ASSOCIATIVE CONES AND INTEGRABLE SYSTEMS

SHENGLI KONG, CHUU-LIAN TERNG, AND ERXIAO WANG

ABSTRACT. We identify $\mathbb{R}^7$ as the pure imaginary part of octonions. Then the multiplication in octonions gives a natural almost complex structure for the unit sphere $S^6$. It is known that a cone over a surface $M$ in $S^6$ is an associative submanifold of $\mathbb{R}^7$ if and only if $M$ is almost complex in $S^6$. In this paper, we show that the Gauss-Codazzi equation for almost complex curves in $S^6$ is the equation for primitive maps associated to the 6-symmetric space $G_2/T^2$, and use this to explain some of the known results. Moreover, the equation for $S^1$-symmetric almost complex curves in $S^6$ is the periodic Toda lattice associated to $G_2$, and a discussion of periodic solutions is given.

1. INTRODUCTION

We identify $\mathbb{R}^7$ as the pure imaginary part of the octonions $\mathbb{O}$. It is known that the group of automorphism of $\mathbb{O}$ is the compact simple Lie group $G_2$, and the constant 3-form on $\mathbb{R}^7$,

$$\phi(u_1, u_2, u_3) = (u_1 \cdot u_2, u_3),$$

is invariant under $G_2$. A 3-dimensional submanifold $M$ in $\mathbb{R}^7$ is associative if $\mathbb{R}1+TM_x$ is an associative subalgebra of $\mathbb{O}$ for all $x \in M$, i.e., it is isomorphic to the quaternions. It is easy to see that a 3-dimensional submanifold of $\mathbb{R}^7$ is associative if and only if it is calibrated by the 3-form $\phi$.

The multiplication of octonions defines an almost complex structure on the unit sphere $S^6$ by $J_x(v) = x \cdot v$. An immersion $f$ from a Riemann surface $\Sigma$ to $S^6$ is called almost complex if the differential of $f$ is complex linear, i.e.,

$$df_x(iv) = J_x(df_x(v)) = x \cdot df_x(v).$$

It is known that ([11]) a surface $\Sigma$ is an almost complex curve in $S^6$ if and only if the cone over $\Sigma$ is an associative submanifold of $\mathbb{R}^7$.

An immersion $f$ from a Riemann surface to $S^n$ is called totally isotropic if

$$((\nabla_{\frac{\partial}{\partial z^i}})^*f_*(\frac{\partial}{\partial z^j}), (\nabla_{\frac{\partial}{\partial z^i}})^*f_*(\frac{\partial}{\partial z^j})) = 0$$

for all $i \geq 1$, where $(X, Y) = \sum_{i=1}^{n+1} X_i Y_i$ is the complex bilinear form on $\mathbb{C}^{n+1}$. A surface in $S^n$ is said to be full if it does not contain in any hypersphere. Bolton, Vrancken, and Woodward ([4]) used harmonic sequences to prove that if $f : \Sigma \to S^6$ is an immersed almost complex curve, then $f$ must be one of the following:

(i) full in $S^6$ and totally isotropic,
(ii) full in $S^6$ and not totally isotropic,
(iii) full in some totally geodesic $S^5$ in $S^6$,
(iv) a totally geodesic $S^2$.

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Bryant ([5]) used twistor theory to construct type (i) almost complex curves of any genus in \(S^6\). Cones over a type (iii) almost complex curves in \(S^6\) are special Lagrangian submanifolds, which have been studied by several authors ([8, 12, 13, 16, 15]). To state known results for type (ii) almost complex curves, we need to recall Burstall and Pedit’s definition of primitive maps ([6]). Let \(\sigma\) be an order 6 inner automorphism of \(G_2\) such that the fixed point set of \(\sigma\) is a maximal torus \(T^2\), i.e., \(G_2/T^2\) is a 6-symmetric space. Let \(h_j\) denote the eigenspace of the complexified \(d\sigma_c\) on \(g^C_2 = g_2 \otimes \mathbb{C}\). A map \(f : \mathbb{C} \rightarrow G_2/T^2\) is primitive if there is a lift \(F : \mathbb{C} \rightarrow G_2\) such that \(F^{-1}F_z \in h_0 + h_{-1}\). We will call any smooth map \(F : \mathbb{C} \rightarrow G_2\) satisfying the condition that \(F^{-1}F_z \in h_0 + h_{-1}\) a \(\sigma\)-primitive \(G_2\)-frame. Bolton, Pedit, and Woodward ([3]) proved that if \(f : \Sigma \rightarrow S^6\) is a type (ii) almost complex curve, then there exists a \(\sigma\)-primitive \(G_2\)-frame \(\psi\). Conversely, they show that if \(\psi\) is a \(\sigma\)-primitive \(G_2\)-frame, then the first column of \(\psi\) gives an almost complex curve. The equation for \(\sigma\)-primitive \(G_2\)-frame is an elliptic integrable system, so techniques from integrable systems can be used to study almost complex surfaces in \(S^6\).

