Math 162B: Lecture Notes on Curves and Surfaces in $\mathbb{R}^3$, Part II

by Chuu-Lian Terng, Spring quarter 2005
Department of Mathematics, University of California at Irvine

1. Review

A parametrized surface is a smooth map $f : \mathcal{O} \to \mathbb{R}^3$ such that $f_{x_1}(p), f_{x_2}(p)$ are linearly independent for all $p \in \mathcal{O}$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^3$. The tangent plane Tf_p at $f(p)$ is spanned by $f_{x_1}(p)$ and $f_{x_2}(p)$. The *first fundamental form* is

$$\mathrm{I} = g_{11}dx_1^2 + 2g_{12}dx_1dx_2 + g_{22}dx_2^2,$$

where

$$g_{ij} = f_{x_i} \cdot f_{x_j},$$

the dot product.

The map $N : \mathcal{O} \to \mathbb{R}^3$ defined by

$$N = \frac{f_{x_1} \times f_{x_2}}{||f_{x_1} \times f_{x_2}||},$$

is smooth, has unit length, and is perpendicular to the tangent plane of f, and we call N the *unit normal vector field* of f. The *shape operator* A_{p_0} of f at $p_0 \in \mathcal{O}$ is the linear map from Tf_{p_0} to Tf_{p_0} defined by

$$A_{p_0}(f_{x_i}(p_0)) = -N_{x_i}(p_0), \quad i = 1, 2.$$

So If $v = af_{x_1}(p_0) + bf_{x_2}(p_0)$, then

$$A_{p_0}(v_1) = -aN_{x_1}(p_0) - bN_{x_2}(p_0).$$

We proved in 162A that

Proposition 1.0.1. *The shape operator $A_{p_0} : Tf_{p_0} \to Tf_{p_0}$ is a self-adjoint operator, i.e.,*

$$A_{p_0}(v_1) \cdot v_2 = v_1 \cdot A_{p_0}(v_2)$$

for all $v_1, v_2 \in Tf_{p_0}$.

The eigenvalues and unit eigenvectors of the shape operator are called the *principal curvatures* and *principal directions* of the parametrized surface f. We use k_1, k_2 to denote the principal curvature functions. The Gaussian curvature and mean curvature are defined by

$$K = k_1k_2, \quad H = k_1 + k_2.$$

A point $f(p_0)$ is *umbilic* if the principal curvatures $k_1(p_0) = k_2(p_0)$, i.e., the shape operator A_{p_0} is equal to a scalar times the identity map of the tangent plane Tf_{p_0}.

The *second fundamental form* II_p is a symmetric bilinear form on Tf_p associated to the shape operator A_p, i.e.,

$$\mathrm{II}_p(v_1, v_2) = A_p(v_1) \cdot v_2$$

for $v_1, v_2 \in Tf_p$. We denote II by

$$\mathrm{II} = \ell_{11} dx_1^2 + 2\ell_{12} dx_1 dx_2 + \ell_{22} dx_2^2,$$

where

$$\ell_{ij} = A(f_{x_i}) \cdot f_{x_j} = -N_{x_i} \cdot f_{x_j} = N \cdot f_{x_i x_j}.$$

This means that if $v_i = a_i f_{x_1} + b_i f_{x_2}$ for $i = 1, 2$, then

$$\mathrm{II}(v_1, v_2) = \ell_{11} a_1 a_2 + \ell_{12}(a_1 b_2 + a_2 b_1) + \ell_{22} b_1 b_2.$$

The Gaussian curvature and mean curvature written in terms of I, II are

$$K = \frac{\det(\ell_{ij})}{\det(g_{ij})},$$
$$H = \sum_{i,j=1}^{2} g^{ij} \ell_{ij} = \frac{1}{\det(g_{ij})}(g_{22}\ell_{11} - 2g_{12}\ell_{12} + g_{11}\ell_{22}),$$

where $(g^{ij}) = (g_{ij})^{-1} = \frac{1}{\det(g_{ij})} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix}$.

A surface $f : \mathcal{O} \to \mathbb{R}^3$ is said to be parametrized by *lines of curvature coordinates* if $g_{12} = \ell_{12} = 0$, in other words, f_{x_1}, f_{x_2} are perpendicular and are eigenvectors of the shape operator. We know from 162A that

Theorem 1.0.2. *If p_0 is not an umbilic point of a surface $f : \mathcal{O} \to \mathbb{R}^3$, then we can change coordinate locally so that the surface is parametrized by line of curvature coordinates, i.e., there exists an open subset $\mathcal{O}_0$ of $\mathcal{O}$ containing p_0, an open subset $\mathcal{O}_1$ of $\mathbb{R}^2$, a diffeomorphism $\phi : \mathcal{O}_1 \to \mathcal{O}_0$ such that h_{x_1}, h_{x_2} are eigenvectors of the shape operator, where $h = f \circ \phi$.*

Suppose $f : \mathcal{O} \to \mathbb{R}^3$ is parametrized by line of curvature coordinates. We write $g_{11} = A_1^2$, $g_{22} = A_2^2$. Then we have

$$\mathrm{I} = A_1^2 dx_1^2 + A_2^2 dx_2^2, \quad \mathrm{II} = \ell_{11} dx_1^2 + \ell_{22} dx_2^2.$$

The principal curvature k_1, k_2, Gaussian curvature K, and the mean curvature H written in line of curvature coordinates are given as follows:

$$k_1 = \frac{\ell_{11}}{A_1^2}, \quad k_2 = \frac{\ell_{22}}{A_2^2},$$
$$K = \frac{\ell_{11}\ell_{22}}{A_1^2 A_2^2} = k_1 k_2,$$
$$H = \frac{\ell_{11}}{A_1^2} + \frac{\ell_{22}}{A_2^2} = k_1 + k_2.$$

We have proved that $A_1, A_2, \ell_{11}, \ell_{22}$ must satisfy the Gauss-Codazzi equations:

$$\begin{cases} \left(\frac{(A_1)_{x_2}}{A_2}\right)_{x_2} + \left(\frac{(A_2)_{x_1}}{A_1}\right)_{x_1} = -\frac{\ell_{11}}{A_1}\frac{\ell_{22}}{A_2}, \\ (\frac{\ell_{11}}{A_1})_{x_2} = \frac{(A_1)_{x_2}}{A_2}\frac{\ell_{22}}{A_2} \\ (\frac{\ell_{22}}{A_2})_{x_1} = \frac{(A_2)_{x_1}}{A_1}\frac{\ell_{11}}{A_1}. \end{cases} \tag{1.0.1}$$

If we set

$$r_1 = \frac{\ell_{11}}{A_1}, \quad r_2 = \frac{\ell_{22}}{A_2},$$

then the Gauss-Codazzi equations (1.0.1) can be written as

$$\begin{cases} \left(\frac{(A_1)_{x_2}}{A_2}\right)_{x_2} + \left(\frac{(A_2)_{x_1}}{A_1}\right)_{x_1} = -r_1 r_2, \\ (r_1)_{x_2} = \frac{(A_1)_{x_2}}{A_2} r_2, \\ (r_2)_{x_1} = \frac{(A_2)_{x_1}}{A_1} r_1. \end{cases} \tag{1.0.2}$$

The first equation of the above system is called the *Gauss equation*, and the second and third equations are called the *Codazzi equations*.

The *Fundamental Theorem of Surfaces in* $\mathbb{R}^3$ states that: Suppose $A_1, A_2, \ell_{11}, \ell_{22}$ are smooth functions from $\mathcal{O}$ to $\mathbb{R}$ satisfying the Gauss-Codazzi equations (1.0.1). Then given $p_0 \in \mathcal{O}$, $q_0 \in \mathbb{R}^3$, and v_1, v_2, v_3 an orthonormal basis of $\mathbb{R}^3$, there exists an open subset $\mathcal{O}_0$ containing p_0 and a unique smooth immersion $f : \mathcal{O}_0 \to \mathbb{R}^3$ such that the first and second fundamental form of f are

$$\mathrm{I} = A_1^2 dx_1^2 + A_2^2 dx_2^2, \quad \mathrm{II} = \ell_{11} dx_1^2 + \ell_{x_2} dx_2^2,$$

$f(p_0) = q_0$, $v_1 = \frac{f_{x_1}(p_0)}{A_1}$, and $v_2 = \frac{f_{x_2}(p_0)}{A_2}$. Moreover, if two surfaces $f, h : \mathcal{O} \to \mathbb{R}^3$ have the same I, II, then they are congruent, i.e., there is a rigid motion ϕ of $\mathbb{R}^3$ such that $h = \phi \circ f$.

2. Surfaces in $\mathbb{R}^3$ with $K = -1$ and SGE

In geometry we are often interested in understanding geometric objects whose invariants are of simplest kind. For example, in plane geometry we have many theorems for equilateral, isoceles, and right triangles. The Guassian and mean curvatures are the simplest kind of invariants for surfaces in $\mathbb{R}^3$, so it is natural to study the geometry of surfaces in $\mathbb{R}^3$ whose K or H are constants.

In this section, we will show the existence of Tchbyshef line of curvature coordinates for surfaces in $\mathbb{R}^3$ with $K = -1$, and we also show that there is a one-to-one correspondence between solutions of the sine-Gordon equation

$$q_{x_1 x_1} - q_{x_2 x_2} = \sin q \cos q \qquad \text{SGE}$$

and surfaces in $\mathbb{R}^3$ with $K = -1$ up to rigid motions.

Theorem 2.0.3. *If a surface in* $\mathbb{R}^3$ *has* $K = -1$*, then locally there exists line of curvature parametrization such that the two fundamental forms are*

$$\mathrm{I} = \cos^2 q \; dx_1^2 + \sin^2 q \; dx_2^2, \quad \mathrm{II} = \sin q \cos q \, (dx_1^2 - dx_2^2)$$

for some function q*. Moreover, the Gauss-Codazzi equation is the sine-Gordon equation*

$$q_{x_1 x_1} - q_{x_2 x_2} = \sin q \cos q. \qquad \text{SGE} \qquad (2.0.3)$$

Proof. Since $K = k_1 k_2 = -1$, k_1 is never equal to k_2. So there exists line of curvature parametrization locally. Thus we may assume that

$$\mathrm{I} = A_1^2 dx_1^2 + A_2^2 dx_2^2, \quad \mathrm{II} = \ell_{11} dx_1^2 + \ell_{22} dx_2^2.$$

Since $K = k_1 k_2 = -1$, so there exists a function q such that

$$k_1 = \tan q, \quad k_1 = -\cot q$$

($q = \tan^{-1} k_1$). But $k_1 = \frac{\ell_{11}}{A_1^2}$, $k_2 = \frac{\ell_{22}}{A_2^2}$, so

$$r_1 = \frac{\ell_{11}}{A_1} = k_1 A_1 = \tan q \; A_1,$$
$$r_2 = \frac{\ell_{22}}{A_2} = k_2 A_2 = -\cot q \; A_2.$$

We will use the first Codazzi equation to prove that $\frac{A_1}{\ell_{11}}$ is a function of x_1 alone. It suffices to prove that $\left(\frac{A_1}{\ell_{11}}\right)_{x_2} = 0$. But by the second equation of (1.0.2), we have

$$(r_1)_{x_2} = (A_1 \tan q)_{x_1} = (A_1)_{x_2} \tan q + A_1 \sec^2 q \; q_{x_2}$$
$$= \frac{(A_1)_{x_2}}{A_2} r_2 = -\frac{(A_1)_{x_2}}{A_2} A_2 \cot q = -(A_1)_{x_2} \cot q.$$

Thus

$$(A_1)_{x_2} (\tan q + \cot q) = -A_1 \sec^2 q \; q_{x_2},$$

But $\tan q + \cot q = \frac{1}{\sin q \cos q}$, so we get

$$\frac{(A_1)_{x_2}}{A_1} = -\frac{\sin q}{\cos q} \; q_{x_2}.$$

This implies that $(\ln A_1 - \ln \cos q)_{x_2} = 0$, hence $\left(\ln \frac{A_1}{\cos q}\right)_{x_2} = 0$. It follows from calculus that $\ln \frac{A_1}{\cos q}$ is a function of x_1 alone, say $c_1(x_1)$ for some one variable function c_1. So we have proved that

$$\frac{A_1}{\cos q} = e^{c_1(x_1)}. \qquad (2.0.4)$$

Similarly, we use the third equation of (1.0.2) to conclude that

$$\frac{A_2}{\sin q} = e^{c_2(x_2)} \qquad (2.0.5)$$

for some one variable function c_2.

Define $\tilde{x}_i$ as a function of x_i alone for $i = 1, 2$ such that

$$\frac{d\tilde{x}_1}{dx_1} = e^{c_1(x_1)}, \quad \frac{d\tilde{x}_2}{dx_2} = e^{c_2(x_2)}.$$

Then $(x_1, x_2) \mapsto (\tilde{x}_1, \tilde{x}_2)$ is a local diffeomorphism, and there is a local inverse, i.e., we can write x_1, x_2 as functions of $\tilde{x}_1, \tilde{x}_2$. Next we compute the two fundamental forms in $(\tilde{x}_1, \tilde{x}_2)$ coordinate. Since

$$\frac{\partial f}{\partial \tilde{x}_i} = \frac{\partial f}{\partial x_i}\frac{dx_i}{d\tilde{x}_i} = f_{x_i} e^{-c_i(x_i)},$$

$\tilde{g}_{ij} = f_{\tilde{x}_i} \cdot f_{\tilde{x}_j} = e^{-c_i(x_i) - c_j(x_j)} g_{ij}$.

Use (2.0.4) and (2.0.5) in the following computations:

$$\begin{aligned}
\tilde{g}_{11} &= g_{11} e^{-2c_1(x_1)} = A_1^2 e^{-2c_1(x_1)} = \left(\frac{A_1}{e^{c_1(x_1)}}\right)^2 = \cos^2 q, \\
\tilde{g}_{12} &= 0, \\
\tilde{g}_{22} &= g_{22} e^{-2c_2(x_2)} = A_2^2 e^{-2c_2(x_2)} = \left(\frac{A_2}{e^{c_2(x_2)}}\right)^2 = \sin^2 q.
\end{aligned}$$

This shows that $\mathrm{I} = \cos^2 q d\tilde{x}_1^2 + \sin^2 q d\tilde{x}_2^2$.

To compute II, we note that

$$\begin{aligned}
\tilde{\ell}_{11} &= k_1 \tilde{A}_1^2 = \tan q \cos^2 q = \sin q \cos q, \\
\tilde{\ell}_{12} &= 0 \\
\tilde{\ell}_{22} &= k_2 \tilde{A}_2^2 = -\cot q \sin^2 q = -\sin q \cos q.
\end{aligned}$$

So $\mathrm{II} = \sin q \cos q\, (\tilde{d}x_1^2 - d\tilde{x}_2^2$ and (x_1, x_2) are line of curvature coordinate system.

To check the G-C equation, we compute

$$\begin{aligned}
\tilde{r}_1 &= \frac{\tilde{\ell}_{11}}{\tilde{A}_1} = \frac{\sin q \cos q}{\cos q} = \sin q, \\
\tilde{r}_2 &= \frac{\tilde{\ell}_{22}}{\tilde{A}_2} = \frac{-\sin q \cos q}{\sin q} = -\cos q, \\
\frac{(A_1)_{x_2}}{A_2} &= \frac{-\sin q \ q_{x_2}}{\sin q} = -q_{x_2}, \\
\frac{(A_2)_{x_1}}{A_1} &= \frac{\cos q \ q_{x_1}}{\cos q} = q_{x_1}.
\end{aligned}$$

The Codazzi equations (the second and the third equations of (1.0.2)) do not give any extra condition on q. The Gauss equation (the first equation of (1.0.2) gives SGE. □

As a consequence of the Fundamental Theorem of Surfaces, we have

Theorem 2.0.4. *Let $\mathcal{O}$ be an open disk in $\mathbb{R}^2$, and $q : \mathcal{O} \to \mathbb{R}$ a solution of SGE such that $q(x_1, x_2) \in (0, \frac{\pi}{2})$. Then given $x_0 \in \mathcal{O}$, $p_0 \in \mathbb{R}^3$, and an orthonormal basis $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$, there exists a unique surface $f : \mathcal{O} \to \mathbb{R}^3$ such that its fundamental forms are*

$$\mathrm{I} = \cos^2 q \; dx_1^2 + \sin^2 q \; dx_2^2,$$

and $f(x_0) = p_0$, $f_{x_1}(x_0) = \cos q(x_0) v_1$, $f_{x_2}(x_0) = \sin q(x_0) v_2$.

In other words, we have proved that there is a one-to-one correspondence between solutions of SGE whose image lies in the interval $(0, \frac{\pi}{2})$ and a surface in $\mathbb{R}^3$ with $K = -1$ modulo rigid motions.

Next we review the proof of the Fundamental Theorem of Surfaces here, and will see that when we do not assume any condition on the image of a solution q we still can get a smooth map f, but now f fails to be an immersion at points p where $\sin q(p) \cos q(p) = 0$, i.e., when $q = \frac{n\pi}{2}$ for some integer n. To get the immersion f, we need to solve the following first order equation (see 162A Lecture Notes):

$$\begin{cases} (f, e_1, e_2, e_3)_{x_1} = (e_1, e_2, e_3) \begin{pmatrix} A_1 & 0 & \frac{(A_1)_{x_2}}{A_2} & -r_1 \\ 0 & -\frac{(A_1)_{x_2}}{A_2} & 0 & 0 \\ 0 & r_1 & 0 & 0 \end{pmatrix} \\ (f, e_1, e_2, e_3)_{x_2} = (e_1, e_2, e_3) \begin{pmatrix} 0 & 0 & -\frac{(A_2)_{x_1}}{A_1} & 0 \\ A_2 & \frac{(A_2)_{x_1}}{A_1} & 0 & -r_2 \\ 0 & 0 & r_2 & 0 \end{pmatrix}. \end{cases} \tag{2.0.6}$$

In general, if A_i vanishes at x_0, then the right hand side is not continuous at x_0, so the above equation can not be solved in a neighborhood of x_0. But in the case for surfaces with $K = -1$, the trouble terms $\frac{(A_1)_{x_2}}{A_2} = -q_{x_2}$ and $\frac{(A_2)_{x_1}}{A_1} = q_{x_1}$ are both well-defined smooth maps even when A_1 or A_2 vanish at some points. So (2.0.6) becomes

$$\begin{cases} (f, e_1, e_2, e_3)_{x_1} = (e_1, e_2, e_3) \begin{pmatrix} \cos q & 0 & -q_{x_2} & -\sin q \\ 0 & q_{x_2} & 0 & 0 \\ 0 & \sin q & 0 & 0 \end{pmatrix} \\ (f, e_1, e_2, e_3)_{x_2} = (e_1, e_2, e_3) \begin{pmatrix} 0 & 0 & -q_{x_1} & 0 \\ \sin q & q_{x_1} & 0 & \cos q \\ 0 & 0 & -\cos q & 0 \end{pmatrix}. \end{cases} \tag{2.0.7}$$

Since the right hand side is smooth, by Frobenius Theorem this system is solvable. So we obtain a unique solution (f, e_1, e_2, e_3) such that the initial data at x_0 is (p_0, v_1, v_2, v_3). Since $f_{x_1} = \cos q \, e_1$ and $f_{x_2} = \sin q \, e_2$, the map f is an immersion if and only if $\sin q \cos q \neq 0$, i.e., when $q(p) \neq \frac{k\pi}{2}$ for some integer k. At points where $q(p) = \frac{k\pi}{2}$, the map f is still smooth at those points, but the rank of the Jacobian matrix of f at those points is 1. We call such points a *cusp singularity* of f.

