{E}[X_k] = 0 \quad \text{and} \quad \lambda_{\max}(X_k) \leq R.
\]
Let
\[
\sigma^2 = \frac{1}{n} \left\| \sum_{k=1}^{n} \text{Var}(X_k) \right\| = \frac{1}{n} \left\| \sum_{k=1}^{n} \mathbb{E}(X_k^2) \right\|.
\]
Then for all \( \epsilon \geq 0 \)
\[
\Pr \left\{ \lambda_{\max} \left( \frac{1}{n} \sum_{k=1}^{n} X_k \right) \geq \epsilon \right\} \leq d \exp \left( -\frac{n\epsilon^2}{2R^2 h\left( \frac{Re}{\sigma^2} \right)} \right) \leq d \exp \left( -\frac{ne^2}{2\sigma^2 + 2Re/3} \right),
\]
where the function \( h(u) = (1 + u) \log(1 + u) - u \) for \( u \geq 0 \).

Proof. The first one is Bennett and the second is Bernstein. The Bennett inequality is a combination of Lemma 4.4 and Corollary 5.4. From Bennett to Bernstein is from the bound
\[
h(u) \geq \frac{u^2}{2 + 2u/3},
\]
which can be proved using calculus. \(\square\)

If replace the boundedness of \( \lambda_{\max}(X_k) \) to the boundedness of \( \|X_k\| \leq R \), by applying the Bernstein estimate to \( X_k \) and \( -X_k \), we could obtain the tail bound for the spectral norm.
Corollary 5.6. Let \( \{X_k\} \) be a sequence of independent random matrices in \( \mathcal{H}_d \). Assume
\[
E[X_k] = 0 \quad \text{and} \quad \|X_k\| \leq R.
\]
Let
\[
\sigma^2 = \frac{1}{n} \left\| \sum_{k=1}^{n} \text{Var}(X_k) \right\| = \frac{1}{n} \left\| \sum_{k=1}^{n} E(X_k^2) \right\|.
\]
Then for all \( \epsilon \geq 0 \)
\[
(17) \quad \Pr\left\{ \frac{1}{n} \sum_{k=1}^{n} \|X_k\| \geq \epsilon \right\} \leq 2d \exp\left( -\frac{n\sigma^2}{2\epsilon^2 + 2R\epsilon/3} \right).
\]
We present a concentration result for sum of rank-1 matrices which is quite useful in the randomized numerical linear algebra.

Corollary 5.7 (Sum of Rank-1 Matrices). Let \( y_1, y_2, \ldots, y_n \) be i.i.d. random column vectors in \( \mathbb{C}^d \) with
\[
\|y_i\| \leq M \quad \text{and} \quad \|E[y_1y_1^*]\| \leq 1.
\]
Then for all \( 0 \leq \epsilon \leq 1 \)
\[
\Pr\left\{ \frac{1}{n} \sum_{k=1}^{n} y_ky_k^* - E[y_1y_1^*] \geq \epsilon \right\} \leq 2d \exp\left( -\frac{3n\epsilon^2}{8(M^2 + 1)} \right).
\]

Proof. Let \( A = E[y_1y_1^*], Y_k = y_ky_k^*, \) and \( X_k = Y_k - A \). Then \( E[X_k] = 0 \). We bound the spectral norm of \( X_k \) as
\[
\|X_k\| \leq \|Y_k\| + \|A\| \leq M^2 + 1,
\]
where we use the fact \( \|y_ky_k^*\| = \|y_k\|^2 \leq M^2 \). We then compute the variance
\[
\|E[X_k^2]\| = \|\text{Var}(Y_k)\| = \|E[Y_k^2] - A^2\| \leq \|E[\|y_k\|^2 Y_k]\| + \|A^2\| \leq M^2 + 1.
\]
Here we compute \( Y_k^2 = y_ky_k^*y_ky_k^* = \|y_k\|^2 Y_k \).

Then we can apply matrix Bernstein inequality with bound \( \epsilon \leq 1 \) to get the desired estimate. \( \square \)

Remark 5.8. In [4], the bound is: for all \( \epsilon \geq 0 \)
\[
\Pr\left\{ \frac{1}{n} \sum_{k=1}^{n} y_ky_k^* - E[y_1y_1^*] \geq \epsilon \right\} \leq (2 \min(d, n))^2 \exp\left( -\frac{n\epsilon^2}{16M^2 \min(\epsilon^2, 4\epsilon - 4)} \right),
\]
which leads to meaningful results even the ambient dimension \( d \) is arbitrarily large. \( \square \)

To control the spectral norm of rectangular matrices, we can use the dilation to form a larger matrix, i.e., for \( B_{d_1 \times d_2} \), we let
\[
\mathcal{F}(B) := \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}.
\]
Then \( \mathcal{F}(B) \in H_{d_1 + d_2} \) and
\[
\mathcal{F}(B)^2 := \begin{pmatrix} BB^* & 0 \\ 0 & B^*B \end{pmatrix}.
\]
Consequently
\[
\lambda_{\max}(\mathcal{F}(B)) = \|\mathcal{F}(B)\| = \|B\|.
Corollary 5.9 (Rectangular Matrix Bernstein). Let $\{Z_k\}$ be a sequence of independent random matrices of size $d_1 \times d_2$. Assume

$$\mathbb{E}[Z_k] = 0 \quad \text{and} \quad \|Z_k\| \leq R.$$ 

Let

$$\sigma^2 = \max \left\{ \left\| \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}(Z_k Z_k^*) \right\|, \left\| \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}(Z_k^* Z_k) \right\| \right\}.$$ 

Then for all $\epsilon \geq 0$

$$\Pr \left\{ \left\| \frac{1}{n} \sum_{k=1}^{n} Z_k \right\| \geq \epsilon \right\} \leq d \exp \left( -\frac{n\epsilon^2}{2\sigma^2 + 2R\epsilon/3} \right), \tag{18}$$

where $d = d_1 + d_2$.

REFERENCES