In this paper, we prove that if \(\Sigma\) is an immersed almost complex surface in \(S^6\) such that the second fundamental form \(\Pi\) is not zero at \(p_0\), then there exist an open neighbor \(O\) of \(p_0\) and a \(\sigma\)-primitive \(G_2\)-frame \(\psi : O \rightarrow G_2\) such that the first column is the immersion. In other words, the Gauss-Codazzi equation for the associative cones in \(\mathbb{R}^7\) is the equation for \(\sigma\)-primitive \(G_2\)-frames. Then we use this elementary submanifold geometry set up to derive some of the known properties of almost complex curves in \(S^6\). We also formulate the equation for \(S^1\)-symmetric almost complex curves in \(S^6\) as a Toda type equation and use the AKS (Adler-Kostant-Symes) theory (cf. [1, 6, 2]) to construct \(S^1\)-symmetric almost complex curves.

This paper is organized as follows. We review basic properties of \(G_2\) ([14]) in section 2, prove the existence of a \(\sigma\)-primitive \(G_2\)-frame on an almost complex surface with non-vanishing second fundamental form in section 3. The equation for \(\sigma\)-primitive \(G_2\)-frame is a system of first order PDEs for 5 complex functions, we explain in section 4 the necessary and sufficient conditions on these 5 functions corresponding to the four types of almost complex curves. In section 5, we explain how periodic Toda lattice arises from \(S^1\)-symmetric almost complex curves in \(S^6\), and finally in section 6, we use the AKS theory to construct all \(S^1\)-symmetric almost complex curves.

2. The octonions and Lie group \(G_2\)

Let \(\mathbb{H} = \mathbb{R}\{1, i, j, k\}\) be the quaternions, where \(i, j, k\) satisfy the condition \(i \cdot j = k, j \cdot k = i, k \cdot i = j, i^2 = j^2 = k^2 = -1\). The conjugate of \(a = a_0 + a_1i + a_2j + a_3k\) is \(\overline{a} = a_0 - a_1i - a_2j - a_3k\). The quaternions \(\mathbb{H}\) equipped with the standard norm of \(\mathbb{R}^4\) is an associative normed algebra, i.e., \(\|a \cdot b\| = \|a\| \cdot \|b\|\). The octonions are defined to be \(\mathbb{O} = \mathbb{H} \oplus \mathbb{He}\) with
the multiplication

\[(a + be) \cdot (c + de) = (a \cdot c - d \cdot b) + (d \cdot a + b \cdot c)e\]

The octonions \(\mathbb{O}\) equipped with the standard norm of \(\mathbb{R}^8\) is a non-associative normed algebra. Let \(\{e_1, \cdots, e_7\}\) be the standard basis of \(\mathbb{R}^7\). We identify \(\mathbb{R}^7\) with \(\text{Im}\mathbb{O}\) as follows:

\[e_1 \rightarrow i, \ e_2 \rightarrow j, \ e_3 \rightarrow k, \ e_4 \rightarrow e, \ e_5 \rightarrow ie, \ e_6 \rightarrow je, \ e_7 \rightarrow ke.\]

The multiplication table of octonions is:

<table>
<thead>
<tr>
<th></th>
<th>(e_1)</th>
<th>(e_2)</th>
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<th>(e_4)</th>
<th>(e_5)</th>
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<tbody>
<tr>
<td>(e_1)</td>
<td>-1</td>
<td>(e_3)</td>
<td>-(e_2)</td>
<td>(e_5)</td>
<td>-(e_4)</td>
<td>-(e_7)</td>
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<tr>
<td>(e_2)</td>
<td>-(e_3)</td>
<td>-1</td>
<td>(e_1)</td>
<td>(e_6)</td>
<td>(e_7)</td>
<td>-(e_4)</td>
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</tr>
<tr>
<td>(e_3)</td>
<td>(e_2)</td>
<td>(e_1)</td>
<td>-1</td>
<td>(e_7)</td>
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<td>-(e_4)</td>
</tr>
<tr>
<td>(e_4)</td>
<td>-(e_5)</td>
<td>-(e_6)</td>
<td>(e_7)</td>
<td>-1</td>
<td>(e_1)</td>
<td>(e_2)</td>
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<td>(e_5)</td>
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<td>-1</td>
<td>(e_1)</td>
</tr>
</tbody>
</table>

The Lie group \(G_2\) is defined by

\[G_2 = \text{Aut}(\mathbb{O}) = \{g \in \text{GL}(\mathbb{O}) \mid g(x \cdot y) = g(x) \cdot g(y)\}\]

We list below some basic properties of the Lie group \(G_2\) we need in this paper:

1. Let \(f_1, f_2\) be two orthonormal column vectors in \(\mathbb{R}^7\). If \(f_3 = f_1 \cdot f_2\), then \(f_3\) is a unit vector and perpendicular to \(f_1, f_2\). Let \(f_4\) be a unit column vector which is perpendicular to \(f_1, f_2, f_3\) and denote \(f_5 = f_1 \cdot f_4, \ f_6 = f_2 \cdot f_4, \ f_7 = f_3 \cdot f_4\). Then \((f_1, \cdots, f_7) \in G_2\) Such \(\{f_1, \cdots, f_7\}\) is called a \(G_2\)-frame.

2. Any element of \(G_2\) can be realized by a \(G_2\)-frame.

3. \(G_2\) is a compact, simply-connected, simple Lie group, \(G_2 \subset \text{SO}(\text{Im}\mathbb{O})\), and \(\dim(G_2) = 14\).