Remark. If we modify the MatLab project for the Fundamental Theorem of Surfaces in line of curvature coordinates by using (2.0.7), then the program should generate surfaces with $K = -1$ with cusp singularities when we input a solution of SGE.

3. Tchbyshef asymptotic coordinates for $K = -1$ surfaces

The asymptotic lines for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$. A quadratic curve $ax^2 + 2bxy + cy^2 = 1$ represents a hyperbola if $b^2 - 4ac > 0$, i.e., if $\det \begin{pmatrix} a & b \\ b & c \end{pmatrix} < 0$. We call the directions of the two lines given by $ax^2 + 2bxy + cy^2 = 0$ the *asymptotic directions* for the quadratic form $ax^2 + 2bxy + cy^2$. The second fundamental form of a surface f at point p is a quadratic form on the tangent plane Tf_p. Since $K(p) = \frac{\det(\ell_{ij})}{\det(g_{ij})}$ and $\det(g_{ij}) > 0$, $\det(\ell_{ij}) < 0$ if and only if $K(p) < 0$. In particular, if $K(p_0) < 0$, then there exists two linearly independent aysmptotic directions for the quadratic form II_{p_o}.

Definition 3.0.5. A non-zero tangent vector $v \in Tf_{p_0}$ is *asymptotic* if $\mathrm{II}_{p_0}(v, v) = 0$. A parametrized surface $f : \mathcal{O} \to \mathbb{R}^3$ is parametrized by *asymptotic coordinates* if $\ell_{11} = \ell_{22} = 0$, i.e., both f_{x_1} and f_{x_2} are asymptotic vectors.

Suppose $f : \mathcal{O} \to \mathbb{R}^3$ is the Tchbyshef line of curvature parametrization for a surface with $K = -1$ with

$$\mathrm{I} = \cos^2 q \, dx_1^2 + \sin^q \, dx_2^2, \quad \mathrm{II} = \sin q \cos q \, (dx_1^2 - dx_2^2).$$

A tangent vector $v = a_1 f_{x_1} + a_2 f_{x_2}$ is *asymptotic* if

$$\mathrm{II}(v, v) = \sin q \cos q \, (a_1^2 - a_2^2) = 0,$$

or equivalently, $a_1^2 = a_2^2$. So $v_1 = f_{x_1} + f_{x_2}$ and $v_2 = f_{x_1} - f_{x_2}$ are asymptotic vectors. If we make the following change of coordinates

$$\begin{cases} s = \frac{x_1 + x_2}{2}, \\ t = \frac{x_1 - x_2}{2}, \end{cases}$$

then by the Chain rule $f_s = f_{x_1} + f_{x_2}$ and $f_t = f_{x_1} - f_{x_2}$. So

$$\tilde{f}(s, t) = f(x_1(s, t), x_2(s, t)) = f(s + t, s - t)$$

is an asymptotic parametrization. We compute I, II next. Since

$$f_{x_1} \cdot f_{x_1} = \cos^2 q, \qquad f_{x_1} \cdot f_{x_2} = 0, \quad f_{x_2} \cdot f_{x_2} = \sin^2 q,$$

$$\tilde{g}_{11} = f_s \cdot f_s = (f_{x_1} + f_{x_2}) \cdot (f_{x_1} + f_{x_2}) = \cos^2 q + \sin^2 q = 1.$$

Similar computation gives

$$\tilde{g}_{12} = f_s \cdot f_t = \cos 2q, \quad \tilde{g}_{22} = 1.$$

As a consequence, we see that *the angle between the asymptotic vectors is* $2q$. We have proved that f_s, f_t are asymptotic vectors, hence $\tilde{\ell}_{11} = \tilde{\ell}_{22} = 0$. It remains to compute $\tilde{\ell}_{12}$. Since

$$f_{x_1x_1} \cdot N = -f_{x_2x_2} \cdot N = \sin q \cos q, \quad f_{x_1x_2} \cdot N = 0,$$

we have

$$\tilde{\ell}_{12} = f_{st} \cdot N = (f_{x_1} + f_{x_2})_{x_1} - (f_{x_1} + f_{x_2})_{x_2} = (f_{x_1x_1} - f_{x_2x_2}) \cdot N = 2 \sin q \cos q.$$

So the fundamental form in (s,t) coordinate becomes

$$\mathrm{I} = ds^2 + 2\cos(2q)\, ds\ dt + dt^2, \quad \mathrm{II} = 4 \sin q \cos q\, ds\ dt = 2\sin(2q) ds\ dt.$$

We call (s,t) the *Tchbyshef asymptotic coordinate system* (parametrization).

Note that q satisfies the SGE. We want to write the SGE in (s,t) coordinates. By Chain rule, we have

$$u_{x_1} = \frac{1}{2}(u_s + u_t), \quad u_{x_2} = \frac{1}{2}(u_s - u_t),$$

so $q_{x_1x_1} - q_{x_2x_2} = q_{st}$. Thus

$$q_{st} = \sin q \cos q.$$

To summarize, we have proved that

Theorem 3.0.6. *Locally there exists an asymptotic parametrization for a surface with* $K = -1$ *such that*

$$\mathrm{I} = ds^2 + 2\cos(2q)\, ds\ dt + dt^2, \quad \mathrm{II} = 4 \sin q \cos q\, ds\ dt = 2\sin(2q)\, ds\ dt,$$

where $2q$ *is the angle between the asymptotic directions. Moreover, the Gauss-Codazzi equation is*

$$q_{st} = \sin q \cos q. \qquad \text{SGE}$$

4. Change of parametrizations

We will show that although the two fundamental forms look different when we use different parametrization of the same surface, they are the same symmetric bilinear forms. To see this, we review some linear algebra and material we taught in 162A, Let V be a vector space, and $\{v_1, \ldots, v_n\}$ a basis of V. It is easy to check that the space V^* of all linear maps from V to $\mathbb{R}$ is again a vector space. Now let v_i^* denote the linear map from V to $\mathbb{R}$ such that

$$v_i^*(v_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

for $1 \leq i \leq n$.

Exercise 4.0.1. Prove that $\{v_1^*, \ldots, v_n^*\}$ is a basis of V^*.

Suppose $\ell_1, \ell_2 : V \to \mathbb{R}$ are linear maps. We define $\ell_1 \otimes \ell_2 : V \times V \to \mathbb{R}$ by

$$\ell_1 \otimes \ell_2(\xi, \eta) = \ell_1(\xi)\ell_2(\eta)$$

for all $\xi, \eta \in V$. Let $\ell_1\ell_2 : V \times V \to \mathbb{R}$ denote the bilinear map defined by

$$\ell_1\ell_2 = \frac{1}{2}(\ell_1 \otimes \ell_2 + \ell_2 \otimes \ell_1),$$

i.e., $\ell_1\ell_2(\xi, \eta) = \frac{1}{2}(\ell_1(\xi)\ell_2(\eta) + \ell_2(\xi)\ell_1(\eta))$.

Exercise 4.0.2. Prove that

(1) $\ell_1 \otimes \ell_2$ is bilinear.
(2) $\ell_1\ell_2$ is symmetric bilinear.

Exercise 4.0.3. Let $b : V \times V \to \mathbb{R}$ be a bilinear map, $v_1, \ldots, v_n$ a basis of V, $v_1^*, \ldots, v_n^*$ dual basis of V^*, and $b_{ij} = b(v_i, v_j)$.

(1) Prove that $b = \sum_{i,j=1}^n b_{ij} v_i^* \otimes v_j^*$.
(2) Prove that b is symmetric if and only if $b_{ij} = b_{ji}$ for all $1 \le i, j \le n$.
(3) Prove that if b is symmetric then $b = \sum_{i,j=1}^n b_{ij} v_i^* v_j^*$.

Suppose $f : \mathcal{O} \to \mathbb{R}^3$ is a parametrized surface. The first fundamental form $\mathrm{I}_p : Tf_p \times Tf_p \to \mathbb{R}$ is defined by $\mathrm{I}_p(\xi, \eta) = \xi \cdot \eta$ the dot product for all $\xi, \eta \in Tf_p$. It is easy to check that I_p is a symmetric bilinear map. Note that $\{f_{x_1}(p), f_{x_2}(p)\}$ is a basis of the tangent plane Tf_p. Let dx_1, dx_2 be the dual basis of Tf_p^*. It follows from the above exercise that

$$\mathrm{I} = g_{11} dx_1^2 + 2g_{12} dx_1 dx_2 + g_{22} dx_2^2,$$

where $g_{ij} = f_{x_i} \cdot f_{x_j}$. Since I_p is defined without parametrization $\mathrm{I}_p(\xi, \eta) = \xi \cdot \eta$, different parametrizations will give the same I.

The unit normal vector field to f is the map from $\mathcal{O}$ to the unit sphere S^2:

$$N(x_1, x_2) = \frac{f_{x_1} \times f_{x_2}}{||f_{x_1} \times f_{x_2}||}.$$

(N is also called the *Gauss-map* of the surface. The shape operator A_p is the self-adjoint operator from Tf_p to Tf_p defined by $A_p(f_{x_i}) = -N_{x_i}$. Now we change parametrization of the surface by a diffeomorphism $x = x(y) : \mathcal{O}_1 \to \mathcal{O}$, i.e., we use $\tilde{f}(y) = f(x(y))$ as a parametrization of the same surface. By definition, the shape operator $\tilde{A}$ for the new parametrization is the linear operator defined by

$$\tilde{A}(\tilde{f}_{y_i}) = -N_{y_i}.$$

We will show that A and $\tilde{A}$ are the same. Because by Chain rule, we have

$$\frac{\partial u}{\partial y_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial y_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial y_i}.$$

So

$$\tilde{A}(\tilde{f}_{y_i}) = \frac{\partial N}{\partial y_i} = -\left(\frac{\partial x_1}{\partial y_i}\ N_{x_1} + \frac{\partial x_2}{\partial y_i} N_{x_2}\right).$$

However, $A(f_{y_i}) = A\left(\frac{\partial x_1}{\partial y_i}\ f_{x_1} + \frac{\partial x_2}{\partial y_i} f_{x_2}\right)$ and A is linear, we see that

$$A(f_{y_i}) = -\left(\frac{\partial x_1}{\partial y_i}\ N_{x_1} + \frac{\partial x_2}{\partial y_i} N_{x_2}\right),$$

which is equal to $\tilde{A}(f_{y_i})$. This shows that the shape operator does not depend on the choice of parametrization of a surface.

The second fundamental form is the bilinear form associated to the shape operator:

$$\mathrm{II}_p(\xi, \eta) = A_p(\xi) \cdot \eta$$

for all $\xi, \eta \in Tf_p$. Since the shape opeartor is independent of parametrization of the surface, so is II. Note that the mean curvature H and the Gaussian curvature K are the trace and the determinant of the shape operator. This shows that H and K are well-defined function on the surface, independent of parametrizations.

5. Calculus of variations of one variable

In this section, C^1 means continuously differentiable. Let $C^1([a,b], \mathbb{R}^2)$ denote the space of all C^1 maps $x : [a,b] \to \mathbb{R}^2$ (i.e., x is a differentiable and its derivative x' is continuous). Fix $p_0, q_0 \in \mathbb{R}^2$, let $C^1([a,b], \mathbb{R}^2)_{p_0,q_0}$ denote the set of all $x \in C^1([a,b], \mathbb{R}^2)$ such that $x(a) = p_0$ and $x(b) = q_0$.

A function $J : C^1([a,b], \mathbb{R}^2)_{p_0,q_0} \to \mathbb{R}$ is called a a *calculus of variations functional* if it has the form

$$J(x) = J^L(x) = \int_a^b L(t, x(t), x'(t))\ dt,$$

where

$$L : [a,b] \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

is a C^1 map. We call L the *Lagrangian function associated to the functional* J.

To motivate the definitions of directional derivative and critical points of J, we review these definitions for $f : \mathbb{R}^n \to \mathbb{R}$. The directional derivative of f at p_0 in the direction v is $\frac{d}{ds}\big|_{s=0} f(p_0 + sv)$. A point p_0 is a critical point of f if $\frac{\partial f}{\partial x_i}(p_0) = 0$ for all $1 \leq i \leq n$. Given $v = (v_1, \ldots, v_n)$, the directional derivative

$$Df_v(p_0) = \frac{d}{ds}\bigg|_{s=0} f(p_0 + sv) = \sum_{i=1}^n \frac{\partial}{\partial x_i}(p_0) v_i.$$

So the following statements are equivalent:

(i) p_0 is a critical point of f,
(ii) all directional derivatives of f at p_0 are zero,

(iii) $\frac{d}{ds}\big|_{s=0} f(\sigma(s)) = 0$ for all smooth curve $\sigma : (-\epsilon, \epsilon) \to \mathbb{R}^n$ with $\sigma(0) = p_0$. equation

The functional J is a function on the vector space $C^1([a,b], \mathbb{R}^2)_{p_0,q_0}$, so we can define directional derivatives and critical points of J in the same manner as for $f : \mathbb{R}^n \to \mathbb{R}$.

The directional derivative of J^L.

Let $x^0 = (x_1^0, x_2^0) \in C^1([a,b], \mathbb{R}^2)_{p_0,q_0}$, and $h = (h_1, h_2) \in C^1([a,b], \mathbb{R}^2)_{0,0}$, i.e., $h(a) = h(b) = 0$. Note that $x^s = x^0 + sh$ is in $C^1([a,b], \mathbb{R}^2)_{p_0,q_0}$ for all $s \in \mathbb{R}$. Let us find the derivative of $J^L(x^s)$ with respect to s at $s = 0$:

$$\begin{aligned}
\frac{d}{ds}\bigg|_{s=0} J^L(x^s) &= \int_a^b L(t, x^s(t), (x^s)'(t))\, dt \\
&= \int_a^b \frac{d}{ds}\bigg|_{s=0} L(t, x^0(t) + sh(t), (x^0)'(t) + sh'(t))\, dt, \quad \text{by chain rule,} \\
&= \int_a^b \frac{\partial L}{\partial x_1}(t, x^0(t), (x^0)'(t)) h_1(t) + \frac{\partial L}{\partial x_2}(t, x^0(t), (x^0)'(t)) h_2(t) \\
&\quad + \frac{\partial L}{\partial x_1'}(t, x^0(t), (x^0)'(t)) h_1'(t) + \frac{\partial L}{\partial x_2'}(t, x^0(t), (x^0)'(t)) h_2'(t)\, dt.
\end{aligned}$$

Next we want to use integration by part, $\int_a^b f'(t)g(t)dt = f(t)g(t)\Big|_a^b - \int_a^b f(t)g'(t)\, dt$, to change h_i' to h_i in the above integration. First note that since $h_i(a) = h_i(b) = 0$ for $i = 1, 2$, we have

$$\int_a^b \frac{d}{dt}\left(\frac{\partial L}{\partial x_i'} h_i(t)\right)\, dt = 0,$$

and hence

$$\int_a^b \frac{\partial L}{\partial x_i'} h_i' + \left(\frac{\partial L}{\partial x_i'}\right)' h_i\, dt = 0.$$

So

$$\begin{aligned}
&\int_a^b \frac{\partial L}{\partial x_1'}(t, x^0(t), (x^0)'(t)) h_1'(t) + \frac{\partial L}{\partial x_2'}(t, x^0(t), (x^0)'(t)) h_2'(t)\, dt \\
&= -\int_a^b \left(\frac{\partial L}{\partial x_1'}(t, x^0(t), (x^0)'(t))\right)' h_1(t) + \left(\frac{\partial L}{\partial x_2'}(t, x^0(t), (x^0)'(t))\right)' h_2(t)\, dt.
\end{aligned}$$

Thus the formula for the derivational derivative of J at x^0 in the direction h is

$$\frac{d}{ds}\bigg|_{s=0} J^L(x^s) = \int_a^b \left(\frac{\partial L}{\partial x_1} - \left(\frac{\partial L}{\partial x_1'}\right)'\right) h_1 + \left(\frac{\partial L}{\partial x_2} - \left(\frac{\partial L}{\partial x_2'}\right)'\right) h_2\, dt. \tag{5.0.8}$$

x^0 is called a *critical point* of J^L if this directional derivatives is zero for *all* choices of $h \in C^1([a,b],\mathbb{R}^2)_{0,0}$. This will certainly be the case if the following so-called *Euler-Lagrange equation* is satisfied:

$$\frac{\partial L}{\partial x_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial x_i'}\right), \quad i = 1,2. \tag{5.0.9}$$

The following is the fundmental lemma of calculus of variations:

Lemma 5.0.7. *Let $f = (f_1,\ldots,f_n) : [a,b] \to \mathbb{R}^n$ be a C^1 map. Suppose that $\int_a^b \sum_{i=1}^n f_i(t)h_i(t)\, dt = 0$ for all C^1 maps $h = (h_1,\ldots,h_n) : [a,b] \to \mathbb{R}^n$ with $h(a) = h(b) = 0$. Then $f = 0$.*

Proof. We prove this by contracdition. It suffices to prove the case when $n = 1$. Suppose f is not the zero function, there there exists $c \in (a,b)$ such that $f(c) \neq 0$. Say $f(c) > 0$. Then there exists $\epsilon > 0$ such that $(c-\epsilon, c+\epsilon) \subset (a,b)$ and $f(t) > 0$ for all $t \in (c-\epsilon, c+\epsilon)$, Choose a C^1 function $g : [a,b] \to \mathbb{R}$ such that $g > 0$ on $(c-\epsilon, c+\epsilon)$, $g = 0$ outside $[c-\epsilon, c+\epsilon]$. Then

$$\int_a^b f(t)g(t)\, dt = \int_{c-\epsilon}^{c+\epsilon} f(t)g(t)\, dt > 0,$$

which contradicts to the assumption that $\int_a^b f(t)h(t)\, dt = 0$ for all C^1 h with $h(a) = h(b) = 0$. □

As a consequence of Lemma 5.0.7 and (5.0.8), we get

Theorem 5.0.8. *$x : [a,b] \to \mathbb{R}^2$ is a critical point of J^L if and only if x satisfies the Euler-Lagrage equation* (5.0.9).

Example 5.0.9. Let

$$J(x) = \int_a^b \frac{1}{2}((x_1'(t))^2 + (x_2'(t))^2)\, dt.$$

Then $L(t, x_1, x_2, x_1', x_2') = \frac{1}{2}((x_1')^2 + (x_2')^2)$. The Euler-Lagrange equation for J is computed as follows:

$$\frac{\partial L}{\partial x_i} = 0 = \left(\frac{\partial L}{\partial x_i'}\right)' = (x_i')' = x_i''.$$

In other words, the Euler-Lagrange equation for J is $x_i'' = 0$ for $i = 1,2$. So $x(t) = c_0 + c_1 t$ for some $c_0, c_1 \in \mathbb{R}^2$, i.e., x is a straight line. Since $x(a) = p_0$ and $x(b) = q_0$, $x(t) = p_0 + \frac{(t-a)}{b-a}(q_0 - p_0)$.

