PROGRAMMING OF WEAK GALERKIN METHOD

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1. POISSON TYPE EQUATIONS

1.1. 2-D $\{P_0, P_0\} - RT_0$. We chose piecewise constant bases for boundary edges and interior of triangles. The four bases are denoted by $\phi_0, \phi_{b1}, \phi_{b2}, \phi_{b3}$ as shown in Fig 1. The weak gradient is $\nabla_w \phi = Q_T(\nabla \phi)$. Here $\nabla \phi$ is understood in the distribution sense and $Q_T$ is the $L^2$ projection to $RT_0$ space. Chose a bases $\{\chi_1, \chi_2, \chi_3\}$ of $RT_0(T)$, the computation of $\nabla_w \phi_i = Q_T(\nabla \phi_i)$ will involve the assembling of the corresponding mass matrix and the evaluation of the action $\langle \nabla \phi_i, \chi_j \rangle$.

Since the inverse of the mass matrix is needed, we chose a $L^2$-orthogonal bases of $RT_0(T)$ as the following

\[
\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \chi_3 = \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix}.
\]

where $(\bar{x}, \bar{y})$ is the barycenter of triangle $T$. The mass matrix is

\[
M = \text{diag}(|T|, |T|, C_T^{-1}|T|),
\]

where $|T|$ is the area of triangle $T$ and

\[
C_T = \left[\frac{1}{|T|} \int_T (x - \bar{x})^2 + (y - \bar{y})^2 \, dx \, dy\right]^{-1}.
\]

The quantity $C_T$ can be computed using numerical quadrature, e.g., three middle points rule.

For a weak function $\phi = (\phi_0, \phi_b)$, we now compute

\[
q = (q_j) = \langle \nabla \phi, \chi_j \rangle := -(\phi_0, \nabla \cdot \chi_j)_T + (\phi_b, \chi_j \cdot n)_{\partial T}.
\]
We write the formulae for different block of the local stiffness matrix:

\[ q^0 = \begin{pmatrix} 0 \\ 0 \\ -2|T| \end{pmatrix}, \quad \nabla \phi^0 = M^{-1} q^0 = \begin{pmatrix} 0 \\ 0 \\ -2C_T \end{pmatrix}. \]

For the basis \( \phi_{b_i}, i = 1, 2, 3 \), only need to compute the boundary part. We compute the first two components as follows

\[ q_1^i = (\phi_{b_i}, \chi_1 \cdot n)_{\partial T} = \int_{e_i} \chi_1 \cdot n_i dS = |e_i| n_i \cdot (1, 0), \]
\[ q_2^i = (\phi_{b_i}, \chi_2 \cdot n)_{\partial T} = \int_{e_i} \chi_2 \cdot n_i dS = |e_i| n_i \cdot (0, 1). \]

Now we use the formula of gradient of barycentric coordinate \( \nabla \lambda_i \)

\[ \nabla \lambda_i = -\frac{n_i}{d_i} = -\frac{n_i |e_i|}{2|T|} \]

to express \((q_1^i, q_2^i) = -2\nabla \lambda_i |T|\). The computation of the third component is a little bit subtle.

\[ q_3^i = (\phi_{b_i}, \chi_3 \cdot n)_{\partial T} = \int_{e_i} \chi_3 \cdot n_i dS = (x_{im} - \bar{x}, y_{im} - \bar{y}) \cdot n_i |e_i| = \frac{1}{3} (x_{im} - x_i, y_{im} - y_i) \cdot n_i |e_i| = \frac{1}{3} d_i |e_i| = -\frac{2}{3} |T|. \]

We summarize as for \( i = 1, 2, 3 \)

\[ \mathbf{q}^i = \begin{pmatrix} -2\nabla \lambda_i |T| \\ \frac{2}{3} |T| \end{pmatrix}, \quad \nabla w \phi_{b_i} = M^{-1} \mathbf{q}^i = \begin{pmatrix} -2\nabla \lambda_i \\ \frac{2}{3} C_T \end{pmatrix}. \]

**Remark 1.1.** Due to the nonlinear term \( C_T \), the weak gradient is not affine invariant. The traditional way of computing gradient and local stiffness matrix using affine map is no longer valid.

**Remark 1.2.** It is interesting to note that the first two components of \( \nabla_w \phi_{b_i} \) corresponds to the gradient of nonconforming CR element. For CR element, the three bases are \( \{1 - 2\lambda_i\} \) and the element-wise gradient is \( \{-2\nabla \lambda_i\} \).

With the formulae of weak gradient, we can compute the local stiffness by the standard formulae

\[ A_{ij} = \langle \nabla_w \phi_i, \nabla_w \phi_j \rangle = \langle \nabla_w \phi_j \rangle^T M \nabla_w \phi_i = \mathbf{q}^j \cdot \nabla_w \phi_i. \]

We write the formulae for different block of the local stiffness matrix:

\[ A_{b_i b_j} = 4\nabla \lambda_i \cdot \nabla \lambda_j |T| + \frac{4}{9} C_T |T|, \]
\[ A_{0, b_i} = -\frac{4}{3} C_T |T| \]
\[ A_{00} = 4C_T |T|. \]

If we eliminate the interior basis \( \phi_0 \) and form the Schur complement

\[ S = A_{bb} - A_{b0} A_{00}^{-1} A_{0b} = A_{CR} \]
which is exactly the stiffness matrix for the CR nonconforming element. The difference will be the right hand side \( \frac{1}{2} \int \phi \) comparing with \( \int f(1 - 2\lambda_1) \).