4. Let \(x^1, \cdots, x^7\) be coordinates of \(\mathbb{R}^7\). The 3-form \(\phi(x, y, z) = (x, y \cdot z)\) can be written as

\[
\phi = dx^{123} + dx^{145} - dx^{167} + dx^{246} - dx^{275} + dx^{347} - dx^{356}
\]

where \(dx^{ijkl} = dx^j \wedge dx^k \wedge dx^l\). Then

\[G_2 = \{g \in \text{GL}(7, \mathbb{R}) \mid g^*\phi = \phi\}\]

5. The Lie algebra \(\mathfrak{g}_2\) of \(G_2\) are the space of matrices

\[
\begin{pmatrix}
0 & -x_2 & -x_3 & -x_4 & -x_5 & -x_6 & -x_7 \\
x_2 & 0 & -y_3 & -y_4 & -y_5 & -y_6 & -y_7 \\
x_3 & y_3 & 0 & -x_5 + y_5 & -x_4 - y_4 & x_4 - y_7 & x_5 + y_6 \\
x_4 & y_4 & x_6 - y_5 & 0 & -z_5 & -z_6 & -z_7 \\
x_5 & y_5 & x_7 + y_4 & z_5 & 0 & -x_2 - z_7 & -x_3 + z_6 \\
x_6 & y_6 & -x_4 + y_7 & z_6 & x_2 + z_7 & 0 & -y_3 - z_5 \\
x_7 & y_7 & -x_5 - y_6 & z_7 & x_3 - z_6 & y_3 + z_5 & 0
\end{pmatrix}
\] (2.1)
where \( x_2, \ldots, x_7, y_3, \ldots, y_7, z_5, z_6, z_7 \) are real numbers. To see this fact, we let \( \{e_1, \ldots, e_7\} \) be the standard bases in \( \mathbb{R}^7 \). We have \( e_3 = e_1 \cdot e_2, e_5 = e_1 \cdot e_4, e_6 = e_2 \cdot e_4, e_7 = (e_1 \cdot e_2) \cdot e_4 \). If \( A \in \mathfrak{g}_2 \), then

\[
A(e_j \cdot e_k) = A(e_j) \cdot e_k + e_j \cdot A(e_k)
\]

So \( A \) is determined by \( A(e_1), A(e_2) \) and \( A(e_4) \). Let \( A(e_1) = x_2e_2 + \cdots + x_7e_7 \). Since \( A \in \mathfrak{g}_2 \subset \mathfrak{so}(7) \), we can write \( A(e_2) = -x_2e_1 + y_3e_3 + \cdots + y_7e_7 \). Then

\[
A(e_3) = A(e_1) \cdot e_2 + e_1 \cdot A(e_2)
= -x_3e_1 - y_3e_2 + (x_6 - y_5)e_4 + (x_7 + y_4)e_5 + (y_7 - x_4)e_6 - (x_5 + x_6)e_7
\]

Since \( A \in \mathfrak{g}_2 \subset \mathfrak{so}(7) \), we can write

\[
A(e_4) = -x_4e_1 - y_4e_2 + (y_5 - x_6)e_3 + z_5e_5 + z_6e_6 + z_7e_7
\]

Similarly \( A(e_5), \ldots, A(e_7) \) are determined. Thus \( A \) is a matrix of type (2.1). Conversely, any matrix of type (2.1) is a element of \( \mathfrak{g}_2 \).

3. \( \sigma \)-primitive \( G_2 \)-frame

Let \( X_2 \) denote the matrix defined by (2.1) with \( x_2 = 1 \), and all other variables being zero. The matrices \( X_3, \ldots, X_7, Y_3, \ldots, Y_7, Z_5, Z_6, Z_7 \) are defined similarly.

Let \( h = \exp(\frac{\pi i}{3}(Y_3 + 2Z_5)) \), and \( \sigma : G_2 \to G_2 \) the order 6 inner automorphism defined by \( \sigma(g) = h^{-1}gh \). The eigenspace \( \mathfrak{h}_j \) with eigenvalue \( \exp(i\frac{\pi j}{3}) \) for the complexified \( d\sigma_e \) on \( \mathfrak{g}_2^C = \mathfrak{g}_2 \otimes \mathbb{C} \) is:

\[
\begin{align*}
\mathfrak{h}_0 & = \{Y_3, Z_5\} \\
\mathfrak{h}_1 & = \{X_2 + iX_3 + \frac{i}{2}(Z_6 + iZ_7), Y_4 + iY_5, Z_6 - iZ_7\} \\
\mathfrak{h}_2 & = \{X_1 + iX_5 - \frac{i}{2}(Y_6 + iY_7), Y_6 - iY_7\} \\
\mathfrak{h}_3 & = \{X_6 - iX_7 + \frac{i}{2}(Y_4 - iY_5), X_6 + iX_7 - \frac{i}{2}(Y_4 + iY_5)\} \\
\mathfrak{h}_4 & = \{X_4 - iX_5 + \frac{i}{2}(Y_6 - iY_7), Y_6 + iY_7\} \\
\mathfrak{h}_5 & = \{X_2 - iX_3 - \frac{i}{2}(Z_6 - iZ_7), Y_4 - iY_5, Z_6 + iZ_7\}
\end{align*}
\]

Here \( \{v_1, \ldots, v_m\} \) means the linear span of \( v_1, \ldots, v_m \). Notice \( \mathfrak{h}_j = \mathfrak{h}_{-j} \) (we use the convention that \( \mathfrak{h}_i = \mathfrak{h}_j \) if \( i \equiv j \) (mod 6)).