Example 5.0.10. Newtonian mechanics in $\mathbb{R}^2$

We consider a particle p of mass m moving in the plane $\mathbb{R}^2$ under Newton's Laws of Motion. In physics it is shown that Newton's Third Law implies that the force $F = (F_1, F_2)$ acting on p is derivable from a potential V, i.e., there is a smooth function $V : \mathbb{R}^2 \to \mathbb{R}$ such that $F_i = -\frac{\partial V}{\partial x_i}$ for $i = 1, 2$ (i.e., $F = -\nabla V$). The kinetic energy function is

$$K(x') = \frac{m}{2}((x_1')^2 + (x_2')^2).$$

Define the Lagrange

$$L = K - V = \frac{m}{2}((x_1')^2 + (x_2')^2) - V(x_1, x_2).$$

The functional $\mathcal{L}(x) = \int_a^b L(x, x')\, dt$ is called the *action* in physics. Since

$$\frac{\partial L}{\partial x_i} = -\frac{\partial V}{\partial x_i}, \quad \frac{\partial L}{\partial x_i'} = m x_i',$$

the Euler-Lagrange equation is

$$-\frac{\partial V}{\partial x_i} = (m x_i')' = m x_i'',$$

or

$$m x_i'' = -\frac{\partial V}{\partial x_i} = F_i.$$

In other words, we have proved that the Euler-Lagrange equation for the action functional $\mathcal{L} = \int_a^b K - V dt$ is the Newton's equation $F = ma$, i.e., the force F is equal to the mass times the acceleration.

6. Initial value problem for second order ODEs

A system of second order ODEs is of the form

$$x_i'' = f_i(t, x_1, \dots, x_n, x_1', \dots, x_n'), \qquad 1 \le i \le n. \tag{6.0.10}$$

This can be solved using the existence and uniqueness of systems of first order ODE by introducting new variables: Consider the following system of first order ODE:

$$\begin{cases} x_i' = y_i, & 1 \le i \le n, \\ y_i' = f_i(t, x_1, \dots, x_n, y_1, \dots, y_n), & 1 \le i \le n. \end{cases} \tag{6.0.11}$$

It is easy to see that if $x(t) = (x_1(t), \dots, x_n(t))$ is a solution of (6.0.10), then $(x, y) = (x, x')$ is a solution of (6.0.11). Conversely, if (x, y) is a solution of (6.0.11), then x is a solution of (6.0.10). By the uniqueness of ODE, we know that given initial condition $p_0, q_0 \in \mathbb{R}^n$, there exists a unique solution (x, y) of (6.0.11) such that $x(0) = p_0$ and $y(0) = q_0$. Thus given $p_0, q_0 \in \mathbb{R}^n$, there exists a unique solution $x(t)$ of the second order system (6.0.10) such that $x(0) = p_0$ and $x'(0) = q_0$. In other words, to solve the initial value problem for the second order ODE system (6.0.10) we need to give the initial position $x(0)$ and initial velocity $x'(0)$.

7. Energy functional

Let $f : \mathcal{O} \to \mathbb{R}^3$ be a parametrized surface, and $\mathrm{I} = \sum_{i,j=1}^{2} g_{ij} dx_i dx_j$ the first fundamental form. The energy functional $\mathcal{E} : C^1([a,b], \mathcal{O})_{p_0,q_0} \to \mathbb{R}$ is

$$\mathcal{E}(x) = \int_a^b \sum_{i,j=1}^{2} g_{ij}(x_1(t), x_2(t)) x_i'(t) x_j'(t)\ dt,$$

i.e.,

$$\mathcal{E}(x) = \int_a^b ||\gamma'(t)||^2 dt, \quad \text{where } \gamma(t) = f(x(t)).$$

We want to compute the Euler-Lagrange equation for $\mathcal{E}$. Before we do the general computation, let us do the following simple example:

Example 7.0.11. (**Energy functional on** S^2) We use the spherical parametrization of S^2:

$$f(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi).$$

Then

$$\mathrm{I} = d\phi^2 + \sin^2\phi\ d\theta^2.$$

The energy Lagrangian for S^2 is $E = (\phi')^2 + \sin^2\phi\ (\theta')^2$. Note

$$\begin{aligned}
&\frac{\partial E}{\partial \phi} = 2\sin\phi\cos\phi\ (\theta')^2, \\
&\left(\frac{\partial E}{\partial \phi'}\right)' = (2\phi')' = 2\phi'', \\
&\frac{\partial E}{\partial \theta} = 0, \\
&\left(\frac{\partial E}{\partial \theta'}\right)' = (2\sin^2\phi\ \theta')' = 4\sin\phi\cos\phi\ \phi'\ \theta' + 2\sin^2\phi\ \theta''.
\end{aligned}$$

So the E-L equation is

$$\begin{cases} \phi'' = \sin\phi\cos\phi(\theta')^2, \\ 4\sin\phi\cos\phi\ \phi'\theta' + 2\sin^2\phi\ \theta'' = 0, \end{cases}$$

i.e.,

$$\begin{cases} \phi'' = \sin\phi\cos\phi\ (\theta')^2, \\ \theta'' = -2\frac{\cos\phi}{\sin\phi}\ \phi'\theta'. \end{cases}$$

Next we compute the E-L for the energy functional for an arbitrary surface f. Since $E = \sum_{ij} g_{ij}(x_1, x_2) x_i' x_j'$,

$$\frac{\partial E}{\partial x_k} = \sum_k g_{ij,k} x_k' x_i' x_j',$$

$$\left(\frac{\partial E}{\partial x_k'}\right)' = \left(2 \sum_i g_{ik} x_i'\right)' = 2 \left(\sum_{i,m} g_{ik,m} x_m' x_i' + \sum_i g_{ik} x_i''\right),$$

where $g_{ij,k} = \frac{\partial g_{ij}}{\partial x_k}$. So the E-L equation is

$$2 \sum_i g_{ik} x_i'' + 2 \sum_{i,m} g_{ik,m} x_i' x_m' - \sum_{i,j} g_{ij,k} x_i' x_j' = 0.$$

But

$$\sum_{i,m} g_{ik,m} x_i' x_m' = \sum_{i,j} g_{ik,j} x_i' x_j' = \sum_{i,j} g_{jk,i} x_j' x_i'.$$

(Here we replace m by j to get the first identity, and then exchange i, j to get the second identity). So

$$2 \sum_{i,m} g_{ik,m} x_i' x_m' = \sum_{i,j} (g_{ik,j} + g_{jk,i}) x_i' x_j',$$

and hence the E-L equation for $\mathcal{E}$ can be written as

$$2 \sum_i g_{ik} x_i'' + \sum_{i,j} (g_{ki,j} + g_{jk,i} - g_{ij,k}) x_i' x_j' = 0,$$

i.e.,

$$\sum_i g_{ik} x_i'' + \frac{1}{2} \sum_{i,j} (g_{ki,j} + g_{jk,i} - g_{ij,k}) x_i' x_j' = 0.$$

Recall that we used the following notation

$$[ij, k] := g_{ki,j} + g_{jk,i} - g_{ij,k}$$

when we derived of the G-C equation in general coordinates in 162A. Use this notation, the E-L equation for the energy functional can be written as

$$\sum_i g_{ik} x_i'' + \frac{1}{2} \sum_{i,j} [ij, k] x_i' x_j' = 0.$$

We want to write this system of ODE in the form of (6.0.11). We can do this as follows: Let G denote the 2×2 matrix (g_{ij}), and g^{ij} the ij-th entry of the inverse matrix G^{-1}. Multiply the above equation by $g^{k\ell}$ then sum over k to get

$$\sum_{k,i} g^{k\ell} g_{ik} x_i'' + \frac{1}{2} \sum_{i,j,k} g^{k\ell} [ij, k] x_i' x_j' = 0.$$

Note that G and G^{-1} are symmetric matrices, and the $i\ell$-th entry of GG^{-1} is $\sum_k g_{ik}g^{k\ell}$. But $GG^{-1} = \mathrm{I}$, so $\sum_k g_{ik}g^{k\ell} = \delta_{i\ell}$. Then we get

$$\sum_i \delta_{i\ell} x_i'' + \frac{1}{2} \sum_{i,j,k} g^{k\ell}[ij,k]x_i' x_j' = 0,$$

Recall that we use the following notation in the G-C equation:

$$\Gamma_{ij}^{\ell} := \frac{1}{2} \sum_k g^{k\ell}[ij,k].$$

So the E-L equation for the energy functional can be written as

$$x_\ell'' + \sum_{i,j} \Gamma_{ij}^{\ell} x_i' x_j' = 0, \qquad \ell = 1, 2. \tag{7.0.12}$$

We summarize what we have proved below:

Theorem 7.0.12. *$x : [a,b] \to \mathbb{R}^2$ is a critical point of the energy functional $\mathcal{E}$ if and only if x satisifes* (7.0.12)*.*

Next we want to prove that if x is a critical point of the energy functional for the surface $f : \mathcal{O} \to \mathbb{R}^3$, then the curve $\gamma(t) = f(x(t))$ is travelled at constant speed:

Theorem 7.0.13. *If x is a critical point of the energy functional $\mathcal{E}$, then*

$$E(x) = \sum_{i,j} g_{ij}(x(t))x_i'(t)x_j'(t) = ||\gamma'(t)||^2$$

is a constant, where $\gamma(t) = f(x(t))$.

Proof. It suffices to prove that $(\sum_{i,j} g_{ij}(x)x_i'x_j')' = 0$. But

$$(\sum_{i,j} g_{ij}x_i'x_j')' = \sum_{i,j,k} g_{ij,k}x_k'x_i'x_j' + \sum_{i,j} g_{ij}x_i''x_j' + \sum_{i,j} g_{ij}x_i'x_j''.$$

But x satisfies the E-L equation

$$x_k'' + \sum_{i,j} \Gamma_{ij}^k x_i' x_j' = 0.$$

So

$$
\begin{aligned}
&\left(\sum_{i,j} g_{ij}x'_i x'_j\right)' \\
&= \sum_{i,j,k} g_{ij,k}x'_k x'_i x'_j - \sum_{i,j,k,m} g_{ij}\Gamma^i_{k,m}x'_k x'_m x'_j - \sum_{i,j} g_{ij}x'_i \sum_{k,m}\Gamma^j_{km}x'_k x'_m \\
&= \sum_{i,j,k} g_{ij,k}x'_i x'_j x'_k - \sum_{i,j,k,m} g_{ij}\Gamma^i_{km}x'_k x'_m x'_j - g_{ij}\Gamma^j_{km}x'_i x'_k x'_m \\
&= \sum_{i,j,k} g_{ij,k}x'_i x'_j x'_k - \sum_{i,j,k,m} g_{mj}\Gamma^m_{ki}x'_k x'_i x'_j - g_{im}\Gamma^m_{kj}x'_i x'_j x'_k \\
&= \sum_{i,j,k}\left(g_{ij,k} - \frac{1}{2}\sum_{m,\ell} g_{mj}g^{m\ell}[ki,\ell] - \frac{1}{2}\, g_{mi}g^{m\ell}[kj,\ell]\right) x'_i x'_j x'_k \\
&= \sum_{i,j,k}(g_{ij,k} - \frac{1}{2}\sum_{\ell}\delta_{j\ell}[ki,\ell] - \frac{1}{2}\,\delta_{i\ell}[kj,\ell])x'_i x'_j x'_k \\
&= \sum_{i,j,k}(g_{ij,k} - \frac{1}{2}[ki,j] - \frac{1}{2}[kj,i])x'_i x'_j x'_k \\
&= \sum_{i,j,k}(g_{ij,k} - \frac{1}{2}(g_{ij,k} + g_{jk,i} - g_{ki,j} + g_{ji,k} + g_{ik,j} - g_{kj,i}))x'_i x'_j x'_k = 0.
\end{aligned}
$$

(From the second line to the third line, we interchange m, i of the second term and m, j of the third term.) □

8. **Arc length functional**

Let $f : \mathcal{O} \to \mathbb{R}^3$ be a parametrized surface, $\Sigma = f(\mathcal{O})$, and $\mathrm{I} = \sum_{ij} g_{ij}dx_i dx_j$ the first fundamental of f. A smooth curve $x : [a,b] \to \mathcal{O}$ gives rise to a curve $\gamma = f \circ x$ on the surface Σ, and the arc length of γ is

$$\mathcal{L}(x) = \int_a^b ||\gamma'(t)||\ dt = \int_a^b \sqrt{\sum_{i,j=1}^{2} g_{ij}(x(t))x'_i(t)x'_j(t)}\ dt.$$

A curve $\gamma = f \circ x$ on Σ is called a *geodesic* if x is a critical point of the arc length functional $\mathcal{L}$. Let $E = \sum_{i,j=1}^{2} g_{ij}(x(t))x'_i(t)x'_j(t)$ denote the Lagrangian of the energy functional $\mathcal{E}$. The E-L equation for the energy functional $\mathcal{E}$ is

$$\frac{\partial E}{\partial x_i} = \left(\frac{\partial E}{\partial x'_i}\right)', \quad i = 1, 2. \tag{8.0.13}$$

Let L denote the Lagrangian for the arc length functional $\mathcal{L}$. Then $L = \sqrt{E}$. The E-L equation for $\mathcal{L}$ is

$$\frac{\partial \sqrt{E}}{\partial x_i} = \left(\frac{\partial \sqrt{E}}{\partial x_i'} \right)',$$

so the E-L equation for $\mathcal{L}$ is

$$\frac{1}{2\sqrt{E}} \frac{\partial E}{\partial x_i} = \left(\frac{1}{2\sqrt{E}} \frac{\partial E}{\partial x_i'} \right)', \quad i = 1, 2. \tag{8.0.14}$$

Proposition 8.0.14. *Let $f : \mathcal{O} \to \mathbb{R}^3$ be a parametrized surface. If x is a critical point of the energy functional $\mathcal{E}$, then x is a critical point of the arc length functional $\mathcal{L}$. Conversely, if x is a critical point of $\mathcal{L}$ such that $||\frac{d}{dt} f(x(t))||$ is constant, then x is a critical point of $\mathcal{E}$.*

Proof. By Theorem 7.0.13, $E = \sum_{i,j=1}^{2} g_{ij}(x(t)) x_i'(t) x_j'(t)$ is a constant c. Since x satisfies (8.0.13) and E is constant, x satisfies (8.0.14), so x is a critical point of $\mathcal{L}$. The converse is proved the same way. □

We want to prove below that if $x : [a, b] \to \mathcal{O}$ is a critical point of $\mathcal{L}$ and $t = t(s)$ is a diffeomorphism from $[a, b]$ to $[a, b]$, then $\tilde{x}(s) = x(t(s))$ is also a critical point of $\mathcal{L}$. This follows from the following theorem:

Theorem 8.0.15. *Let V be a vector space, $\phi : V \to V$ a diffeomorphism, and $F : V \to \mathbb{R}$ a smooth function. Suppose $F \circ \phi^{-1} = F$. Then $p_0 \in V$ is a critical point of F implies that $\phi(p_0)$ is also a critical point of F.*

Proof. Note that p_0 is a critical point of F if and only if $\frac{d}{dt}\big|_{t=0} F(x(t)) = 0$ for all smooth curve $x : (-\delta, \delta) \to V$ such that $x(0) = p_0$. To prove that $\phi(p_0)$ is a critical point, let $y : (-\delta, \delta) \to V$ be a smooth curve such that $y(0) = \phi(p_0)$. Then $x = \phi^{-1} \circ y$ is a curve with $x(0) = p_0$. But

$$F(y(t)) = F(\phi^{-1}(y(t)) = F(x(t)),$$

so

$$\left.\frac{d}{dt}\right|_{t=0} F(y(t)) = \left.\frac{d}{dt}\right|_{t=0} F(x(t)),$$

which is zero because p_0 is a critical point of F and x is a curve with $x(0) = p_0$. This proves that $\phi(p_0)$ is also a critical point of F. □

The condition $F = F \circ \phi^{-1}$ is equivalent to $F = F \circ \phi$ because $F(\phi^{-1}(\phi(x))) = F(\phi(x))$ implies that $F(x) = F(\phi(x))$.

The above Theorem says that if a function F is invariant under a transformation ϕ, then $\phi(p_0)$ is a critical point of F if p_0 is.

Proposition 8.0.16. *Let $t = t(s)$ is a diffeomorphism from $[a, b]$ to $[a, b]$, and $\phi : C([a, b], \mathcal{O}) \to C([a, b], \mathcal{O})$ defined by $\phi(x)(s) = x(t(s))$. Then if x^0 is a critical point of $\mathcal{L}$, then $\phi(x^0)$ is also a critical point of $\mathcal{L}$.*

Proof. Note that $\gamma(t) = f(x(t))$ and $\tilde{\gamma}(s) = f(x(t(s)))$ trace out the same curve on the surface Σ. So the arc length of γ and $\tilde{\gamma}$ are the same. This proves that $\mathcal{L}(x) = \mathcal{L}(\phi(x))$. By Theorem 8.0.15, $\phi(x^0)$ is a critical point of $\mathcal{L}$. □

Suppose x^0 is a critical point of $\mathcal{L}$ and the arc length of $\gamma^0 = f(x^0)$ is ℓ. We claim that we can change parameter of γ^0 to a new parameter s such that $||\frac{d\gamma^0}{ds}||$ is the constant $\frac{\ell}{b-a}$. But

$$\frac{\ell}{b-a} = ||\frac{d\gamma^0}{ds}|| = ||\frac{d\gamma^0}{dt}\frac{dt}{ds}|| = ||\frac{d\gamma^0}{dt}||\ |\frac{dt}{ds}|,$$

so if we choose the new parameter s such that

$$\frac{dt}{ds} = \frac{\ell}{(b-a)||\frac{d\gamma^0}{dt}||},$$

then $||\frac{d\gamma^0}{ds}|| = \frac{\ell}{b-a}$. This proves the claim. By Proposition 8.0.16, $y^0(s) = x(t(s))$ is a critical point of $\mathcal{L}$. But $||\frac{d}{ds}f(y^0(s))||$ is constant, so by Proposition 8.0.14, y^0 is a critical point of $\mathcal{E}$. This shows that to construct all geodesics of the surface Σ, it suffices to solve the E-L equation (7.0.12) for the energy functional $\mathcal{E}$. Therefore we will call (7.0.12) the *geodesic equation*.

A curve $\gamma : [a, b] \to \mathbb{R}^3$ is said to be *parametrized proportional to its arc length* if its speed is constant, i.e., $||\frac{d\gamma}{dt}||$ is a constant for all $t \in [a, b]$.

We like to give a geometric calculation for the geodesic equation. Recall that we proved in 162A that if $f : \mathcal{O} \to \mathbb{R}^3$ is a parametrized surface with $\mathrm{I} = \sum_{ij} g_{ij}dx_idx_j$ and $\mathrm{II} = \sum_{ij} \ell_{ij}dx_idx_j$, then

$$f_{x_ix_j} = \Gamma^1_{ij}f_{x_1} + \Gamma^2_{ij}f_{x_2} + \ell_{ij}N, \tag{8.0.15}$$

where $\Gamma^i_{jk} = \frac{1}{2}\sum_m g^{im}[jk, m]$, and N the unit normal of f. We will use (8.0.15) to give a simple criterion for a curve $\gamma = f \circ x$ on the surface f to be a geodesic.