Locally the weak function space \((P_0, P_0)\) is of dimension 4 and its gradient space \( RT_0 \) is dimension 3. The weak gradient \( \nabla_w : (P_0, P_0) \to RT_0 \) maps a 4 \( \times \) 1 vector to a 3 \( \times \) 1 vector. The matrix representation \( G \) is formed by using \( \nabla_w \phi_i \) as column vectors, i.e.,

\[
G = (\nabla_w \phi_0, \nabla_w \phi_1, \nabla_w \phi_2, \nabla_w \phi_3) = \begin{pmatrix}
0 & -2\nabla \lambda_1 & -2\nabla \lambda_2 & -2\nabla \lambda_3 \\
-2CT & -\frac{2}{3}CT & -\frac{2}{3}CT & -\frac{2}{3}CT
\end{pmatrix}.
\]

It is easy to see the rank of \( G \) is 3 and the null space of \( G \) is the constant vector which reflects to the important property of the weak gradient

\[
\nabla_w \phi = 0 \iff \phi \text{ is constant.}
\]

Evaluation of the weak gradient. Suppose four coefficients \( u = (u_0, u_1, u_2, u_3)^T \) are given, the product \( Gu \) will give the coefficients in the bases \( \chi = (\chi_1, \chi_2, \chi_3)^T \). Then the function \( \nabla_w u = \chi^T Gu \). Using the formulae of \( \chi \), we can write the weak gradient in two parts:

\[
\nabla_w u = \nabla_{CR} u + \text{h.o.t.}
\]

The constant vector \( \nabla_{CR} u = -2 \sum u_i \nabla \lambda_i \) is exactly the gradient of CR element. The h.o.t term corresponds to the contribution of \( \chi_3 \) whose coefficient is

\[
\begin{pmatrix}
\frac{1}{3}(u_1 + u_2 + u_3) - u_0
\end{pmatrix} 2CT \chi_3.
\]

The term \((u_1 + u_2 + u_3)/3 - u_0\) is zero for linear polynomial and interpolant and thus in general it is of order \( O(h^2) \). This term can be safely skipped.

We check the scaling as follows: \( \nabla \lambda_i = O(1/h) \) and \( CT \chi_3 = O(1/h) \), i.e., as a gradient of bases, they are in \((1/h)\) scaling. The coefficient \( u_i \) are \( O(1) \) for \( \nabla_{CR} u \) and the coefficient for linear part is \( O(h^2) \). Two orders higher.

**Remark 1.3.** For general Poisson equation with scalar coefficient \( \int_K |\nabla u|^2 \), since the lowest order scheme is used, we compute the average of \( K \) over \( T \) and multiply to the local stiffness matrix. For highly oscillatory or tensor coefficient, we need to compute the mass matrix

\[
M_K = (\int_T K \chi_i \chi_j dV).
\]

The local stiffness matrix will be given by

\[
A_{4 \times 4} = G_{4 \times 3}^T M_{K,3 \times 3} G_{3 \times 4}.
\]

Note that \( M_K \) may not be diagonal and the formulae of \( A_{ij} \) is not concise and not necessary.

1.2. 3D \((P_0, P_0) - RT_0\). The computation is similar. We collect the computation result and skip details.

- **Bases of weak function**: \( \phi_0, \phi_{b_1}, \phi_{b_2}, \phi_{b_3}, \phi_{b_4} \).
- **Bases of \( RT_0 \)**:

\[
\chi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \chi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \chi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \chi_4 = \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \\ z - \bar{z} \end{pmatrix}.
\]
The mass matrix of $RT_0$ is
\[ M = \text{diag}(|T|, |T|, |T|, C_T^{-1}|T|), \]
where $|T|$ is the area of triangle $T$ and
\[ C_T = \left[ \frac{1}{T} \int_T (x - \bar{x})^2 + (y - \bar{y})^2 + (z - \bar{z})^2 \, dx \, dy \, dz \right]^{-1}. \]

The weak gradient is
\[ (\nabla w \phi_0, \nabla w \phi_1, \nabla w \phi_2, \nabla w \phi_3, \nabla w \phi_4) = \left( 0, -3\nabla \lambda_1, -3\nabla \lambda_2, -3\nabla \lambda_3, -3\nabla \lambda_4 \right). \]

Local stiffness matrix
\[ A_{b_i b_j} = 9\nabla \lambda_i \cdot \nabla \lambda_j |T| + \frac{9}{16} C_T |T|, \]
\[ A_{0, b_i} = -\frac{9}{4} C_T |T|, \]
\[ A_{00} = 9 C_T |T|. \]