A smooth map \( \psi : \mathbb{C} \to G_2 \) is \( \sigma \)-primitive if

\[
\psi^{-1}d\psi = (u_0 + u_{-1})dz + (\bar{u}_0 + \bar{u}_{-1})d\bar{z}.
\]

The flatness of \( \psi^{-1}d\psi \) implies that \( (u_0, u_{-1}) : \mathbb{C} \to \mathfrak{h}_0 \oplus \mathfrak{h}_{-1} \) must satisfy

\[
\begin{align*}
(u_0)_z - (\bar{u}_0)_z & = [u_0, \bar{u}_0] + [u_{-1}, \bar{u}_{-1}], \\
(u_{-1})_z & = [u_{-1}, \bar{u}_0].
\end{align*}
\]

This system has a Lax pair

\[
\theta_\lambda = (u_0 + \lambda^{-1}u_{-1})dz + (\bar{u}_0 + \lambda\bar{u}_{-1})d\bar{z}
\]
i.e., \((u_0, u_{-1})\) is a solution of (3.2) if and only if \(\theta_\lambda\) is flat for all \(\lambda \in \mathbb{C}\setminus\{0\}\). Note that:

1. The Lax pair satisfies the following reality conditions:

\[
\left(\theta_{1/\lambda}\right) = \theta_\lambda, \quad \sigma(\theta_\lambda) = \theta_{e^{2\pi i/\lambda}}
\]  

(3.3)

2. \(\xi(\lambda) = \sum_j \xi_j \lambda^j\) satisfies the above reality condition if and only if \(\xi_j \in \frak{h}_j\) and \(\xi_{-j} = \bar{\xi}_j\) for all \(j\).

The following is well-known:

**Proposition 3.1.** Let \((u_0, u_{-1}) : \mathbb{C} \to \frak{h}_0 \oplus \frak{h}_{-1}\) be smooth maps. The following statements are equivalent:

1. \((u_0, u_{-1})\) satisfies (3.1).
2. \(\theta_\lambda = (u_0 + \lambda^{-1} u_{-1}) \, dz + (\omega_0 + \lambda \omega_1) \, d\bar{z}\) is flat for all \(\lambda \in \mathbb{C}\setminus\{0\}\), i.e.,
   \[d\theta_\lambda = -\theta_\lambda \wedge \theta_\lambda,\]
3. \(\theta_1 = (u_0 + u_{-1}) \, dz + (\omega_0 + \omega_1) \, d\bar{z}\) is flat.
4. There exists \(\psi : \mathbb{C} \to G_2\) such that \(\psi^{-1} \psi_z = u_0 + u_{-1}\), i.e., \(\psi\) is a \(\sigma\)-primitive \(G_2\)-frame.

**Proof.** The only nontrivial part is (3) \(\iff\) (1). To see this, we decompose

\[d\theta + \theta \wedge \theta = \left(- (u_0) \bar{z} + (\omega_0) \bar{z} + [u_{-1}, \bar{u}_{-1}]\right) \, dz \wedge d\bar{z} + \left(- (u_{-1}) z + [u_{-1}, \omega_0]\right) \, dz \wedge d\bar{z} + \left((\bar{u}_{-1}) z + [u_0, \bar{u}_{-1}]\right) \, dz \wedge d\bar{z}\]

according to \(\frak{h}_0 \oplus \frak{h}_1 \oplus \frak{h}_{-1}\). Thus \((u_0, u_{-1})\) satisfies (3.1) if and only if

\[d\theta + \theta \wedge \theta = 0.\]

Suppose \((u_0, u_{-1})\) is a solution of (3.1). Since \(\theta_\lambda\) is flat at \(\lambda = 1\), there exists \(\psi : \mathbb{C} \to G_2\) such that

\[
\psi^{-1} \psi_z = u_0 + u_{-1} = \begin{pmatrix}
0 & -c & -i c \\
-c & 0 & -a & -d & id \\
-a & 0 & -id & -d \\
-d & id & 0 & -b & -e + i \frac{1}{2} c & -ie + i \frac{1}{2} c \\
-i d & b & 0 & -ie - i \frac{1}{2} c & e + i \frac{1}{2} c \\
e^{-i \frac{1}{2} c} & ie + i \frac{1}{2} c & 0 & -a - b \\
ie^{-i \frac{1}{2} c} & -e - i \frac{1}{2} c & a + b & 0 \\
\end{pmatrix}
\]  

(3.4)

System (3.1) written in terms of \(a, \ldots, e\) is

\[
\begin{align*}
  a_z - (\overline{\omega})_z &= i \left(2|c|^2 - 4|d|^2\right) \\
  b_z - (\overline{\theta})_z &= i \left(-|c|^2 + 4|d|^2 - 4|e|^2\right) \\
  c_z &= -i\bar{a}c \\
  d_z &= i (\overline{\theta} - \overline{\omega}) \, d \\
  e_z &= i (\overline{\omega} + 2\overline{\theta}) \, e 
\end{align*}
\]  