Theorem 8.0.17. *Let $\gamma = f \circ x$ be a curve on the surface f that is parametrized proportional to its arc length. Then γ is a geodesic if and only if $\gamma''(t)$ is normal to f at $\gamma(t)$ for all t. Moreover, if γ is a geodesic then*

$$\gamma''(t) = \mathrm{II}(\gamma'(t), \gamma'(t))\ N(\gamma(t)).$$

Proof. Chain rule gives $\gamma' = \sum_i f_{x_i} x_i'$, and

$$\begin{aligned}
\gamma''(t) &= \sum_{i,j} f_{x_i x_j} x_i' x_j' + \sum_i f_{x_i} x_i'' \\
&= \sum_{i,j} \left(\left(\sum_k \Gamma_{ij}^k x_i' x_j' \ f_{x_k} \right) + \ell_{ij} x_i' x_j' \ N \right) + \sum_i f_{x_i} x_i'' \\
&= \sum_{i,j} \left(\left(\sum_k \Gamma_{ij}^k x_i' x_j' \ f_{x_k} \right) + \ell_{ij} x_i' x_j' \ N \right) + \sum_k f_{x_k} x_k'' \\
&= \sum_k \left(x_k'' + \sum_{ij} \Gamma_{ij}^k x_i' x_j' \right) f_{x_k} + \sum_{i,j} \ell_{ij} x_i' x_j' \ N.
\end{aligned}$$

So $\gamma''(t)$ is normal if and only if

$$x_k'' + \sum_{i,j} \Gamma_{ij}^k x_i' x_j' = 0$$

for $k = 1, 2$, i.e., x satisfies (7.0.12), or equivalently, γ is a geodesic. □

Example 8.0.18. We can use Theorem 8.0.17 to get geodesics of the unit sphere in $\mathbb{R}^3$ easily by observing that if α is a great circle, then α'' is the radial vector of the great circle, so $\alpha''(t)$ is the unit normal to S^2 at $\alpha(t)$. This implies that $\alpha''(t)$ is normal to S^2, by Theorem 8.0.17, α is a geodesic.

9. Calculus of variations of two variable

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^2$ such that the boundary $\partial\mathcal{O}$ is a smooth curve, i.e., there is a smooth parametrization $\alpha : [a, b] \to \mathbb{R}^2$ for $\partial\mathcal{O}$. Let $\gamma : \partial\mathcal{O} \to \mathbb{R}$ a fixed smooth function, and $\overline{\mathcal{O}} = \mathcal{O} \cup \partial\mathcal{O}$. Let $C_\gamma(\overline{\mathcal{O}}, \mathbb{R})$ denote the space of smooth functions $u : \overline{\mathcal{O}} \to \mathbb{R}$ such that $u \mid \partial\mathcal{O} = \gamma$. A two variable *Lagrangian* is a smooth function $L : \overline{\mathcal{O}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and the variational functional associated to the Lagrangian L is the map $J = J^L : C_\gamma(\overline{\mathcal{O}}, \mathbb{R}) \to \mathbb{R}$ defined by

$$J(u) = \int\int_{\overline{\mathcal{O}}} L(x, y, u(x, y), u_x(x, y), u_y(x, y)) \ dxdy.$$

We will use x, y, u, p, q to denote the variables of L, i.e., $L = L(x, y, u, p, q)$. A function $u^0 : \overline{\mathcal{O}} \to \mathbb{R}$ is called a *critical point* of J if all the directional derivatives of J at u^0 is zero, i.e.,

$$\left. \frac{d}{ds} \right|_{s=0} J(u^0 + sh) = 0$$

for all smooth $h : \overline{\mathcal{O}} \to \mathbb{R}$ with $h \mid \partial\mathcal{O} = 0$. We will calculate the same way as in the calculus of variations of one variable. Recall that in that calculation, we need the fundamental lemma and the Fundamental Theorem of Calculus of one variable. As we will see below that to calculate the condition that u^0 is a critical point, we will need the two dimensional versions of the fundamental lemma and of the Fundamental Theorem of Calculus.

Given $a \in \mathbb{R}$, let $\phi_a : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\phi_a(x) = \begin{cases} \exp(-(x-a)^{-2}), & x > a, \\ 0, & x \le a. \end{cases}$$

Exercise 9.0.4. Prove that $\lim_{x \to a^+} \phi_a(x) = 0$, and the limits of derivatives of ϕ_a of any order tends to zero as $x \to a^+$. (This proves that ϕ_a is a smooth function).

Exercise 9.0.5. Let $\psi_a : \mathbb{R} \to \mathbb{R}$ be the map defined by $\psi_a(x) = \phi_{-a}(-x)$. Prove that

$$\psi_a(x) = \begin{cases} e^{-(x-a)^{-2}}, & x < a, \\ 0, & x \ge a. \end{cases}.$$

Exercise 9.0.6. Given $a < b$, let $h_{a,b} : \mathbb{R} \to \mathbb{R}$ be the function defined by $h_{a,b}(x) = \phi_a(x)\psi_b(x)$. Prove that $h_{a,b}$ is smooth, $h_{a,b} > 0$ on (a,b) and is zero outside (a,b).

Lemma 9.0.19. *Suppose $f : \overline{\mathcal{O}} \to \mathbb{R}$ is continuous, and*

$$\int\int_{\overline{\mathcal{O}}} f(x,y)h(x,y)\ dxdy = 0$$

for all $h : \overline{\mathcal{O}} \to \mathbb{R}$ with $h \mid \partial\mathcal{O} = 0$. Then $f = 0$.

Proof. Suppose f is not identically zero. Then there exists $(x_0, y_0) \in \mathcal{O}$ such that $f(x_0, y_0) \neq 0$. We may assume that $f(x_0, y_0) > 0$. Since f is continuous, there exists $\epsilon > 0$ such that $f(x,y) > 0$ for all (x,y) in the square

$$D = \{(x,y) \mid |x - x_0| < \epsilon, |y - y_0| < \epsilon\}.$$

Let $h_{a,b}$ be the smooth function constructed in the above Exercise, and $k_D : \mathbb{R}^2 \to \mathbb{R}$ the map defined by

$$k_D(x,y) = h_{x_0-\epsilon, x_0+\epsilon}(x) h_{y_0-\epsilon, y_0+\epsilon}(y).$$

It is easy to see that k_D is smooth, $k_D(x,y) > 0$ if $(x,y) \in D$ and is zero outside D. In particular, $h_D \mid \partial\mathcal{O} = 0$. But fh_D is positive in $\mathcal{O}$, so

$$\int\int_{\overline{\mathcal{O}}} fh_D\ dxdy = \int\int_{\bar{D}} fh_D\ dxdy,$$

which is positive because both f and h_D are positive on D, a contradiction. ∎

Next we recall the Green's formula, which can be viewed as the two dimensional version of the Fundamental Theorem of Calculus. First recall that the line integral

$$\oint_{\partial\mathcal{O}} P(x,y)\ dx + Q(x,y)\ dy$$

is computed as follows: Choose a parametrization of the boundary $\partial\mathcal{O}$, say $\alpha : [a,b] \to \mathbb{R}^2$ with $\alpha(t) = (x(t), y(t))$. Then

$$\oint_{\partial\mathcal{O}} P(x,y)\ dx + Q(x,y)\ dy = \int_a^b P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)\ dt.$$

Theorem 9.0.20. *(***Green's formula***)*
Let $P, Q : \overline{\mathcal{O}} \to \mathbb{R}$ be smooth functions. Then

$$\oint_{\partial\mathcal{O}} P\ dx + Q\ dy = \int\int_{\overline{\mathcal{O}}} (-P_x + Q_y)\ dxdy.$$

Corollary 9.0.21. *If $P \mid \partial\mathcal{O} = 0$ and $Q \mid \partial\mathcal{O} = 0$, then*

$$\int\int_{\overline{\mathcal{O}}} (-P_x + Q_y)\ dxdy = 0.$$

(This is because the line integral is zero).

Next we compute the directional derivative of J at u in the direction h, where $u \in C_\gamma(\overline{\mathcal{O}}, \mathbb{R})$ and $h : \overline{\mathcal{O}} \to \mathbb{R}$ satisfying $h \mid \partial\mathcal{O} = 0$.

$$\begin{aligned}
&\frac{d}{ds}\Big|_{s=0} J(u+sh) = \frac{d}{ds}\Big|_{s=0} \int\int_{\overline{\mathcal{O}}} L(x,y,u+sh,(u+sh)_x,(u+sh)_y)\ dxdy \\
&= \frac{d}{ds}\Big|_{s=0} \int\int_{\overline{\mathcal{O}}} L(x,y,u+sh,u_x+sh_x,u_y+sh_y)\ dxdy \\
&= \int\int_{\overline{\mathcal{O}}} \frac{d}{ds}\Big|_{s=0} L(x,y,u+sh,u_x+sh_x,u_y+sh_y)\ dxdy \\
&= \int\int_{\overline{\mathcal{O}}} \frac{\partial L}{\partial u}(x,y,u,u_x,u_y)h(x,y) + \frac{\partial L}{\partial p}(x,y,u,u_x,u_y)h_x(x,y) \\
&\qquad + \frac{\partial L}{\partial q}(x,y,u,u_x,u_y)h_y(x,y)\ dxdy. \\
&= \int\int_{\overline{\mathcal{O}}} \frac{\partial L}{\partial u}h + \left(\frac{\partial L}{\partial p}h\right)_x - \left(\frac{\partial L}{\partial p}\right)_x h\ dxdy \\
&\qquad + \left(\frac{\partial L}{\partial q}h\right)_y - \left(\frac{\partial L}{\partial q}\right)_y h\ dxdy.
\end{aligned}$$

The Green's formula implies that

$$\int\int_{\overline{\mathcal{O}}} \left(\frac{\partial L}{\partial p}h\right)_x + \left(\frac{\partial L}{\partial q}h\right)_y\ dxdy = \oint_{\partial\mathcal{O}} \frac{\partial L}{\partial p}h\ dy - \frac{\partial L}{\partial q}h\ dx.$$

Since $h \mid \partial\mathcal{O} = 0$, by Corollary 9.0.21 the right hand side is zero. So we get

$$\frac{d}{ds}\bigg|_{s=0} J(u+sh) = \int\int_{\overline{\mathcal{O}}} \left(\frac{\partial L}{\partial u} - \left(\frac{\partial L}{\partial p}\right)_x - \left(\frac{\partial L}{\partial q}\right)_y \right) h \; dxdy.$$

Here

$$\left(\frac{\partial L}{\partial p}\right)_x = \left(\frac{\partial L}{\partial p}(x,y,u,u_x,u_y)\right)_x,$$
$$\left(\frac{\partial L}{\partial q}\right)_y = \left(\frac{\partial L}{\partial q}(x,y,u,u_x,u_y)\right)_y.$$

We will use the conventional notation $\frac{\partial L}{\partial u_x}$ to denote $\frac{\partial L}{\partial p}$ and $\frac{\partial L}{\partial u_y}$ to denote $\frac{\partial L}{\partial q}$. By Lemma 9.0.19, we get

Theorem 9.0.22. *u is a critical point of J if and only if*

$$\frac{\partial L}{\partial u} - \left(\frac{\partial L}{\partial u_x}\right)_x - \left(\frac{\partial L}{\partial u_y}\right)_y = 0. \tag{9.0.16}$$

Equation (9.0.16) is called the *Euler-Lagrange equation* for $J = J^L$.

Example 9.0.23. Let

$$J(u) = \int\int_{\overline{\mathcal{O}}} \frac{1}{2}\left((u_x)^2 + (u_y)^2\right) \; dxdy.$$

Then $L(x,y,u,p,q) = \frac{1}{2}(p^2+q^2)$,

$$\frac{\partial L}{\partial u} = 0, \quad \frac{\partial L}{\partial p} = p, \quad \frac{\partial L}{\partial q} = q.$$

To get the E-L equation, we need to substitute $p = u_x$ and $q = u_y$, so $0 - (u_x)_x - (u_y)_y = 0$, i.e., the E-L equation is the *Laplace equation*

$$u_{xx} + u_{yy} = 0.$$

Example 9.0.24. Let

$$J(u) = \int\int_{\overline{\mathcal{O}}} u_x^2 + u_x u_y + u_y^2 - \cos u + (x^2+y^2)u \; dxdy.$$

So $L(x,y,u,p,q) = p^2 + pq + q^2 - \cos u + (x^2+y^2)u$, and

$$\frac{\partial L}{\partial u} = \sin u - (x^2+y^2), \quad \frac{\partial L}{\partial p} = 2p + q, \quad \frac{\partial L}{\partial q} = p + 2q.$$

Substitue $p = u_x$ and $q = u_y$ to see that the E-L equation is

$$\begin{aligned} 0 &= \sin u - (x^2+y^2) - (2u_x + u_y)_x - (u_x + 2u_y)_y = 0 \\ &= \sin u - (x^2+y^2) - (2u_{xx} + u_{yx} + u_{xy} + 2u_{yy}) \\ &= \sin u - (x^2+y^2) - (2u_{xx} + 2u_{xy} + 2u_{yy}). \end{aligned}$$

Example 9.0.25. Let
$$J(u) = \int\int_{\overline{\mathcal{O}}} (1 + u_x^2 + u_y^2)^{\frac{1}{2}}.$$
So $L(x, y, u, p, q) = (1 + p^2 + q^2)^{\frac{1}{2}}$ and
$$\frac{\partial L}{\partial u} = 0, \quad \frac{\partial L}{\partial p} = p(1 + p^2 + q^2)^{-\frac{1}{2}}, \quad \frac{\partial L}{\partial q} = q(1 + p^2 + q^2)^{-\frac{1}{2}}.$$
The E-L equation is
$$\begin{aligned} 0 &= \frac{\partial L}{\partial u} - \left(\frac{\partial L}{\partial p}\right)_x - \left(\frac{\partial L}{\partial q}\right)_y \\ &= 0 - (u_x(1 + u_x^2 + u_y^2)^{-\frac{1}{2}})_x - (u_y(1 + u_x^2 + u_y^2)^{-\frac{1}{2}})_y \\ &= (1 + u_x^2 + u_y^2)^{-3/2} \left((1 + u_y^2)u_{xx} + (1 + u_x^2)u_{yy} - 2u_x u_y u_{xy}\right). \end{aligned}$$

10. Area functional

Let $u : \overline{\mathcal{O}} \to \mathbb{R}$ be a smooth function, and
$$f(x, y) = (x, y, u(x, y))$$
the graph of u. We have seen in 162A that the two fundamental forms for f are
$$\begin{aligned} \mathrm{I} &= (1 + u_x^2)\ dx^2 + 2u_x u_y\ dxdy + (1 + u_y^2)\ dy^2, \\ \mathrm{II} &= \frac{1}{\sqrt{1 + u_x^2 + y_y^2}} \left(u_{xx}\ dx^2 + 2u_{xy}\ dxdy + u_{yy}\ dy^2,\right) \end{aligned}$$
and the mean curvature is
$$H = \frac{(1 + u_x^2)u_{yy} - 2u_x u_y u_{xy} + (1 + u_y^2)u_{xx}}{(1 + u_x^2 + u_y^2)^{3/2}}.$$
Compute directly to see that
$$\det(g_{ij}) = (1 + u_x^2)(1 + u_y^2) - (u_x u_y)^2 = (1 + u_x^2 + u_y^2).$$
So the area of the graph of u (i.e., surface f) is
$$J(u) = \int\int_{\overline{\mathcal{O}}} \sqrt{\det(g_{ij})}\ dxdy = \int\int_{\overline{\mathcal{O}}} (1 + u_x^2 + u_y^2)^{1/2}\ dxdy.$$
We have computed the E-L equation for this functional in Example 9.0.25 and see that
$$\begin{aligned} \left.\frac{d}{ds}\right|_{s=0} J(u + sh) &= \int\int_{\overline{\mathcal{O}}} -\frac{(1 + u_x^2)u_{yy} - 2u_x u_y u_{xy} + (1 + u_y^2)u_{xx}}{(1 + u_x^2 + u_y^2)^{3/2}} h\ dxdy \\ &= \int\int_{\overline{\mathcal{O}}} -Hh\ dxdy. \end{aligned}$$

So the E-L equation for the area functional is $H = 0$. This gives another geometric meaning of the mean curvature. In particular, the surface that has minimum surface area among all surfaces in $\mathbb{R}^3$ with a fixed boundary must have zero mean curvature.

Definition 10.0.26. A surface in $\mathbb{R}^3$ is *minimal* if its mean curvature is zero.

Remark. It follows from physics that if we dip a closed wire frame into a soap solution, then the soap film surface spanned by the wire frame when we take out the frame must be minimal. So we sometimes call minimal surfaces soap films.

11. Minimal surfaces

11.1. Minimal surfaces and harmonic functions.

Definition 11.1.1. A surface $f : \mathcal{O} \to \mathbb{R}^3$ is said to be parametrized by *isothermal coordinates* if $\mathrm{I} = \lambda^2(x_1, x_2)(dx_1^2 + dx_2^2)$, i.e., $g_{11} = g_{22}$ and $g_{12} = 0$. Or equivalently,

$$f_{x_1} \cdot f_{x_1} = f_{x_2} \cdot f_{x_2}, \quad f_{x_1} \cdot f_{x_2} = 0.$$

Theorem 11.1.2. *Suppose $f : \mathcal{O} \to \mathbb{R}^3$ is parametrized by isothermal coordinates with $\mathrm{I} = \lambda^2(dx_1^2 + dx_2^2)$. Then*

$$f_{x_1x_1} + f_{x_2x_2} = \lambda^2 HN, \tag{11.1.1}$$

where H is the mean curvature and N is the unit normal vector field of f.

Proof. First we want to use the isothermal condition, $f_{x_1} \cdot f_{x_1} = f_{x_2} \cdot f_{x_2} = \lambda^2$ and $f_{x_1} \cdot f_{x_2} = 0$, to conclude that

$$(f_{x_1x_1} + f_{x_2x_2}) \cdot f_{x_i} = 0, \quad i = 1, 2,$$

which implies that $f_{x_1x_1} + f_{x_2x_2}$ is parallel to the unit normal N of the surface. To do this, we take derivatives of the isothermal condition:

$$\begin{aligned}(f_{x_1} \cdot f_{x_1})_{x_1} &= 2f_{x_1x_1} \cdot f_{x_1} = (f_{x_2} \cdot f_{x_2})_{x_1} = 2f_{x_1x_2} \cdot f_{x_2} \\ &= 2((f_{x_1} \cdot f_{x_2})_{x_2} - f_{x_1} \cdot f_{x_2x_2}) = 2(0 - f_{x_1} \cdot f_{x_2x_2}) = -2f_{x_2x_2} \cdot f_{x_1},\end{aligned}$$

so $(f_{x_1x_1} + f_{x_2x_2}) \cdot f_{x_1} = 0$. Similar calculation implies that

$$(f_{x_1x_1} + f_{x_2x_2}) \cdot f_{x_2} = 0,$$

thus $f_{x_1x_1} + f_{x_2x_2}$ must be parallel to N. But

$$f_{x_ix_j} = \Gamma^1_{ij} f_{x_1} + \Gamma^2_{ij} f_{x_2} + \ell_{ij} N,$$

where $\mathrm{II} = \ell_{11}dx_1^2 + 2\ell_{12}dx_1dx_2 + \ell_{22}dx_2^2$ and $\Gamma^i_{jk} = \frac{1}{2}\sum_m g^{im}[jk, m]$. Therefore we have

$$f_{x_1x_1} + f_{x_2x_2} = (\sum_i \Gamma^1_{ii}) f_{x_1} + (\sum_i \Gamma^2_{ii}) f_{x_2} + (\ell_{11} + \ell_{22})N.$$

But we have shown that $f_{x_1x_1} + f_{x_2x_2}$ is parallel to N, so the coefficients of f_{x_i} in $f_{x_1x_1} + f_{x_2x_2}$ are zero and we get

$$f_{x_1x_1} + f_{x_2x_2} = (\ell_{11} + \ell_{22})N. \tag{11.1.2}$$

Recall that the mean curvature H is given by the following formula

$$H = \frac{g_{11}\ell_{22} - 2g_{12}\ell_{12} + g_{22}\ell_{11}}{\det(g_{ij})},$$

so $H = \frac{\lambda^2(\ell_{11}+\ell_{22})}{\lambda^4} = \frac{\ell_{11}+\ell_{22}}{\lambda^2}$, which implies that $\ell_{11} + \ell_{22} = \lambda^2 H$. Substitute this into (11.1.2) to get (11.1.1). $\square$

Definition 11.1.3. A smooth function $u : \mathcal{O} \to \mathbb{R}$ is called *harmonic* if

$$u_{x_1x_1} + u_{x_2x_2} = 0.$$

The operator $\triangle = \frac{\partial^1}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is called the Laplace operator and $\triangle u = u_{x_1x_1} + u_{x_2x_2} = 0$ is called the *Laplace equation.*

Corollary 11.1.4. *Suppose $f = (u, v, w) : \mathcal{O} \to \mathbb{R}^3$ is parametrized by isothermal coordinates. Then the following statements are equivalent:*

(1) *f is minimal,*
(2) *$f_{x_1x_1} + f_{x_2x_2} = (0, 0, 0)$,*
(3) *u, v, w are harmonic functions.*

Remark. We have seen that a surface $f : \overline{\mathcal{O}} \to \mathbb{R}^3$ with $H = 0$ is a critical point of the area functional. But this surface need not have minimum area among all surfaces in $\mathbb{R}^3$ that having the same boudary as $f(\overline{\mathcal{O}})$. However, it is a theorem of PDE that if we take any small enough piece Ω of $f(\mathcal{O})$, the surface Ω has the minimum area among all surfaces that have the same boundary of Ω.

Example 11.1.5. (**Caternoid**)
Let $a > 0$ be a constant, and

$$f(x_1, x_2) = (a \cosh x_2 \cos x_1, a \cosh x_2 \sin x_1, ax_2).$$

A direct computation implies that $\mathrm{I} = a^2 \cosh^2 x_2(dx_1^2 + dx_2^2)$, so f is isothermal parametrization. Compute directly to see that $f_{x_1x_1} + f_{x_2x_2} = (0, 0, 0)$, so by Corollary 11.1.4 f is minimal.

Example 11.1.6. (**Helicoid**)
Let $a > 0$ be a constant, and

$$f(x_1, x_2) = (\sinh x_2 \cos x_1, \sinh x_2 \sin x_1, x_1).$$

A direct computation implies that $\mathrm{I} = a^2 \cosh^2 x_2(dx_1^2 + dx_2^2)$ and $f_{x_1x_1} + f_{x_2x_2} = (0, 0, 0)$. So f is minimal. Note that Helicoid and Caternoid have the same first fundamental form!

Example 11.1.7. (**Enneper's surface**)
Let

$$f(x_1, x_2) = (x_1 - \frac{x_1^3}{3} + x_1 x_2^2, x_2 - \frac{x_2^3}{3} + x_2 x_1^2, x_1^2 - x_2^2).$$

Then $\mathrm{I} = (1 + x_1^2 + x_2^2)(dx_1^2 + dx_2^2)$ and $\triangle f = (0, 0, 0)$. So f is minimal.

11.2. **Linear conformal transformations of $\mathbb{R}^2$.**

The plane $\mathbb{R}^2$ can be viewed as $\mathbb{C}$ by identifying $\begin{pmatrix} x \\ y \end{pmatrix}$ with the complex number $z = x + iy$. Let $\alpha = a + ib$ be a fixed complex constant, and $M_\alpha : \mathbb{C} \to \mathbb{C}$ the map defined by $M_\alpha(z) = \alpha z$, i.e.,

$$M_\alpha(x + iy) = (a + ib)(x + iy) = (ax - by) + i(bx + ay).$$

If we use the identification of $\mathbb{C}$ with $\mathbb{R}^2$, then the map M_α becomes $T_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$, where

$$T_\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In other words, the map M_α given by multiplication by α on $\mathbb{C}$ is the map T_α on $\mathbb{R}^2$, which is given by the multiplication by the matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. We can study the geometry of T_α using complex numbers: First write $\alpha = r_0 e^{i\theta_0}$ in polar coordinate, i.e., $r_0 = \sqrt{a^2 + b^2}$ and $\tan\theta_0 = \frac{b}{a}$. If $z = re^{i\theta}$, then $M_\alpha(z) = az = r_0 r e^{i(\theta_0 + \theta)}$. In other words, the map T_α maps a vector v by first rotating v counterclockwise by angle θ_0, then multiplying the length by the factor r_0.

Let $\measuredangle\ (v_1, v_2)$ denote the angle from v_1 to v_2. If $v_j = r_j e^{i\theta_j}$, then $\measuredangle\ (v_1, v_2) = \theta_2 - \theta_1$.

Exercise 11.2.1. Prove that if v_1, v_2 are orthonormal, then $T_\alpha(v_1) \cdot T_\alpha(v_2) = 0$ and $||T_\alpha(v_1)|| = ||T_\alpha(v_2)||$.

The linear operator T_α preserves angles and stretchs each vector by a fixed amont r_0. We call T_α a *linear conformal transformation* of $\mathbb{R}^2$.

Exercise 11.2.2. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear map defined by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Prove that if $\measuredangle\ (T(u), T(v)) = \measuredangle\ (u, v)$ and $||T(e_1)|| = T(e_2)||$, then $d = a$ and $c = -b$, i.e., T is a linear conformal transformation.

If $|\alpha| = 1$, i.e., $\alpha = e^{i\theta_0} = \cos\theta_0 + i\sin\theta_0$, then

$$T_\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta_0 - y\sin\theta_0 \\ x\sin\theta_0 + y\cos\theta_0 \end{pmatrix}$$

is the rotation by angle θ_0. We call

$$\begin{pmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{pmatrix}$$

the rotation matrix by angle θ_0.

11.3. **Analytic functions, definitions and examples.**

Definition 11.3.1. Let $\mathcal{O}$ be an open subset of $\mathbb{C}$.

(1) A map $f : \mathcal{O} \to \mathbb{C}$ is *analytic at* z_0 if

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

exists. We will use $f'(z_0)$ to denote the limit. The limit $f'(z_0)$ is called the *complex derivative* of f at z_0. Here the limit is taking for complex number $h \to 0$, i.e., given any $\epsilon > 0$, there exists $\delta > 0$ such that if $h \in \mathbb{C}$ and $|h| < \delta$ then $|f(z_0+h) - f(z_0)| < \epsilon$.

(2) The map $f : \mathcal{O} \to \mathbb{C}$ is *analytic* if f is analytic at every point $z_0 \in \mathcal{O}$.

Given a function $f : \mathcal{O} \to \mathbb{C}$, we can write $f(z) = u(x, y) + iv(x, y)$ with real valued functions u, v. We will call u and v the real and imaginary part of f, and write $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$.

Example 11.3.2. Let $f(z) = 2x + iy = \frac{3z+\bar{z}}{2}$. It is easy to see that

$$\frac{f(r+is) - f(0)}{r+is} = \frac{2r+is}{r+is} \to \begin{cases} 2, & \text{if } s = 0, r \to 0, \\ 1, & \text{if } r = 0, s \to 0. \end{cases}$$

So the limit of $\frac{f(h)-f(0)}{h}$ does not exist, i.e., f is not analytic.

Example 11.3.3.

(1) Let $f(z) = z$. Then

$$\frac{f(z+h) - f(z)}{h} = \frac{(z+h) - z}{h} = 1,$$

so $f'(z) = 1$.

(2) Let $f(z) = z^n$. Then

$$\begin{aligned} \frac{f(z+h) - f(z)}{h} &= \frac{(z+h)^n - z^n}{h} \\ &= \frac{z^n + nz^{n-1}h + \frac{n(n-1)}{2}\, z^{n-1}h^2 + \cdots + h^n - z^n}{h} \\ &= nz^{n-1} + \frac{n(n-1)}{2} z^{n-2}h + \cdots + h^{n-1} \to nz^{n-1} \quad \text{as } h \to 0. \end{aligned}$$

So $(z^n)' = nz^{n-1}$.

(3) Let $f(z) = e^z = e^{x+iy} := e^x e^{iy} = e^x(\cos x + i \sin x)$. Then $e^{z_1+z_2} = e^{z_1}e^{z_2}$. Since the Taylor series of e^x, $\cos y$, and $\sin y$ converges absolutely,

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n}.$$

So

$$\frac{e^h - 1}{h} = \frac{(1 + h + \frac{h^2}{2} + \cdots) - 1}{h} = 1 + \frac{h}{2} + \cdots,$$

which implies $\lim_{h\to 0} \frac{e^h-1}{h} = 1$. But

$$\frac{e^{z+h} - e^z}{h} = \frac{e^z e^h - e^z}{h} = \frac{e^z(e^h - 1)}{h},$$

hence its limit is e^z as $h \to 0$. Thus $(e^z)' = e^z$.

The addition formula, product formula, quotient formula, and the chain rule for analytic functions can be proved in a similar way as for calculus of one real variable.

We define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Since e^z is analytic, by the chain rule we have $(e^{iz})' = ie^{iz}$ and $(e^{-iz})' = -ie^{-iz}$.

Exercise 11.3.1. Prove that:

(1) $(\sin z)' = \cos z$ and $(\cos z)' = -\sin z$.
(2) $\cos^2 z + \sin^2 z = 1$,

Define

$$\cosh z = \frac{e^z + e^{-z}}{2}, \qquad \sinh z = \frac{e^z - e^{-z}}{2}.$$

Exercise 11.3.2. Prove that

(1) $\cosh^2 z - \sinh^2 z = 1$,
(2) $(\cosh z)' = \sinh z$, $(\sinh z)' = \cosh z$,
(3) $\cosh(iz) = \cos z$, $\sinh(iz) = i \sin z$.

Theorem 11.3.4. *Suppose $f : \mathcal{O} \to \mathbb{C}$ is analytic, and $\alpha, \beta : (-\delta, \delta) \to \mathcal{O}$ smooth curves intersect at $p_0 = \alpha(0) = \beta(0)$. Then*

$$\measuredangle\, (\alpha'(0), \beta'(0)) = \measuredangle\, ((f \circ \alpha)'(0), (f \circ \beta)'(0)),$$

i.e., f preserves angles.

Proof. By the chain rule, $(f\circ\alpha)'(0) = f'(\alpha(0))\alpha'(0) = f'(p_0)\alpha'(0)$. Similarly, $(f \circ \beta)'(0) = f'(p_0)\beta'(0)$. But $\measuredangle\, (\alpha'(0), \beta'(0)) = \measuredangle\, (f'(p_0)\alpha'(0), f'(p_0)\beta'(0))$. $\square$

11.4. **Cauchy-Rieman equation.**

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^2$, and $k(x,y) = (u(x,y), v(x,y))$ a smooth map from $\mathcal{O}$ to $\mathbb{R}^2$. The degree 1 Taylor expansion of k at (x_0, y_0) is

$$k(x_0,y_0) + k_x(x_0,y_0)x + k_y(x_0,y_0).$$

The map $dk_{(x_0,y_0)} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $(x,y) \mapsto k_x(x_0,y_0)x + k_y(x_0,y_0)y$ is linear, and is called the *differential* of k at (x_0,y_0).

We want to write down the matrix for the differential $dk_{(x_0,y_0)}$. Note that

$$\begin{aligned} & dk_{(x_0,y_0)}(x,y) = k_x(x_0,y_0)x + k_y(x_0,y_0)y \\ & = (u_x(x_0,y_0), v_x(x_0,y_0))x + (u_y(x_0,y_0), v_y(x_0,y_0))y \\ & = (u_x(x_0,y_0)x + u_y(x_0,y_0)y, v_x(x_0,y_0)x + v_y(x_0,y_0)y). \end{aligned}$$

Write $\mathbb{R}^2$ as the space of coloumn vectors. Then $dk_{(x_0,y_0)}$ is

$$dk_{(x_0,y_0)} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u_x(x_0,y_0) & u_y(x_0,y_0) \\ v_x(x_0,y_0) & v_y(x_0,y_0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In other words, the matrix of the linear map $dk_{(x_0,y_0)}$ with respect to the standard basis is the Jacobi matrix

$$\begin{pmatrix} u_x(x_0,y_0) & u_y(x_0,y_0) \\ v_x(x_0,y_0) & v_y(x_0,y_0) \end{pmatrix}.$$

It is known from calculus that

$$\frac{||k(x+x_0, y+y_0) - k(x_0,y_0) - dk_{(x_0,y_0)}(x,y)||}{||(x,y)||} \to 0. \tag{11.4.1}$$

Theorem 11.4.1. *Let* $k(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ *be a smooth map, and* $f(z) = u(x,y) + iv(x,y)$. *Then the following conditions are equivalent:*

(i) $dk_{(x_0,y_0)}$ *is linear conformal,*

(ii) $\begin{cases} u_x(x_0,y_0) = v_y(x_0,y_0) \\ u_y(x_0,y_0) = -v_x(x_0,y_0), \end{cases}$

(iii) $f = u + iv$ *is analytic at* $z_0 = x_0 + iy_0$, *and* $f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0)$.

Proof. A linear map $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ is linear conformal if $d = a$ and $c = -b$. So (i) and (ii) are equivalent. If $dk_{(x_0,y_0)}$ is linear conformal, then when we identify $\mathbb{R}^2$ as $\mathbb{C}$, the linear operator $dk_{(x_0,y_0)}$ on $\mathbb{R}^2$ becomes M_α, where $\alpha = u_x(x_0,y_0) + iv_x(x_0,y_0)$. So (11.4.1) becomes

$$\frac{f(z_0+z) - f(z_0) - \alpha z}{z} \to 0$$

as $z \to 0$, i.e., f is analytic at z_0 and $f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0)$. $\square$

As a consequence, we get

Theorem 11.4.2. *Suppose $u, v : \mathcal{O} \to \mathbb{R}$ are smooth. Then $f(z) = u(x, y) + iv(x, y)$ is analytic if and only if u, v satisfies the Cauchy-Rieman equation:*

$$u_x = v_y, \quad u_y = -v_x. \tag{11.4.2}$$

Theorem 11.4.3. *Suppose $f : \mathcal{O} \to \mathbb{C}$ is analytic, and $f(z) = u(x, y) + iv(x, y)$, where u, v are real valued functions. Then*

(a) $f'(z) = f_x = -if_y$,

(b) *u, v satisfy the Cauchy-Rieman equation* (11.4.2),

(c)

$$f'(z) = u_x + iv_x = u_x - iu_y = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f,$$

(d) *u and v are harmonic functions, i.e.,*

$$u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0.$$

Proof. We will give another proof of (b). The function f is analytic means that

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = f'(z).$$

Here $h = r + is$ is a complex number and the limit is equal to $f'(z)$ whenever $h \to 0$. If we choose $h = r \to 0$, then the limit is $f_x = u_x + iv_x$. If we choose $h = is \to 0$, then

$$\frac{f(x+iy+is) - f(x+iy)}{is} = \frac{u(x, y+s) + iv(x, y+s) - u(x, y) - iv(x, y)}{is}$$
$$= \frac{1}{i}\left(\frac{u(x, y+s) - u(x, y)}{s} + i\,\frac{v(x, y+s) - v(x, y)}{s}\right),$$

so as $s \to 0$, the limit is

$$\frac{1}{i}(u_y + iv_y) = v_y - iu_y.$$

But the assumption is that no matter how the complex number $h \to 0$, the limit is always the same, and is equal to $f'(z)$. This shows that

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

In other words, $f'(z) = f_x = -if_y$, which proves (a). This also shows that $u_x = v_y$ and $u_y = -v_x$, which is (b).

Since

$$\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f = \frac{1}{2}(f_x - if_y) = f_x = -if_y,$$

it is equal to $f'(z)$, which gives the last part of (c).

For (d), we use (b) to compute $u_{xx} + u_{yy} = u_{xx} - (v_x)_y = u_{xx} - (v_y)_x = u_{xx} - (u_x)_x = 0$. Similarly, $v_{xx} + v_{yy} = 0$. ☐

Part (c) of the above theorem explains the notation in complex variable:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right).$$

Corollary 11.4.4. *If $f : \mathcal{O} \to \mathbb{C}$ is analytic, then f' is also analytic.*

Proof. Suppose $f = u + iv$. Then $u_x = v_y$ and $u_y = -v_x$. We know that $f' = u_x + iv_x$. We claim that $U = u_x$ and $V = v_x$ satisfy the Cauchy-Rieman euqation. To see this, we compute $U_x = u_{xx}$, $V_y = v_{xy} = (v_y)_x = (u_x)_x = u_{xx}$, so $U_x = V_y$. Also $U_y = u_{xy}$ and $V_x = v_{xx} = (-u_y)_x = -u_{xy}$, thus $U_y = -V_x$. Hence U, V satisfy the Cauchy-Rieman equation. By Theorem 11.4.2, f' is analytic. □

11.5. **Harmonic conjugate of a harmonic function.**

Given a harmonic function $u : \mathcal{O} \to \mathbb{R}$, is there a harmonic function $v : \mathcal{O} \to \mathbb{R}$ such that $f(z) = u(x, y) + iv(x, y)$ is an analytic function? We know that such v must satisfy the Cauchy-Rieman equation

$$\begin{cases} v_x = -u_y, \\ v_y = u_x. \end{cases} \tag{11.5.1}$$

Since u is given, the right hand sides are given functions. By Frobenius Theorem, the compatibility condition that the above system is solvable is that $(-u_y)_y = (u_x)_x$, i.e., $u_{xx} + u_{yy} = 0$. Since u is harmonic, the compatibility condition is satisfied, so we can solve v. In fact, such v are uniquely determined up to a constant. Now let $f(z) = u(x, y) + iv(x, y)$. Because u, v satisfy the Cauchy-Rieman equation, f is analytic. We call such v a *harmonic conjugate* of u.

Next we give more detail of construction of solution v of (11.5.1). Given $P(x, y)$ and $Q(x, y)$, the following initial value problem has a unique $v(x, y)$,

$$\begin{cases} v_x = P \\ v_y = Q, \\ v(0,0) = c_0 \end{cases} \tag{11.5.2}$$

if $P_y = Q_x$. The solution v can be constructed using integration. Fix y, the first equation implies that $v(x, y) = c_0 + \int_0^x P(s, y)\ ds + g(y)$ for some g. But

$$\begin{aligned} v_y &= \frac{\partial}{\partial y} \int_0^x P(s, y)\ ds + g'(y) = \int_0^x \frac{\partial P}{\partial y}(s, y)\ dy + g'(y) \\ &= \int_0^x Q_x(s, y)\ dx + g'(y) = Q(x, y) - Q(0, y) + g'(y), \end{aligned}$$

which is equal to Q if $-Q(0, y) + g'(y) = 0$. So $g'(y) = Q(0, y)$, hence $g(y) = \int_0^y Q(0, t)\ dt$. In other words, we have proved:

Theorem 11.5.1. *The solution for* (11.5.2) *is*

$$v(x, y) = c_0 + \int_0^y P(s, y)\ ds + \int_0^y Q(0, t)\ dt.$$

Theorem 11.5.2. *If u is harmonic, then*

(i)

$$v(x,y) = c - \int u_y(x,y)\ dx + \int u_x(0,y)\ dy$$

is a harmonic conjugate of u*,*

(ii) $f = u + iv$ *is an analytic function.*

Proof. A harmonic conjugate of u must satisfy the Cauchy-Rieman equation

$$\begin{cases} v_x = -u_y, \\ v_y = u_x. \end{cases}$$

Part (i) follows from Theorem 11.5.1, and (ii) follows from Theorem 11.4.2. □

Example 11.5.3.
Let $u(x,y) = \cosh y \cos x$. It is easy to check that u is harmonic. The harmonic conjugates of u is obtained by solving

$$\begin{cases} v_x = -u_y = -\sinh y \cos x = P, \\ v_y = u_x = -\cosh y \sin x = Q. \end{cases}$$

So $Q(0,y) = 0$. By Theorem 11.5.2,

$$v(x,y) = \int -\sinh y \cos x\ dx + \int 0 dy = c - \sinh y \sin x$$

is a harmonic conjugate of u. Note that

$$u + iv = \cosh y \cos x - i \sinh y \sin x + ic,$$

which is equal to $\cosh(-iz) + ic$ because

$$\begin{aligned} \cosh(-iz) = \cosh(-i(x+iy)) = \cosh(y - ix) &= \frac{e^{y-ix} + e^{-(y-ix)}}{2} \\ &= \frac{e^y(\cos x - i\sin x) + e^{-y}(\cos x + i \sin x)}{2} \\ &= \frac{e^y + e^{-y}}{2} \cos x - i\frac{e^y - e^{-y}}{2} \sin x = \cosh y \cos x - i \sinh y \sin x. \end{aligned}$$

Hence $\cosh y \cos x$ is the real part of the analytic function $\cosh(-iz)$.