(3.5)
Let \( f_1, \ldots, f_7 \) denote the columns of \( \psi \). Then (3.4) written in columns gives
\[
\begin{align*}
(f_1)_e &= cf_2 - icf_3, \\
(f_2)_e &= -cf_1 + af_3 + d f_4 - id f_5, \\
(f_3)_e &= icf_1 - af_2 + id f_4 + d f_5, \\
(f_4)_e &= -d f_2 - id f_3 + bf_5 + (e - \frac{i}{2}) f_6 + (ie - \frac{c}{2}) f_7, \\
(f_5)_e &= id f_2 - d f_3 - bf_4 + (ie + \frac{c}{2}) f_6 - (e + \frac{i c}{2}) f_7, \\
(f_6)_e &= (-e + \frac{i}{2} c) f_4 - (ie + \frac{c}{2}) f_5 + (a + b) f_7, \\
(f_7)_e &= (-i e + \frac{c}{2}) f_4 + (e + \frac{ic}{2}) f_5 - (a + b) f_6.
\end{align*}
\]  

(3.6)

4. ASSOCIATIVE CONES AND ALMOST COMPLEX CURVES

The following well-known Proposition relates almost complex curves to associative cones:

**Proposition 4.1.** ([11]) Let \( \Sigma \) be a 2-dimensional surface in \( S^6 \), and \( C(\Sigma) = \{t_0 \mid t > 0, x \in M\} \) the cone of \( \Sigma \) in \( \mathbb{R}^7 \). Then \( C(\Sigma) \) is an associative submanifold in \( \mathbb{R}^7 \) if and only if \( \Sigma \) is a almost complex curve in \( S^6 \).

**Proof.** Let \( \{e_1, e_2\} \) be an orthonormal basis of \( T_x \Sigma \). Then \( \{x, e_1, e_2\} \) is an orthonormal basis of \( T_x C(\Sigma) \). Lemma follows from the fact that \( \mathbb{R} \{1, x, e_1, e_2\} \) is an associative subalgebra if and only if \( x \cdot e_1 = e_2 \). \( \square \)

So the study of associative cones in \( \mathbb{R}^7 \) reduces to the study of almost complex curves in \( S^6 \).

Since associative cones are calibrated by the 3-form \( \phi \), they are minimal. But a cone \( C(\Sigma) \) in \( \mathbb{R}^7 \) is minimal if and only if \( \Sigma \) is minimal in \( S^6 \), so almost complex curves in \( S^6 \) are minimal.

**Theorem 4.2.** ([3]) If \( \psi = (f_1, \ldots, f_7) : \mathbb{C} \rightarrow G_2 \) satisfies
\[
\psi^{-1} \psi_z \in \mathfrak{h}_0 \oplus \mathfrak{h}_{-1}
\]  
(4.1)

Then \( f_1 : \mathbb{C} \rightarrow S^6 \) is almost complex. Conversely, if \( f : \mathbb{C} \rightarrow S^6 \) is a type (ii) almost complex curve, i.e., \( f \) is full and not totally isotropic, then there exists a \( \sigma \)-primitive map \( \psi : \mathbb{C} \rightarrow G_2 \) such that the first column of \( \psi \) is \( f \).

The first part of the above theorem is easy to see: Write \( \psi = (f_1, \ldots, f_7) \), and
\[
\psi^{-1} \psi_z = u_0 + u_{-1}.
\]

Then \( u_0 + u_{-1} \) is given by (3.4), so
\[
(f_1)_e = cf_2 - icf_3.
\]

By the definition of almost complex structure \( J \) on \( S^6 \), we have
\[
J(f_1)_e = f_1 \cdot (f_1)_e = cf_3 + icf_2 = i (f_1)_e
\]

So \( f_1 \) is almost complex.

Next we prove that a \( \sigma \)-primitive \( G_2 \)-frame exists on any almost complex curve in \( S^6 \) with non-vanishing second fundamental forms.
Theorem 4.3. Suppose \( f_1 : \Sigma \to S^6 \) is an almost complex curve such that the second fundamental form \( \Pi \) is non-zero at some \( p_0 \in \Sigma \). Then there exists a neighborhood \( \mathcal{O} \) of \( p_0 \) and a \( \sigma \)-primitive \( G_2 \)-frame \( \psi = \{ f_1, \ldots, f_7 \} \) on \( \mathcal{O} \) such that \( f_2 \) and \( f_3 \) are tangent to the immersion, \( \psi^{-1}\psi_z \) is given by 5 functions \( a, \ldots, e \) and is given by (3.4), and (3.5) is the Gauss-Codazzi equation for \( f_1 \). Moreover, the first and second fundamental forms of \( f_1 \) are

\[
I = 2|c|^2|dz|^2, \\
\Pi \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) = 2cd(f_4 - if_5),
\]

and the normal connection is given by the lower \( 5 \times 5 \) matrices (3.4).

Proof. Locally we can choose orthonormal tangent frame \( \{ f_2, f_3 \} \) such that \( f_3 = f_1 \cdot f_2 \). Let \( f_4 \) be an arbitrary unit vector such that \( f_4 \perp \text{span}_\mathbb{R}\{ f_1, f_2, f_3 \} \). Then we have a \( G_2 \)-frame \( \psi = \{ f_1, \ldots, f_7 \} \) where \( f_5 = f_1 \cdot f_4, f_6 = f_2 \cdot f_4, f_7 = f_3 \cdot f_4 \). Therefore we obtain a \( g_2 \)-valued flat connection 1-form \( \omega = (\omega_{ij}) = \psi^{-1}d\psi \).

Write

\[
df_1 = f_2 \otimes \theta_2 + f_3 \otimes \theta_3,
\]

where \( \theta_j \) is the dual 1-form of \( f_j \) for \( j = 2, 3 \). Therefore

\[
\omega_{21} = \theta_2, \quad \omega_{31} = \theta_3 \quad \omega_{\alpha 1} = 0, \quad 4 \leq \alpha \leq 7
\]

Since \( \omega \) is \( g_2 \)-valued, we have

\[
\omega_{43} = -\omega_{52}, \quad \omega_{53} = \omega_{42}, \quad \omega_{63} = \omega_{72}, \quad \omega_{73} = -\omega_{62}
\]

Let

\[
\omega_{52} = a_2 \theta_2 + a_3 \theta_3, \quad \omega_{62} = b_2 \theta_2 + b_3 \theta_3
\]

It follows from the flatness of \( (\omega_{ij}) \) that

\[
d\omega_{\alpha 1} + \sum_{j=1}^{7} \omega_{\alpha j} \wedge \omega_{j 1} = 0, \quad (\alpha = 4, 5),
\]

so we have

\[
(a_2 \theta_2 + a_3 \theta_3) \wedge \theta_2 + \omega_{42} \wedge \theta_3 = 0
\]

\[
\omega_{42} \wedge \theta_2 - (a_2 \theta_2 + a_3 \theta_3) \wedge \theta_2 = 0
\]

Thus

\[
\omega_{53} = \omega_{42} = a_3 \theta_2 - a_2 \theta_3
\]

Similarly,

\[
\omega_{63} = \omega_{72} = b_3 \theta_2 - b_2 \theta_3
\]

Then the second fundamental form of immersion is given by

\[
\Pi = \sum_{\alpha=4}^{7} f_\alpha \otimes (\omega_{\alpha 2} \otimes \theta_2 + \omega_{\alpha 3} \otimes \theta_3)
\]

\[
= v_1 \otimes (\theta_2 \otimes \theta_2 - \theta_3 \otimes \theta_3) - v_2 \otimes (\theta_2 \otimes \theta_3 + \theta_3 \otimes \theta_2)
\]
where \( v_1 = a_3 f_4 + a_2 f_5 + b_3 f_7 + b_2 f_6 \) and \( v_2 = a_2 f_4 - a_3 f_5 + b_2 f_7 - b_3 f_6 \). Note that
\[
(v_1, v_1) = (v_2, v_2), \quad (v_1, v_2) = 0
\]
Since \( II(p_0) \neq 0 \), there exists a neighborhood \( U \) of \( p \) such that \( v \) and \( w \) are nonzero. Let \( \tilde{f}_j = f_j, \ j = 1, 2, 3, \)
\[
\tilde{f}_4 = \frac{v_1}{||v_1||}
\]
and
\[
\tilde{f}_5 = \tilde{f}_1 \cdot \tilde{f}_4, \quad \tilde{f}_6 = \tilde{f}_2 \cdot \tilde{f}_4, \quad \tilde{f}_7 = \tilde{f}_3 \cdot \tilde{f}_4
\]
Then \( \tilde{\psi} = \{\tilde{f}_1, \ldots, \tilde{f}_7\} \) is a \( G_2 \)-frame, and a computation using the octonion multiplication implies that \( \tilde{f}_5 = v_2 \). Let \( \tilde{\omega} = (\tilde{\omega}_{ij}) = \tilde{\psi}^{-1} d\tilde{\psi} \). Since \( (II, \tilde{f}_6) = (II, \tilde{f}_7) = 0 \), we have
\[
\tilde{\omega}_{02} = \tilde{\omega}_{63} = \tilde{\omega}_{72} = \tilde{\omega}_{73} \equiv 0.
\]
So \( \tilde{\omega} \) lies in \( \mathfrak{h}_0 + \mathfrak{h}_1 + \mathfrak{h}_{-1} \), where \( \mathfrak{h}_j \) is the eigenspace of \( d\sigma \) on \( g_2 \otimes \mathbb{C} \) with eigenvalue \( e^{2\pi i j/3} \). Or equivalently, \( \psi^{-1} \psi_\omega \) is of the form \( (3.4) \), i.e., \( \psi \) is a \( \sigma \)-primitive \( G_2 \)-frame. In particular, this shows that the Gauss-Codazzi equation for almost complex curves is \( (3.5) \). It follows from \( (3.6) \) and a computation that the two fundamental forms for \( f_1 \) are given as in the Theorem. 

As a consequence of the Fundamental Theorem of submanifolds in space forms and the above theorem, we get

**Corollary 4.4.** Every simply connected immersed almost complex curve in \( (S^6, J) \) with non-vanishing second fundamental form has a \( \sigma \)-primitive \( G_2 \)-frame such that the first column is the immersion. Conversely, the first column of a \( \sigma \)-primitive \( G_2 \)-frame is an almost complex surface in \( S^6 \). 

Next, we use Theorem 4.3 to give conditions on \( a, \ldots, e \) to determine the four types of almost complex curves mentioned in the introduction.

**Corollary 4.5.** Let \( (a, \ldots, e) \) be a solution of \( (3.5) \), \( \psi \) a solution of \( (3.4) \), and \( f_1 \) the first column of \( \psi \). Then \( f_1 \) is almost complex in \( S^6 \) and is

(i) full in \( S^6 \) and totally isotropic if and only if \( e \equiv 0 \) and \( d \neq 0 \),
(ii) full in \( S^6 \) and not totally isotropic if and only if \( de \neq 0 \),
(iii) full in \( S^5 \) if and only if \( de \neq 0 \) and \( a + b \equiv 0 \),
(iv) totally geodesic two sphere if and only if \( d \equiv 0 \), i.e., \( II \equiv 0 \).