Exercise 11.5.1.

(1) Prove that $u(x,y) = \frac{1}{2}\ln(x^2 + y^2)$ is harmonic for $x > 0$, and find a harmonic conjugate for u. (The analytic function obtained this way is $\ln(z)$).

(2) Prove that $u(x,y) = x - \frac{x^3}{3} + xy^2$ is harmonic, and find the harmonic conjugate v for u with $v(0,0) = 0$.

11.6. **Weierstrass representations of minimal surfaces.**

Proposition 11.6.1. *Let* $f = (f_1, f_2, f_3) : \mathcal{O} \to \mathbb{R}^3$ *be a parametrized surface. Set*

$$\phi_j = (f_j)_x - i(f_j)_y, \quad 1 \le j \le 3.$$

Then f *is isothermal paramtrization if and only if* $\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$.

Proof. Note that

$$\begin{aligned}
\phi_1^2 + \phi_2^2 + \phi_3^2 &= \sum_{j=1}^{3} ((f_j)_x - i(f_j)_y)^2 \\
&= \sum_{j=1}^{3} (f_j)_x^2 - (f_j)_y^2 - 2i(f_j)_x(f_j)_y \\
&= ||f_x||^2 - ||f_y||^2 - 2i f_x \cdot f_y,
\end{aligned}$$

which is zero if and only if both the real part and imaginary part are zero, i.e.,

$$||f_x||^2 - ||f_y||^2 = 0, \quad f_x \cdot f_y = 0,$$

which is equivalent to $f(x, y)$ is isothermal parametrization. □

Proposition 11.6.2. *Let* $f = (f_1, f_2, f_3) : \mathcal{O} \to \mathbb{R}^3$ *be a surface parametrized by isothermal parametrization, and* $\phi_j = (f_j)_x - i(f_j)_y$. *Then* f *is minimal if and only if* ϕ_j *is an analytic function for* $j = 1, 2, 3$.

Proof. Since the parametrization is isothermal, by Corollary 11.1.4 f is minimal if and only if each f_j is harmonic for $1 \le j \le 3$. Let g_j be a harmonic conjugate of f_j. Then $F_j = f_j + ig_j$ is an analytic function for $1 \le j \le 3$. By Theorem 11.4.3 (c), $F_j'(z) = (f_j)_x - i(f_j)_y = \phi_j$. By Corollary 11.4.4, ϕ_j is analytic. □

Given an analytic function g, let $\int g \; dz$ denote an anti-derivative of g, i.e., $(\int g \; dz)' = g$.

Example 11.6.3.

(1) $\int z \; dz = \frac{z^2}{2} + c$,
(2) $\int z^n \; dz = \frac{z^{n+1}}{n+1} + c$,
(3) $\int e^z \; dz = e^z + c$,
(4) $\int \sin z \; dz = -\cos z + c$.

Given a complex value function $f = u + iv$, we use the notation $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$.

Corollary 11.6.4. *Suppose* $\phi_j : \mathcal{O} \to \mathbb{C}$ *are analytic such that* $\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$ *and* $|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2$ *never vanishes on* $\mathcal{O}$. *Set*

$$F_j(z) = \int \phi_j(z) \; dz, \qquad f_j = \mathrm{Re}(F_j).$$

Then $f = \mathrm{Re}(f_1, f_2, f_3)$ *is a minimal surface.*

Therefore given any two analytic functions ϕ_1, ϕ_2 from $\mathcal{O}$ to $\mathbb{C}$, we may choose $\phi_3 = \sqrt{1 - \phi_1^2 - \phi_2^2}$ and let

$$F = \left(\int \phi_1(z)dz, \int \phi_2(z)dz, \int \phi_3(z)dz \right), \qquad f = \mathrm{Re}(F).$$

If $||\phi(z)||$ is not zero for all $z \in \mathcal{O}$, then $f := \mathrm{Re}(F) : \mathcal{O} \to \mathbb{R}^3$ is a minimal surface.

Example 11.6.5. Let

$$\phi_1(z) = \sinh z, \quad \phi_2(z) = -i \cosh z, \quad \phi_3(z) = 1.$$

Then $\phi_1^2 + \phi_2^2 + \phi_3^2 = \sinh^2 z - \cosh^2 z + 1 = 0$, and

$$F(z) = \left(\int \sinh z \ dz, \int -i \cosh z \ dz, \int 1 \ dz \right) = (\cosh z, -i \sinh z, z).$$

But

$$\cosh z = \frac{e^z + e^{-z}}{2} = \cosh x \cos y + i \sinh x \sin y,$$
$$\sinh z = \frac{e^z - e^{-z}}{2} = \sinh x \cos y + i \cosh x \sin y.$$

So the real part $f(z)$ of $F(z)$ is

$$f(z) = f(x, y) = (\cosh x \cos y, \cosh x \sin y, x),$$

which is a Catenoid. Note that $|| \sum_{j=1}^3 \phi_j(z)||^2||$ is always positive. So f is an immersion for $(x, y) \in \mathbb{R}^2$.

Exercise 11.6.1. Given $\phi = (\phi_1, \phi_2, \phi_3)$ with ϕ_j analytic for $j = 1, 2, 3$:

(a) $\phi(z) = (i \sinh z, \cosh z, i)$,
(b) $\phi(z) = (1 - z^2, i(1 + z^2), 2z)$,
(c) $\phi(z) = \left(\frac{2}{1+z^2}, \frac{2i}{1-z^2}, \frac{4z}{1-z^4} \right)$
(d) $\phi(z) = (1 - \cosh(-iz), i \sinh(-iz), 2 \sinh(-iz/2))$.

Then:

(1) Prove that $\sum_{j=1}^3 \phi_j^2 = 0$,
(2) Compute $F(z) = \int \phi(z) \ dz = \left(\int \phi_1(z) \ dz, \int \phi_2(z) \ dz, \int \phi_3(z) \ dz \right)$.
(3) Find the region of (x, y) such that $||\phi(z)|| > 0$, where $z = x + iy$.
(4) Compute the real part of $F(z)$ to get a minimal surface. (For (a)-(d) the minimal surfaces are helicoid, Ennper, Scherk, and Catalan surfaces respectively.

Theorem 11.6.6. *Suppose $\phi_1(z), \phi_2(z), \phi_3(z)$ are analytic functions and*

$$\phi_1(z)^2 + \phi_2(z)^2 + \phi_3(z)^2 = 0.$$

Let $\phi(z) = (\phi_1(z), \phi_2(z), \phi_3(z))$, and $F(z) = \int \phi(z) \ dz = f(z) + ig(z)$. Let $0 \le \theta \le 2\pi$ be a constant, and set

$$\psi_j(z) = e^{i\theta}\phi_j(z), \quad \psi(z) = (\psi_1(z), \psi_2(z), \psi_3(z)) = e^{i\theta}\phi(z)$$

Then

(a) $\sum_{j=1}^{3} \psi_j(z)^2 = 0$,
(b) $\int \psi(z)\ dz = e^{i\theta} F(z) = (f\cos\theta - g\sin\theta) + i(f\sin\theta + g\cos\theta)$,
(c) $f_\theta(x, y) = f(x, y)\cos\theta - g(x, y)\sin\theta$ *is a minimal surface.*

Proof. (a) Since $\psi_j = e^{i\theta}\phi_j$,

$$\sum_{j=1}^{3} \psi_j(z)^2 = e^{2i\theta} \sum_{j=1}^{3} \phi_j^2 = 0.$$

(b) Note $\int \psi(z)\ dz = \int e^{i\theta}\phi(z)\ dz = e^{i\theta} \int \phi(z)\ dz = e^{i\theta} F(z)$.
(c) is a consequence of Corollary 11.6.4. □

The above Theorem can be used as follows: Given a minimal surface $f = (f_1, f_2, f_3)$ in isothermal parametrization, let $\phi = f_x - if_y$, and $F = \int \phi(z)\ dz$. Then the real part of F is f. Let g denote the imaginary part of F. Theorem 11.6.6 says that $f\cos\theta - g\sin\theta$ is also a minimal surface for each constant $0 \le \theta < 2\pi$. So each minimal surface comes in a family, which is called the *associated family* of f.

Exercise 11.6.2.

(1) Find the associated families of the minimal surfaces given in Example 11.6.5 and Exercise 11.6.1.
(2) Use MathLab to plot these minimal surfaces (for each family, plot the surface f_θ with $\theta = 0, \pi/6, \pi/4, \pi/3, \pi/2$).

12. DIFFERENTIAL FORMS

Let $f : \mathcal{O} \to \mathbb{R}^3$ be a parametrized surface in $\mathbb{R}^3$. Given $p_0 \in \mathcal{O}$, $(dx_1)_{p_0}$ and $(dx_2)_{p_0}$ are linear functionals on the tangent plane Tf_{p_0} defined by

$$(dx_i)_{p_0}(f_{x_j}(p_0)) = \delta_{ij}.$$

A smooth 1*-form* on the surface f is

$$\theta = a_1(x_1, x_2)\ dx_1 + a_2(x_1, x_2)\ dx_2,$$

where $a_1, a_2 : \mathcal{O} \to \mathbb{R}$ are smooth functions. In particular, $\theta(f_{x_i}) = a_i$ for $i = 1, 2$.

Wedge product

Let θ_1, θ_2 be 1-forms. The wedge product $\theta_1 \wedge \theta_2$ is defined as follows:

$$(\theta_1 \wedge \theta_2)(v_1, v_2) = \frac{1}{2}(\theta_1(v_1)\theta_2(v_2) - \theta_1(v_2)\theta_2(v_1)).$$

Exercise 12.0.3. Suppose θ_1, θ_2 are 1-forms. Prove that

(1) $\theta_1 \wedge \theta_2$ is skew-symmetric, i.e., $(\theta_1 \wedge \theta_2)(v_2, v_1) = -(\theta_1 \wedge \theta_2)(v_1, v_2)$.
(2) $\theta_1 \wedge \theta_1 = 0$,
(3) $\theta_2 \wedge \theta_1 = -\theta_2 \wedge \theta_1$,

(4) if $\theta_i = a_i dx_1 + b_i dx_2$ for $i = 1, 2$, then

$$\theta_1 \wedge \theta_2 = (a_1 b_2 - a_2 b_1)\ dx_1 \wedge dx_2.$$

A smooth 2-form is

$$\tau = h(x_1, x_2) dx_1 \wedge dx_2$$

for some smooth function $h : \mathcal{O} \to \mathbb{R}$.

Exterior differentiation

If $\phi : \mathcal{O} \to \mathbb{R}$ is a smooth function, then the exterior differentiation of ϕ is the 1-form defined by

$$d\phi = \phi_{x_1}\ dx_1 + \phi_{x_2}\ dx_2.$$

The exterior differentiation of a smooth 1-form $\theta = a\ dx_1 + b\ dx_2$ is defined by

$$d\theta = da \wedge dx_1 + db \wedge dx_2.$$

Exercise 12.0.4. Prove that if $\theta = a\ dx_1 + b\ dx_2$, then

$$d\theta = (-a_{x_2} + b_{x_1}) dx_1 \wedge dx_2.$$

Proposition 12.0.7. *If $\phi : \mathcal{O} \to \mathbb{R}$ is a smooth function, then the exterior differentiation of the 1-form $d\phi$ is zero, i.e., $d(d\phi) = 0$.*

Proof. Since the $(\phi_{x_1})_{x_2} = (\phi_{x_2})_{x_1}$,

$$d(d\phi) = d(\phi_{x_1} dx_1 + \phi_{x_2} dx_2) = (-(\phi_{x_1})_{x_2} + (\phi_{x_1})_{x_2})\ dx_1 \wedge dx_2 = 0.$$

□

Exercise 12.0.5. Suppose $h, k : \mathcal{O} \to \mathbb{R}$ are smooth functions. Prove that $d(hk) = (dh)k + h\ dk$.

Exercise 12.0.6. Let $h : \mathcal{O} \to \mathbb{R}$ be a smooth function, and θ a smooth 1-form. Prove that

(1) $d(h\theta) = dh \wedge \theta + h\ d\theta$,
(2) $d(h\theta) = d(\theta h) = (d\theta)h - \theta \wedge dh$.

Suppose Ω is a domain in $\mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$. The 2-form $dx_1 \wedge dx_2$ can be viewed as the counterclockwise orientation of Ω (so $dx_2 \wedge dx_1$ is the clockwise orientation of Ω). Suppose we choose the counterclockwise orientation for Ω, then this induces an orientation on the boundary by requiring that from the outward normal of Ω to the orientation of the boundary is counter-clockwise. We call such orientation on the boundary $\partial\Omega$ the *induced orientation*.

We recall the Green's formula:

Green's formula

If $P, Q : \Omega \to \mathbb{R}$ are smooth functions, then

$$\int_{\partial\Omega} P(x_1, x_2)\ dx_1 + Q(x_1, x_2)\ dx_2 = \int\int_{\Omega} (-P_{x_2} + Q_{x_1})\ dx_1 \wedge dx_2,$$

where $\partial\Omega$ has the induced orientation.

Note that

$$d(P\ dx_1 + Q\ dx_2) = (-P_{x_2} + Q_{x_1})\ dx_1 \wedge dx_2,$$

so the Green's formula can be rewritten as

$$\int_{\partial\Omega} \theta = \int\int_{\Omega} d\theta, \tag{12.0.1}$$

where θ is the 1-form $\theta = P\ dx_1 + Q\ dx_2$. Here the line integral $\int_{\partial\Omega} P\ dx_1 + Q\ dx_2$ can be computed by choosing a parametrization $t \in [0, c] \mapsto (x_1(t), x_2(t))$ that parametrized the boundary curve $\partial\Omega$ with the induced orientation. Then

$$\int_{\partial\Omega} P\ dx_1 + Q\ dx_2 = \int_0^c P(x_1(t), x_2(t))x_1'(t) + Q(x_1(t), x_2(t))x_2'(t)dt.$$

Formula (12.0.1) holds for a domain in $\mathbb{R}^n$ if we replace the 1-form by an $n-1$ form, and is called the *Stoke's formula*. We explain the 3-dimensional case: Let D be a domain in $\mathbb{R}^3$. A smooth 1-form on D is

$$\theta = a_1\ dx_1 + a_2\ dx_2 + a_3\ dx_3$$

for some smooth functions $a_1, a_2,\ a_3$ on D. Let $\theta_1, \theta_2, \theta_3$ be 1-forms on D. The wedge product $\theta_1 \wedge \theta_2$ is defined as above, and the wedge product $\theta_1 \wedge \theta_2 \wedge \theta_3$ is the 3-form defined by

$$\theta_1 \wedge \theta_2 \wedge \theta_3 = \frac{1}{3!} \sum_{s \in S_3} \mathrm{sgn}(s)\theta_1(v_{s(1)})\theta_2(v_{s(2)})\theta_3(v_{s(3)}).$$

Here S_3 denote the set of all permutations of $\{1, 2, 3\}$, i.e., the set of all bijective maps from $\{1, 2, 3\}$ to itself, and $\mathrm{sgn}(s) = (-1)^m$ if s can be written as product of m permutations of two letters. It follows from the definition that $(\theta_1 \wedge \theta_2 \wedge \theta_3)_p$ is an alternating multi-linear functional on $\mathbb{R}^3$ for each $p \in D$. A smooth 2-form can be written uniquely as

$$b_1 dx_2 \wedge dx_3 + b_2 dx_3 \wedge dx_1 + b_3 dx_1 \wedge dx_2,$$

and a smooth 3-form on D is

$$h(x_1, x_2, x_3)\ dx_1 \wedge dx_2 \wedge dx_3,$$

where $h : D \to \mathbb{R}$ is some smooth function. The exterior differentiation is defined similarly:

(1) For $h : D \to \mathbb{R}$,

$$dh = h_{x_1} dx_1 + h_{x_2} dx_2 + h_{x_3} dx_3.$$

(2) For a 1-form $\theta = a_1 dx_1 + a_2 dx_2 + a_3 dx_3$, $d\theta$ is the 2-form defined by

$$d\theta = da_1 \wedge dx_1 + da_2 \wedge dx_2 + da_3 \wedge dx_3.$$

(3) For a 2-form $\tau = b_1 dx_2 \wedge dx_3 + b_2 dx_3 \wedge dx_1 + b_3 dx_1 \wedge dx_2$, $d\tau$ is the 3-form defined by

$$d\tau = db_1 \wedge dx_2 \wedge dx_3 + db_2 \wedge dx_3 \wedge dx_1 + db_3 \wedge dx_1 \wedge dx_2.$$

Exercise 12.0.7. Let D be an open subset of $\mathbb{R}^3$.

(1) Suppose $\theta = a_1 dx_1 + a_2 dx_2 + a_3 dx_3$. Prove that

$$d\theta = ((a_2)_{x_1} - (a_1)_{x_2})\ dx_1 \wedge dx_2 + ((a_3)_{x_2} - (a_2)_{x_3})\ dx_2 \wedge dx_3 \\ + ((a_1)_{x_3} - (a_3)_{x_1}) dx_3 \wedge dx_1.$$

(2) Prove that if $h : D \to \mathbb{R}$ is smooth then $d(dh) = 0$.

Exercise 12.0.8. Prove that

(1) $dx_2 \wedge dx_3 \wedge dx_1 = dx_3 \wedge dx_1 \wedge dx_2 = dx_1 \wedge dx_2 \wedge dx_3$,
(2) if $\tau = b_1 dx_2 \wedge dx_3 + b_2 dx_3 \wedge dx_1 + b_3 dx_1 \wedge dx_2$, then

$$d\tau = ((b_1)_{x_1} + (b_2)_{x_2} + (b_3)_{x_3})\ dx_1 \wedge dx_2 \wedge dx_3.$$

The divergence formula is

$$\int\int\int_D ((b_1)_{x_1} + (b_2)_{x_2} + (b_3)_{x_3})\ dx_1 \wedge dx_2 \wedge dx_3 \\ = \int\int_{\partial D} b_1\ dx_2 \wedge dx_3 + b_2\ dx_3 \wedge dx_1 + b_3\ dx_1 \wedge dx_2,$$

which can be written as the Stoke's Theorem:

$$\int\int\int_D d\tau = \int\int_{\partial D} \tau,$$

where $\tau = b_1\ dx_2 \wedge dx_3 + b_2\ dx_3 \wedge dx_1 + b_3\ dx_1 \wedge dx_2$.