Moreover, the cone over the curve of type (iii) is a special Lagrangian cone in \( \mathbb{R}^6 \) with the appropriate complex structure.

*Proof.* The first fundamental form is positive definite, so \( c \neq 0 \). A surface is full then \( II \) can not be zero, so \( d \neq 0 \). Let \( \psi \) satisfy \( \psi^{-1} d\psi = (u_0 + u_{-1})dz + (\bar{u}_0 + \bar{u}_{-1})d\bar{z} \), and \( f_1 \) denote the first column of \( \psi \), where \( u_0 + u_{-1} \in \mathbb{C} \)
Assume $\mathfrak{h}_0 + \mathfrak{h}_{-1}$ is given by (3.4). Then $f_1$ is almost complex. Use (3.6) and a direct computation to see that

$$((\nabla_{\bar{\partial}_z})^2 f_2(\frac{\partial}{\partial z}), (\nabla_{\bar{\partial}_z})^2 f_2(\frac{\partial}{\partial z})) = -32i\alpha^3 d^2 e,$$

where $(Y, Z) = \sum_j y_j z_j$ is the complex bilinear form on $\mathbb{C}^7$. If $f$ is totally isotropic, then

$$((\nabla_{\bar{\partial}_z})^2 f_2(\frac{\partial}{\partial z}), (\nabla_{\bar{\partial}_z})^2 f_2(\frac{\partial}{\partial z})) = 0$$

for all other $0 \leq j \leq 2$, so $e = 0$.

Next we prove that if an almost complex curve is of type (III), then $a + b \equiv 0$. Since there is a constant unit normal vector field on the curve, there exists real functions $\lambda_i$ ($4 \leq i \leq 7$) on the curve such that this normal vector is $\sum_{i=4}^7 \lambda_i f_i$. Then

$$\sum_{i=4}^7 \lambda_i f_i = \sum_{i=4}^7 (\lambda_i z f_i + \sum_{i=4}^7 \lambda_i (f_i) z)
= \sum_{i=4}^7 (\lambda_i) z f_i + \lambda_4 [-d f_2 - id f_3 + b f_5 + (e - ic/2) f_6 + (ie - c/2) f_7] + \cdots = 0.$$

So the coefficient of $f_i$ must be zero for $2 \leq i \leq 7$. Since $d \neq 0$, it implies that $\lambda_4 = 0$ and $\lambda_5 = 0$. The coefficients for $f_6$ and $f_7$ are $(\lambda_6) z - (a + b)$ and $(\lambda_7) z + (a + b)$ respectively. Therefore $(\lambda_6 + \lambda_7) z = 0$, i.e., $\lambda_6 + \lambda_7$ is anti-holomorphic. Since $\lambda_6 + \lambda_7$ is also real, it must be a constant. Finally both $\lambda_6$ and $\lambda_7$ have to be constant because their square sum is $1$. Thus $a + b = (\lambda_6) z = 0$.

Conversely, if $a + b \equiv 0$, then the system (3.5) implies that

$$c_\xi = -i\bar{\alpha} c, \quad e_\xi = -i\bar{\alpha} e$$

and $i (|\alpha|^2 - 4|\xi|^2) = (a + b) z - (\bar{a} + \bar{b}) z = 0$. Let $\alpha = \frac{\alpha}{2\xi}$. Then $\alpha z = 0$ and $|\alpha| = 1$. So $\alpha \in S^1$ is a constant and $\beta = \frac{-1 + i \alpha}{-1 + \bar{\alpha}}$ is a real constant. It follows from (3.6) that $(f_6 - \beta f_7) z = 0$. Thus $n = \frac{1}{\sqrt{1 + \beta^2}} (f_6 - \beta f_7)$ is a unit constant normal vector. So the image of the immersion lies in the hyperplane $V$ which is orthogonal to $n$. Note $J(x) = n \cdot x$ defines a complex structure on the hyperplane $V$ and $J(f_1) = \frac{1}{\sqrt{1 + \beta^2}} (\beta f_6 + f_7)$, $J(f_2) = \frac{1}{\sqrt{1 + \beta^2}} (f_4 + \beta f_5)$, and $J(f_3) = \frac{1}{\sqrt{1 + \beta^2}} (-\beta f_4 - f_5)$. Thus $J(\text{span}_{\mathbb{R}} \{f_1, f_2, f_3\}) = \text{span}_{\mathbb{R}} \{f_4, f_5, \beta f_6 + f_7\}$, so the cone over the image of $f_1$ is Lagrangian in $(\mathbb{R}^6, J)$. We know it is minimal, so by Proposition 2.17 of [11] that it is $\theta$-special Lagrangian for some $\theta$. \hfill \Box

Next we use Theorem 4.3 to give a proof of one of Bryant’s results on almost complex curves in $S^6$. First recall that the 5-dimensional complex quadric $Q_5$ is defined by

$$Q_5 = \{[z_1 : \cdots : z_7] \in \mathbb{CP}^6 \mid z_1^2 + \cdots + z_7^2 = 0\}.$$
Theorem 4.6. [5] If \( f : \Sigma \rightarrow S^6 \) is a totally isotropic almost complex curve that is not totally geodesic, then it can be lifted to a horizontal holomorphic map to \( Q_5 \).