$\mathbb{R}^n$-valued form 1-form

Let $h = (h_1, h_2, h_3) : \mathcal{O} \to \mathbb{R}^3$ be a smooth map. We define

$$dh := (dh_1, dh_2, dh_3).$$

Then dh is a $\mathbb{R}^3$-valued 1-form. In fact,

$$dh = h_{x_1} dx_1 + h_{x_2} dx_2.$$

Note that here h_{x_1}, h_{x_2} are $\mathbb{R}^3$-valued maps.

13. Cartan's method of moving frame

Let $f : \mathcal{O} \to \mathbb{R}^3$ be a parametrized surface, and e_1, e_2, e_3 a smooth orthonormal frame on the surface such that $e_3 = N$ is the unit normal vector field of the surface. We can write $(e_i)_{x_1}$ and $(e_i)_{x_2}$ as linear combinations of e_1, e_2, e_3:

$$\begin{cases} (e_i)_{x_1} = e_1 p_{1i} + e_2 p_{2i} + e_3 p_{3i}, \\ (e_i)_{x_2} = e_1 q_{1i} + e_2 q_{2i} + e_3 q_{3i}, \end{cases}.$$

where

$$p_{ji} = (e_i)_{x_1} \cdot e_j, \quad q_{ji} = (e_i)_{x_2} \cdot e_j.$$

If we use the differential form notation, then the above equation becomes

$$\begin{aligned} de_i &= \sum_{j=1}^{3} p_{ji} e_j \; dx_1 + \sum_{j=1}^{3} q_{ji} e_j dx_2 \\ &= \sum_{j=1}^{3} p_{ji} dx_1 e_j + \sum_{j=1}^{3} q_{ji} dx_2 e_j = \sum_{j=1}^{3} (p_{ji} dx_1 + q_{ji} dx_2) e_j. \end{aligned}$$

Set

$$w_{ji} = p_{ji} dx_1 + q_{ji} dx_2,$$

then we can rewrite de_i as

$$de_i = \sum_{j=1}^{3} w_{ji} e_j, \qquad 1 \leq i \leq 3. \tag{13.0.2}$$

Recall that the Gauss-Codazi equation for surface f is given by the condition that $(e_i)_{x_1 x_2} = (e_i)_{x_2 x_1}$, which is also the condition that $d(de_i) = 0$. But

$$\begin{aligned} 0 = d(de_i) = d\left(\sum_{j=1}^{3} w_{ji} e_j\right) &= \sum_{j=1}^{3} dw_{ji} e_j - w_{ji} \wedge de_j \\ &= \sum_{j=1}^{3} dw_{ji} e_j - \sum_{j=1}^{3} w_{ji} \wedge \sum_{k=1}^{3} w_{kj} e_k \\ &= \sum_{k=1}^{3} dw_{ki} e_k - \sum_{k=1}^{3} \sum_{j=1}^{3} w_{ji} \wedge w_{kj} \; e_k \\ &= \sum_{k=1}^{3} dw_{ki} e_k + \sum_{k=1}^{3} \sum_{j=1}^{3} w_{kj} \wedge w_{ji} e_k \\ &= \sum_{k=1}^{3} \left(dw_{ki} + \sum_{j=1}^{3} w_{kj} \wedge w_{ji} \right) e_k, \end{aligned}$$

so each coefficient of e_k must be zero, i.e.,

$$dw_{ki} + \sum_{j} w_{kj} \wedge w_{ji} = 0. \tag{13.0.3}$$

This is the condition that $d(de_i) = 0$, which is the same as $(e_i)_{x_1 x_2} = (e_i)_{x_2 x_1}$, so the above equation is the Gauss-Codazzi equation. Take the dot product of (13.0.2) with e_j to get

$$w_{ji} = de_i \cdot e_j.$$

But $e_i \cdot e_j = \delta_{ij}$ implies that $de_i \cdot e_j + e_i \cdot de_j = 0$, so

$$w_{ij} + w_{ji} = 0, \quad w_{ii} = 0.$$

Thus

$$\sum_{k=1}^{3} w_{1k} \wedge w_{k2} = w_{11} \wedge w_{12} + w_{12} \wedge w_{22} + w_{13} \wedge w_{32} = 0 + 0 + w_{13} \wedge w_{32}.$$

Similarly, we get $\sum_{k=1}^{3} w_{2k} \wedge w_{k3} = w_{21} \wedge w_{13}$, and $\sum_{k=1}^{3} w_{1k} \wedge w_{k3} = w_{12} \wedge w_{23}$. So the Gauss-Codazzi equation (13.0.3) becomes

$$\begin{cases} dw_{12} = -w_{13} \wedge w_{32}, \\ dw_{23} = -w_{21} \wedge w_{13}, \\ dw_{13} = -w_{12} \wedge w_{23}. \end{cases} \tag{13.0.4}$$

The first equation is the Gauss equation and the second and third are the Codazzi equations.

Set

$$w_1 = df \cdot e_1, \quad w_2 = df \cdot e_2.$$

Then we have

$$df = w_1 e_1 + w_2 e_2.$$

We claim that

$$\begin{aligned} \mathrm{I} &= w_1^2 + w_2^2, \\ \mathrm{II} &= -(w_1 \otimes w_{13} + w_2 \otimes w_{23}). \end{aligned}$$

To see this, we assume the surface f is parametrized by orthogonal coordinates, and

$$A_1^2 = f_{x_1} \cdot f_{x_1}, \quad A_2^2 = f_{x_2} \cdot f_{x_2}, \quad \ell_{ij} = f_{x_i x_j} \cdot e_3.$$

So

$$\begin{aligned} \mathrm{I} &= A_1^2 dx_1^2 + A_2^2 dx_2^2, \\ \mathrm{II} &= \ell_{11} dx_1^2 + 2\ell_{12} dx_1 dx_2 + \ell_{22} dx_2^2. \end{aligned}$$

We choose

$$e_1 = \frac{f_{x_1}}{A_1}, \quad e_2 = \frac{f_{x_2}}{A_2}, \quad e_3 = e_1 \times e_2.$$

Then

$$w_1 = df \cdot e_1 = (f_{x_1} dx_1 + f_{x_2} dx_2) \cdot \frac{f_{x_1}}{A_1} = \frac{f_{x_1} \cdot f_{x_1} dx_1}{A_1} = \frac{A_1^2 dx_1}{A_1} = A_1 dx_1.$$

Similarly,

$$w_2 = A_2 dx_2,$$

so

$$\mathrm{I} = w_1^2 + w_2^2.$$

Write

$$w_{13} = h_{11}w_1 + h_{21}w_2 = h_{11}A_1dx_1 + h_{21}A_2dx_2 = \sum_{j=1}^{2} h_{j1}A_jdx_j,$$

$$w_{23} = h_{12}w_1 + h_{22}w_2 = h_{21}A_1dx_1 + h_{22}A_2dx_2 = \sum_{j=1}^{2} h_{j2}A_jdx_j.$$

Since $w_{i3} = de_3 \cdot e_i$,

$$(e_3)_{x_i} \cdot e_1 = w_{13}(f_{x_i}) = h_{i1}A_i,$$

Similar computation implies that

$$(e_3)_{x_i} \cdot e_2 = h_{i2}A_i.$$

But

$$(e_3)_{x_i} \cdot e_j = N_{x_i} \cdot \frac{f_{x_j}}{A_j} = -\frac{\ell_{ij}}{A_j}.$$

So $h_{ij}A_i = -\frac{\ell_{ij}}{A_j}$, which implies that

$$\ell_{ij} = -h_{ij}A_iA_j.$$

Since $\ell_{12} = \ell_{21}$, $h_{12} = h_{21}$. Next we compute

$$\begin{aligned} & w_1 \otimes w_{13} + w_2 \otimes w_{23} = w_1 \otimes (h_{11}w_1 + h_{21}w_2) + w_2 \otimes (h_{12}w_1 + h_{22}w_2) \\ & = h_{11}w_1^2 + 2h_{12}w_1w_2 + h_{22}w_2^2 = h_{11}A_1^2dx_1^2 + 2h_{12}A_1A_2dx_1dx_2 + h_{22}A_2^2dx_2^2 \\ & = -(\ell_{11}dx_1^2 + 2\ell_{12}dx_1dx_2 + \ell_{22}dx_2^2) = -\mathrm{II}. \end{aligned}$$

This proves the claim. Recall that

$$K = \frac{\ell_{11}\ell_{22} - \ell_{12}^2}{g_{11}g_{22} - g_{12}^2}, \quad H = \frac{g_{22}\ell_{11} - 2g_{12}\ell_{12} + g_{11}\ell_{22}}{g_{11}g_{22} - g_{12}^2}.$$

Since $\ell_{ij} = h_{ij}A_iA_j$, we have

$$K = \det(h_{ij}), \quad H = h_{11} + h_{22}.$$

The geometric meaning of the 2-form $w_{13} \wedge w_{23}$

We compute

$$w_{13} \wedge w_{23} = (h_{11}w_1 + h_{12}w_2) \wedge (h_{12}w_1 + h_{22}w_2) = (h_{11}h_{22} - h_{12}^2)w_1 \wedge w_2.$$

But the Gaussian curvature is $K = h_{11}h_{22} - h_{12}^2$, so the first equation of (13.0.4) (the Gauss-equation) gives

$$dw_{12} = -w_{13} \wedge w_{32} = w_{13} \wedge w_{23} = Kw_1 \wedge w_2,$$

i.e.,

$$dw_{12} = Kw_1 \wedge w_2. \tag{13.0.5}$$

Note that K is a function on the surface and $w_1 \wedge w_2 = A_1A_2\ dx_1 \wedge dx_2 = \sqrt{\det(g_{ij})}\ dx_1 \wedge dx_2$ is the area element of the surface.

Structure equations

We will show that w_{12} can be computed in terms of the first fundamental form. Since

$$df = w_1 e_1 + w_2 e_2,$$

$$\begin{aligned} 0 = d(df) = d(w_1e_1 + w_2e_2) &= dw_1 e_1 - w_1 \wedge de_1 + dw_2 e_2 - w_2 \wedge de_2 \\ &= dw_1 e_1 - w_1 \wedge (w_{21}e_2 + w_{31}e_3) + dw_2 e_2 - w_2 \wedge (w_{12}e_1 + w_{32}e_3) \\ &= (dw_1 - w_2 \wedge w_{12})e_1 + (dw_2 - w_1 \wedge w_{21})e_2 - (w_1 \wedge w_{31} + w_2 \wedge w_{32})e_3 \\ &= (dw_1 + w_{12} \wedge w_2)e_1 + (dw_2 + w_{21} \wedge w_1)e_2 + (w_1 \wedge w_{13} + w_2 \wedge w_{23})e_3. \end{aligned}$$

Since e_1, e_2, e_3 form a basis, the coefficients of e_i in the above formula must be zero. Use $w_i \wedge w_i = 0$, $w_1 \wedge w_2 = -w_2 \wedge w_1$ to compute the coefficient of e_3 to get

$$\begin{aligned} &w_1 \wedge w_{13} + w_2 \wedge w_{23} = w_1 \wedge (h_{11}w_1 + h_{21}w_2) + w_2 \wedge (h_{12}w_1 + h_{22}w_2) \\ &= h_{21}w_1 \wedge w_2 + h_{12}w_2 \wedge w_1 = (h_{21} - h_{12})w_1 \wedge w_2, \end{aligned}$$

which is zero, so we obtain $h_{12} = h_{21}$. This gives another proof that the shape operator is self-adjoint. The coefficients of e_1, e_2 are zero give the so called *structure equation* of the surface:

$$\begin{cases} dw_1 + w_{12} \wedge w_2 = 0, \\ dw_2 + w_{21} \wedge w_1 = 0, \end{cases} \qquad w_{21} = -w_{12}. \tag{13.0.6}$$

Exercise 13.0.9. Suppose $w_1 = A_1 dx_1$, $w_2 = A_2 dx_2$, and $w_{12} = u\ dx_1 + v\ dx_2$. Use the structure equations (13.0.6) to prove that

$$u = \frac{(A_1)_{x_2}}{A_2}, \quad v = -\frac{(A_2)_{x_1}}{A_1}.$$

In other words, w_{12} can be solved from I.

We summarize the moving frame method: Let $f : \mathcal{O} \to \mathbb{R}^3$ be a parametrized surface. We

(i) choose an orthonormal tangent moving frame e_1, e_2 on the surface,
(ii) set $e_3 = e_1 \times e_2$, $w_i = df \cdot e_i$ for $i = 1, 2$, and $w_{ij} = de_i \cdot e_j$ for $1 \le i, j \le 3$.

Then w_{ij} satisfy the Gauss-Codazzi equation (13.0.4), and w_1, w_2, w_{12} satisfy the structure equation (13.0.6).

Conversely, given $\mathrm{I} = E\ dx_1^2 + 2F\ dx_1 dx_2 + G\ dx_2^2$ and $\mathrm{II} = L\ dx_1^2 + 2M\ dx_1 dx_2 + N\ dx_2^2$ such that $E > 0$, $EG - F^2 > 0$

(1) write I as $w_1^2 + w_2^2$ by the method of completing squares to get w_1, w_2,
(2) construct w_{12} by solving the structure equation (13.0.6),
(3) w_{13}, w_{23} can be obtained by writing II as $-(w_1 w_{13} + w_2 w_{23})$, (this can be done by writing $-\mathrm{II}$ as $h_{11}w_1^2 + 2h_{12}w_1 w_2 + h_{22}w_2^2$, then $w_{13} = h_{11}w_1 + h_{12}w_2$ and $w_{23} = h_{12}w_1 + h_{22}w_2$),
(4) set $w_{ii} = 0$ for $1 \le i \le 3$ and $w_{ji} = -w_{ij}$.

The Fundamental Theorem of surfaces in $\mathbb{R}^3$ can be stated as follows: If w_{ij} satisfies (13.0.3) (or equivalently (13.0.4)), then there exists a surface in $\mathbb{R}^3$ unique up to rigid motion with I, II as the fundamental forms.

Exercise 13.0.10. (1) Let $\mathrm{I} = \sum_{i,j=1}^2 g_{ij} dx_i dx_j$ be the first fundamental form of some surface. Prove that

$$\begin{aligned} & g_{11} dx_1^2 + 2g_{12} dx_1 dx_2 + g_{22} dx_2^2 \\ & = \left(\sqrt{g_{11}}\left(dx_1 + \frac{g_{12}}{g_{11}}\, dx_2\right)\right)^2 + \left(\sqrt{\frac{g_{11}g_{22} - g_{12}^2}{g_{11}}}\, dx_2\right)^2, \end{aligned}$$

(2) let $w_1 = \sqrt{g_{11}}(dx_1 + \frac{g_{12}}{g_{11}}\, dx_2)$ and $w_2 = \sqrt{\frac{g_{11}g_{22}-g_{12}^2}{g_{11}}}\, dx_2$, prove that

$$w_1 \wedge w_2 = \sqrt{g_{11}g_{22} - g_{12}^2}\, dx_1 \wedge dx_2,$$

which is the surface area element.

Exercise 13.0.11. Suppose

$$\mathrm{I} = e^{2x_2}\, dx_1^2 + dx_2^2, \quad \mathrm{II} = -e^{x_2}\sqrt{1 - e^{2x_2}}\, dx_1^2 + \frac{e^{x_2}}{\sqrt{1 - e^{2x_2}}}\, dx_2^2.$$

Find w_1, w_2, w_{ij} and prove that w_{ij} satisfy the Gauss-Codazzi equation (13.0.4).

14. Geodesic curvature

Let $\alpha(s) = f(x_1(s), x_2(s))$ be a smooth curve on the parametrized surface $f : \mathcal{O} \to \mathbb{R}^3$, and s the arc length parameter of α, i.e., $||\alpha'(s)|| = 1$. Let $v_1(s) = \alpha'(s)$, $v_3(s) = N(\alpha(s))$, the unit normal of the surface at $\alpha(s)$, and $v_2(s) = v_3(s) \times v_1(s)$. Note that $v_2(s)$ is perpendicular to $\alpha'(s)$ and is tangent to the surface at $\alpha(s)$. Since (v_1, v_2, v_3) is orthonormal,

$$(v_1' v_2', v_3') = (v_1, v_2, v_3) \begin{pmatrix} 0 & -k_g & -k_n \\ k_g & 0 & \tau_n \\ k_n & -\tau_n & 0 \end{pmatrix}$$

for some smooth function k_g, k_n and τ_n. In fact,

$$k_g = v_1' \cdot v_2, \quad k_n = v_1' \cdot v_3, \quad \tau_n = v_3' \cdot v_2,$$

which are called the *geodesic curvature*, *normal curvature*, and *normal torsion* respectively. Note that (v_1, v_2, v_3) is not the Frenet frame of α.

Proposition 14.0.8. *Suppose e_1, e_2 is an orthonormal tangent frame for the surface $f : \mathcal{O} \to \mathbb{R}^3$, $e_3 = e_1 \times e_2$, and $w_{ij} = de_j \cdot e_i$ for $1 \le i, j \le 3$. Let $\alpha(s) = f(x_1(s), x_2(s))$ be a smooth curve on the surface parametrized by the arc-length, and $\phi(s)$ the angle from $e_1(x_1(s), x_2(s))$ to $\alpha'(s)$. Then*

$$k_g = -w_{12}(\alpha'(s)) + \phi'(s). \tag{14.0.7}$$

Proof. Since the angle from $e_1(s)$ to $\alpha'(s)$ is $\phi(s)$,

$$v_1(s) = \cos\phi(s)e_1(s)+\sin\phi(s)e_2(s), \quad v_2(s) = -\sin\phi(s)e_1(s)+\cos\phi(s)e_2(s).$$

But

$$\begin{aligned} k_g &= v_1' \cdot v_2 = (\cos\phi\ e_1 + \sin\phi\ e_2)' \cdot (-\sin\phi\ e_1 + \cos\phi\ e_2) \\ &= (-\phi'\sin\phi\ e_1 + \phi'\cos\phi\ e_2 + \cos\phi\ e_1' + \sin\phi\ e_2') \cdot (-\sin\phi\ e_1 + \cos\phi\ e_2) \\ &= \phi' + \cos^2\phi\ e_1' \cdot e_2 - \sin^2\phi\ e_2' \cdot e_1 \\ &= \phi' - w_{12}(\alpha'(s)). \end{aligned}$$

In the above computation we use the fact that $e_i' \cdot e_i = 0$ and $e_i' \cdot e_j = -e_i \cdot e_j'$. □

Proposition 14.0.9. *Suppose $\alpha(s) = f(x_1(s), x_2(s))$ is a smooth curve on the parametrized surface $f : \mathcal{O} \to \mathbb{R}^3$, and α is parametrized by arc-length. Then α is a geodesic if and only if $k_g = 0$.*

Proof. We had a result that if α is parametrized by arc-length, then α is a geodesic if and only if $\alpha''(s)$ is parallel to the normal vector of the surface at $\alpha(s)$ (i.e., $e_3(x_1(s), x_2(s))$). But $v_1 = \alpha'$, $v_3 = e_3$, $v_2 = v_3 \times v_1$,

$$v_1' = \alpha'' = k_g v_2 + k_n v_3,$$

So α'' is parallel to e_3 if and only if $k_g = 0$. □

Corollary 14.0.10. *Suppose $\alpha(s) = f(x_1(s), x_2(s))$ is a geodesic for f, and α is parametrized by arc-length, e_1, e_2 is an orthonormal tangent frame, and $\phi(s)$ is the angle from $e_1(s)$ to $\alpha'(s)$. Then $\phi'(s) = w_{12}(\alpha'(s))$.*

15. Theorem of turning tangents

A smooth curve $\alpha : [a,b] \to \mathbb{R}^3$ is a *closed curve* if $\alpha(a) = \alpha(b)$ and $\alpha^{(j)}(a) = \alpha^{(j)}(b)$ for all $j > 0$, where $\alpha^{(j)} = \frac{d^j\alpha}{dt^j}$. A smooth closed curve α is *simple* if α is one to one.

Suppose the image of a simple closed curve $\alpha : [a,b] \to \mathbb{R}^3$ lies in an open ball B in $\mathbb{R}^3$, and $e_1 : B \to \mathbb{R}^3$ is a smooth unit vector field. Given $v_1, v_2 \in \mathbb{R}^3$, let $\measuredangle\ (v_1, v_2)$ denote the angle from v_1 to v_2. Let

$$\phi(s) = \measuredangle\ (e_1(\alpha(s)), \alpha'(s)).$$

Intuitively, we see that

$$\phi(b) - \phi(a) = 2\pi,$$

which is the total turning of the tangents.

A continuous curve $\alpha : [0,c] \to \mathbb{R}^3$ is called a *piecewise smooth k-gon* if there exists $0 = t_1 < t_1 < \cdots < t_{k+1} = c$ such that

(1) the restriction of α to $[t_i, t_{i+1}]$ is smooth, and let $\alpha_i = \alpha \mid [t_i, t_{i+1}]$ denote the restriction of α to the interval $[t_i, t_{i+1}]$ for $1 \leq i \leq k$.
(2) $\alpha(c) = \alpha(0)$,

(3) $\measuredangle\, (\alpha_i'(t_{i+1}), \alpha_{i+1}'(t_{i+1})) \neq 0$ or π for all $1 \leq i \leq k$ (here $\alpha_{k+1} = \alpha_1$). We call $\alpha(t_i)$ a *vertex* of the piecewise smooth k-gon, $\alpha_i \mid [t_i, t_{i+1}]$ an *edge* of the k-gon. A piecewise smooth k-gon is a geodesic k-gon if each smooth edge α_i is a geodesic. The angle

$$\theta_{i+1} = \measuredangle\, (\alpha_i'(t_{i+1}), \alpha_{i+1}'(t_{i+1}))$$

is the *exterior angle* at the vertex $\alpha(t_i)$, and

$$\beta_i = \pi - \theta_i$$

is the *interior angle* at $\alpha(t_i)$. Set

$$\phi_i(t) = \measuredangle\, (e_1(\alpha(t)), \alpha_i'(t)).$$

Then the total turning of the tangents along α_i is $\phi_i(t_{i+1}) - \phi_i(t_i)$. The total turning of the tangents for the piecewise smooth k-gon is the sum of the turning on every edge plus the jump at the vertex $\alpha(t_i)$, so the total turning of tangents is

$$\sum_{i=1}^{k} (\phi_i(t_{i+1}) - \phi_i(t_i)) + \theta_i.$$

The following is a theorem in topology:

Theorem 15.0.11. *(Theorem of turning tangents)*

$$\sum_{i=1}^{k} (\phi_i(t_{i+1}) - \phi_i(t_i)) + \theta_i = 2\pi.$$

16. Local Gauss-Bonnet formula

Let $d\sigma$ denote the area element of the surface $f : \mathcal{O} \to \mathbb{R}^3$, i.e.,

$$d\sigma = \sqrt{g_{11}g_{22} - g_{12}^2}\; dx_1 \wedge dx_2 = w_1 \wedge w_2.$$

The Gauss equation (13.0.5), $dw_{12} = Kw_1 \wedge w_2$, implies that

$$\int\int_\Omega K\; d\sigma = \int\int_\Omega K w_1 \wedge w_2 = \int\int_\Omega dw_{12}.$$

By the Green's formula (or the Stoke's formula), we have

$$\int\int_\Omega dw_{12} = \int_{\partial\Omega} w_{12},$$

so

$$\int\int_\Omega K\; w_1 \wedge w_2 = \int_{\partial\Omega} w_{12}$$

Suppose $\triangle$ is a geodesic triangle, i.e., $\partial\triangle$ can be parametrized by a piecewise smooth $\alpha : [0, c] \to \mathbb{R}^3$ with $0 = t_1 < t_2 < t_3 < t_4 = c$ such that $\alpha_i = \alpha \mid [t_i, t_{i+1}]$ is a smooth geodesic for $1 \leq i \leq 3$ and $\alpha(0) = \alpha(c)$. Let

$\theta_i = \measuredangle\ (\alpha_i'(t_{i+1}), \alpha_i(t_{i+1}))$ denote the exterior anlge at the vertex $\alpha(t_i)$, and $\beta_i = \pi - \theta_i$ the interior angle at $\alpha(t_i)$.

Theorem 16.0.12.
(1) $\int\int_\triangle K\ d\sigma = 2\pi - (\theta_1 + \theta_2 + \theta_3)$,
(2) $\beta_1 + \beta_2 + \beta_3 = \pi + \int\int_\triangle K\ d\sigma$.

Proof. We have shown that $\int\int_\triangle K\ d\sigma = \int_{\partial\triangle} w_{12}$. But

$$\int_{\partial\triangle} w_{12} = \int_0^c w_{12}(\alpha'(t))\ dt = \sum_{i=1}^3 \int_{t_i}^{t_{i+1}} w_{12}(\alpha_i'(t)\ dt.$$

Since α_i is a geodesic, by Corollary 14.0.10,

$$w_{12}(\alpha') = \phi_i',$$

where $\phi_i(t) = \measuredangle\ (\alpha_i'(t_i), \alpha_i'(t_{i+1}))$. So

$$\int\int_\triangle K\ d\sigma = \sum_{i=1}^3 \int_{t_i}^{t_{i+1}} \phi_i'(t)\ dt = \sum_{i=1}^3 \phi_i(t_{i+1}) - \phi_i(t_i).$$

The Theorem of turning tangents implies that the above term is equal to $2\pi - (\theta_1 + \theta_2 + \theta_3)$, which proves the first formula. Since $\beta_i = \pi - \theta_i$, the second formula follows. ☐

Corollary 16.0.13. *The sum of interior angles of a geodesic triangle*
(1) *in the plane is* 2π*,*
(2) *in the unit sphere is* 2π *plus the area of the triangle,*
(3) *in a surface with* $K = -1$ *is* 2π *minus the area of the triangle.*

17. Closed surface in $\mathbb{R}^3$

Given a smooth function $u : \mathcal{O} \to \mathbb{R}$, the graph of u over the xy-, yz-, and xz- plane are $\{(x, y, u(x, y) \mid (x, y) \in \mathcal{O}\}$, $\{(u(y, z), y, z) \mid (y, z) \in \mathcal{O}\}$, and $\{(x, u(x, z), z) \mid (x, z) \in \mathcal{O}\}$ respectively.

A subset M of $\mathbb{R}^3$ is called an *embedded surface* if given any point $p \in M$ there exist an open subset $\mathcal{U}$ of $\mathbb{R}^3$ containing p, an open subset $\mathcal{O}$ of $\mathbb{R}^2$, and a smooth function $u : \mathcal{O} \to \mathbb{R}$ such that $\mathcal{U} \cap M$ is the graph of u over xy-plane, yz-plane, or xz-plane.

An emebedded surface M is *closed* if M is bounded.

Example 17.0.14. The unit sphere S^2 is an embedded surface. For given $p = (p_1, p_2, p_3) \in S^2$, we have $p_1^2 + p_2^2 + p_3^2 = 1$, so not all p_1, p_2 p_3 are zero. If $p_1 < 0$, then $\mathcal{U} = \{(x, y, z) \mid x < 0$, and $\mathcal{U} \cap S^2$ is the graph of

$$u : \{(x, y) \mid x^2 + y^2 < 1\} \to \mathbb{R}, \quad \text{defined by } u(x, y) = -\sqrt{1 - x^2 - y^2}.$$

Given $S \subset \mathbb{R}^3$, a subset A of S is *open in S* if there exists an open subset U of $\mathbb{R}^3$ such that $A = S \cap U$.

Example 17.0.15. Suppose $f : \mathcal{O} \to \mathbb{R}^3$ is a parametrized surface, f is $1-1$, and $f(U)$ is open in $f(\mathcal{O})$ if U is open in $\mathcal{O}$. Then $M = f(\mathcal{O})$ is an embedded surface. For f is a parametrized surface, $f_{x_1} \times f_{x_2} \neq 0$. Let $f = (f_1, f_2, f_3)$. $f_{x_1} \times f_{x_2}(p_0) \neq 0$ implies that one of the following determinants at p_0 is not zero:

$$d_1 = \begin{vmatrix} (f_1)_{x_1} & (f_1)_{x_2} \\ (f_2)_{x_1} & (f_2)_{x_2} \end{vmatrix}, \quad d_2 = \begin{vmatrix} (f_1)_{x_1} & (f_1)_{x_2} \\ (f_3)_{x_1} & (f_3)_{x_2} \end{vmatrix}, \quad d_3 = \begin{vmatrix} (f_2)_{x_1} & (f_2)_{x_2} \\ (f_3)_{x_1} & (f_3)_{x_2} \end{vmatrix}.$$

If $d_1(p_0) \neq 0$, then by the Inverse Function Theorem the map

$$(x_1, x_2) \mapsto (y_1, y_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$$

is a local diffeomorphism, i.e., locally there is an inverse $(x_1, x_2) = g(y_1, y_2)$. Then $(y_1, y_2, f_3(g(y_1, y_2))$ is the graph over y_1y_2-plane. Similar argument shows that if $d_2(p_0) \neq 0$, then near $f(p_0)$, $f(\mathcal{O})$ is a graph over the y_1y_3-plane; if $d_3(p_0) \neq 0$, then near $f(p_0)$, $f(\mathcal{O})$ is a graph over y_2y_3-plane.

Given a smooth function $\phi : \mathbb{R}^3 \to \mathbb{R}$ and $c \in \mathbb{R}$, the level set $\phi^{-1}(c) = \{p \in \mathbb{R}^3 \mid \phi(p) = c\}$. The gradient of ϕ at p is

$$\nabla\phi(p) = (\phi_{x_1}(p), \phi_{x_2}(p), \phi_{x_3}(p)).$$

We call $c \in \mathbb{R}$ a *regular value of* ϕ if for all $p \in \phi^{-1}(c)$, $\nabla\phi(p) \neq 0$.

Theorem 17.0.16. *Suppose $\phi : \mathbb{R}^3 \to \mathbb{R}$ is smooth, and c is a regular value of ϕ. Then $\phi^{-1}(c)$ is an embedded surface.*

Proof. Given $p \in \phi^{-1}(c)$, since c is a regular value for ϕ, $(\phi_{x_1}(p), \phi_{x_2}(p), \phi_{x_3}(p)) \neq (0, 0.0)$. So one of the component must be non-zero, say $\phi_{x_3}(p) \neq 0$. Then the Implicit Function Theorem says that near p, we can find a function u such that $\phi(x_1, x_2, u(x_1, x_2)) = c$, so near p, $\phi^{-1}(c)$ is a graph. $\square$

Exercise 17.0.12.

(1) Prove that $\{(x, y, z) \mid x^2 + y^2 - z^2 = 1\}$ is an embedded surface.
(2) Is $\{(x, y, z) \mid x^2 + y^2 - z^2 = 0\}$ an embedded surface?

18. Euler characteristic

Let $\triangle_0$ denote the triangle in $\mathbb{R}^2$ with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. A triangle on an embedded surface M is the image of a one to one continuous map $h : \triangle_0 \to M$. Note that we do not require the restriction of h to the boundary to be piecewise smooth, and h is only assumed to be continuous.

A triangulation on an embedded surface M in $\mathbb{R}^3$ is a collection of triangles $\{\triangle_i \mid i \in I\}$ satisfying the following conditions:

(1) $\cup_{i\in I}\triangle_i = M$,
(2) $\triangle_i \cap \triangle_j$ is either an empty set, a vertex, or an edge,

By the definition of surfaces, we see that there are exactly two triangles contains each edge of the triangulation.

It is a theorem in topology that every embedded closed surface has a triangulation. Given a triangulation $\mathcal{D} = \{\triangle_i \mid i \in I\}$ on M, let

$V =$ the number of vertices in $\mathcal{D}$,

$E =$ the number of edges in $\mathcal{D}$,

$F =$ the number of faces in $\mathcal{D}$.

Another theorem in topology states that the number $F - E + V$ is independent of triangulations (for a proof of this theorem see []). The *Euler characteristic of* M is

$$\mathcal{X}(M) = F - E + V.$$

To give some reason to see why$\mathcal{X}(M)$ is independent of choices of triangulations, let us start with a triangluation $\mathcal{D}$ of a surface M. Suppose s is a common edge of triangles $\triangle_1, \triangle_2$ in $\mathcal{D}$. We want to construct a new triangluation on M that adds the mid point of s as a vertex. Then we must joint the third vertices of $\triangle_1$ and $\triangle_2$ to this mid point in order to make it a triangulation, which will be denoted by $\mathcal{D}'$. Let $F(\mathcal{D}), E(\mathcal{D}), V(\mathcal{D})$ denote the number of faces, edges, and vertices in $\mathcal{D}$, and similar notations for $\mathcal{D}'$. Note that $\mathcal{D}'$ has one more vertex, two more faces, and three move edges than $\mathcal{D}$, so

$$F(\mathcal{D}') - E(\mathcal{D}') + V(\mathcal{D}') = F(\mathcal{D}) - E(\mathcal{D}) + V(\mathcal{D}).$$

Let M_1 and M_2 be two embedded surfaces in $\mathbb{R}^3$. A map $\psi : M_1 \to M_2$ is *continuous* if the preimage of an open subset in M_2 is an open subset in M_1. A continuous 1-1 and onto map $\psi : M_1 \to M_2$ is called a *homemorphism* if the inverse ψ^{-1} is also continuous. Note that if $\mathcal{D} = \{\triangle_i \mid i \in I\}$ is a triangluation of M_1 and $\psi : M_1 \to M_2$ is a homeomorphism, then

$$\psi(\mathcal{D}) = \{\psi(\triangle_i) \mid i \in I\}$$

is a triangulation of M_2. Moreover, $\mathcal{X}(\mathcal{D}) = \mathcal{X}(\psi(\mathcal{D}))$, hence $\mathcal{X}(M_1) = \mathcal{X}(M_2)$. A property P of embedded surfaces is said to be *invariant under homeomorphism* if M_1 has property P then so is $\psi(M)$ for any homeomorphism ψ. We call such property a *topological invariant* of surfaces. So the Euler-characteristic is a topological invariant of surfaces.

Exercise 18.0.13. Let M be the unit sphere S^2. We use three great circles in the xy-, yz-, and xz-plane to cut S^2 into 8 pieces. Each piece is a geodesic triangle. It can be checked easily that these 8 triangles give a triangulation for S^2. Then $F = 8$, $E = 12$, and $V = 6$, so $\mathcal{X}(S^2) = 2$.

Exercise 18.0.14. Attaching a handle to a closed surface

Suppose $\mathcal{D} = \{\triangle_i \mid i \in I\}$ is a triangulation on M. Choose any two disjoint triangles $\triangle_1, \triangle_2$ in $\mathcal{D}$. We attach a handle to M as follows:

(1) take out the interiors of $\triangle_1, \triangle_2$ from M,

(2) attach a triangular cylinder without base and the top to $M \setminus \{\triangle_1 \cup \triangle_2\}$ by gluing the boundary of the top of the cylinder to $\partial\triangle_1$ and the boundary of the bottom of the cylinder to $\partial\triangle_2$.

To compute the Euler characteristic of Σ_1, we choose the triangulation $\mathcal{E}$ on Σ_1 as follows:

(1) use the same triangulation on M,
(2) the cylinder part has three sides, each side is a rectangle, add one diagonal to each rectangular side.

Then it is easy to see that

$$F(\mathcal{E}) = F(\mathcal{D}) - 2 + 6, \quad E(\mathcal{E}) = E(\mathcal{D}) + 6, \quad V(\mathcal{E}) = V(\mathcal{D}).$$

So $\mathcal{X}(\Sigma_1) = \mathcal{X}(M) - 2$. If $M = S^2$, then the resulting surface Σ_1 is homemorphic to a torus T^2 and hence the Euler characteristic of a torus is 0. In general, if we attach g handles to S^2, then the resulting surface Σ_g is a torus with g holes and the above argument implies that

$$\mathcal{X}(M_g) = 2 - 2g.$$

19. Gauss-Bonnet Theorem

Theorem 19.0.17. *(Gauss-Bonnet Theorem) If M is an embedded closed surface in $\mathbb{R}^3$, then*

$$\int\int_M K\, d\sigma = 2\pi\mathcal{X}(M).$$

Proof. We may choose a triangulation of M such that each triangle $\triangle_i$ is a geodesic triangle. Since M is bounded, the triangulation has only finitely many triangles, say m triangles, $\triangle_1, \triangle_2, \ldots, \triangle_m$. Let $\theta_j(i)$ and $\beta_j(i)$ be the exterior angle and interior angle of $\triangle_i$ at the j-th vertex. Then $\theta_j(i)+\beta_j(i) = \pi$. Note

$$\int\int_M K\, d\sigma = \sum_{i=1}^m \int\int_{\triangle_i} K\, d\sigma = \sum_{i=1}^m \int_{\partial\triangle_i} w_{12}.$$

By Theorem 16.0.12, we have

$$\int_{\partial\triangle_i} w_{12} = 2\pi - (\theta_1(i) + \theta_2(i) + \theta_3(i))$$

so

$$\int\int_M K\, d\sigma = 2\pi m - \sum_{i=1}^m \sum_{j=1}^3 \theta_j(i).$$

But exterior angle is computed on each edge of the triangle, and each edge is contained in exactly two triangle, so

$$\sum_{i=1,j=1}^{m,3} \theta_j(i) = 2\pi E - \sum_{i=1,j=1}^{m,3} \beta_j(i).$$

In order to get the sum of all interior angles of all m triangles, we can add the interior angles at each vertex, which is 2π, hence

$$\sum_{i=1,j=1}^{m,3} \beta_j(i) = 2\pi V.$$

We assume the triangulation has m triangles, so $F = m$ and $\int\int_M K \, d\sigma$ is equal to $2\pi F - (2\pi E - 2\pi V)$, which is equal to $2\pi \mathcal{X}(M)$. □

We call $\int\int_M K \, d\sigma$ the *total curvature* of M. It is clear that K depends on the first fundamental form of M, so the total curvature is a geometric invariant, but $2\pi\mathcal{X}(M)$ is a topological invariant. One consequence of the above theorem is that the total curvature is a topological invariant. For example, if M is a closed surface in $\mathbb{R}^3$ with g holes, then $\int\int_M K \, d\sigma = 2\pi(2 - 2g) = 4\pi(1 - g)$ regardless of how M is embedded in $\mathbb{R}^3$.

The Gauss Bonnet Theorem is often viewed as the beginning of global differential geometry and topology, and also the first indication of the relation between curvature and characteristic classes, which plays an very important role in modern mathematics and mathematical physics.