Proof. Let \( \psi = (f_1, \ldots, f_7) : \Sigma \rightarrow G_2 \) denote the \( \sigma \)-primitive \( G_2 \)-frame obtained in Theorem 4.3. So \( \psi^{-1}\psi \) is of the form (3.4). Let \( \Phi : \Sigma \rightarrow Q_5 \) be the map defined by

\[
\Phi = [f_6 + if_7]
\]

Clearly \( \Phi \) is well-defined and is independent of choice of the frame. By (3.6), we have

\[
(f_6 + if_7)z = -2\bar{a}(f_4 - if_5) - i(\bar{a} + \bar{b})(f_6 + if_7)
\]

But we have shown in Corollary 4.5 that if \( f \) is totally isotropic then \( e = 0 \), so \( \Phi \) is holomorphic. \( \square \)

5. \( S^1 \)-symmetric solutions and periodic Toda lattice

By the maximal torus theorem, given \( A \in G_2 \), there exists \( k \in G_2 \) and real numbers \( \lambda_1, \lambda_2 \) such that \( A = k^{-1}(\lambda_1 Y_3 + \lambda_2 Z_5)k \). Note

\[
\lambda_1 Y_3 + \lambda_2 Z_5 = \begin{pmatrix}
0 & -\lambda_1 \\
\lambda_1 & -\lambda_2 \\
\lambda_2 & -\lambda_3 \\
-\lambda_3 & 0
\end{pmatrix}
\]

where \( \lambda_3 = -(\lambda_1 + \lambda_2) \). We say \( A = k^{-1}(\lambda_1 Y_3 + \lambda_2 Z_5)k \) is rational if \( \lambda_1, \lambda_2 \) are linearly dependent over the rationals. It is easy to see that \( A \) is rational if and only if \( \{\exp(sA) \mid s \in \mathbb{R}\} \) is periodic.

To construct a \( S^1 \)-symmetric almost complex curve in \( S^6 \), we need to construct \( \psi = e^A g(t) \) with rational \( A \) and \( g(t) \in G_2 \) such that

\[
\psi^{-1}\psi z = u_0 + u_{-1} \in \mathfrak{h}_0 + \mathfrak{h}_1,
\]

where \( z = s + it \) and \( u_0 + u_{-1} \) is given by (3.4) and \( a, b, c, d, e \) are complex valued functions of \( t \) only. A simple computation gives

\[
\psi^{-1}d\psi = (g^{-1}Ag) \, ds + (g^{-1}g_t) \, dt.
\]

The flatness of \( \psi^{-1}d\psi \) implies that

\[
(g^{-1}Ag)_t = [g^{-1}Ag, g^{-1}g_t]. \tag{5.1}
\]

Write \( a = a_1 + ia_2, \ b = b_1 + ib_2, \ldots, \ e = e_1 + ie_2 \) in real and imaginary part, and \( c = r_1e^{i\alpha_1}, \ d = r_2e^{i\alpha_2}, \ e = r_3e^{i\alpha_3} \) in polar coordinates. Since
\( \psi^{-1}\psi_z = g^{-1}Ag = \psi^{-1}\psi_z + \psi^{-1}\psi_t, \psi^{-1}\psi_t = g^{-1}gt = i (\psi^{-1}\psi_z - \psi^{-1}\psi_t), \)

and \( \psi^{-1}\psi_z \) is given by (3.4), we have

\[
g^{-1}Ag = \begin{pmatrix}
0 & -2c_1 & -2c_2 \\
2c_1 & 0 & -2a_1 & -2d_1 & -2d_2 \\
2c_2 & 2a_1 & 0 & 2d_2 & -2d_1 \\
2d_1 & -2d_2 & 0 & -2b_1 & -2e_1 - c_2 & 2e_2 + c_1 \\
2d_2 & 2d_1 & 2b_1 & 0 & 2e_2 - c_1 & 2e_1 - c_2 \\
2e_1 + c_2 & -2e_2 + c_1 & 0 & -2a_1 - 2b_1 \\
2e_2 - c_1 & -2e_1 - c_2 & 2a_1 + 2b_1 & 0
\end{pmatrix}
\]

\[
g^{-1}gt = \begin{pmatrix}
0 & 2c_2 & -2c_1 \\
-2c_1 & 0 & 2a_2 & 2d_2 & -2d_1 \\
2c_2 & -2a_2 & 0 & 2d_1 & 2d_2 \\
-2d_2 & -2d_1 & 0 & 2b_2 & 2e_2 - c_1 & 2e_1 - c_2 \\
2d_1 & -2d_2 & -2b_2 & 0 & 2e_1 + c_2 & -2e_2 - c_1 \\
-2e_2 + c_1 & -2e_1 - c_2 & 0 & 2a_2 + 2b_2 \\
-2e_1 + c_2 & 2e_2 + c_1 & -2a_2 - 2b_2 & 0
\end{pmatrix}
\]

System (5.1) written in \( a, b, r, \beta \) gives the following two separable systems

\[
\begin{align*}
\dot{a}_1 &= 2r_1^2 - 4r_2^2, \\
\dot{b}_1 &= -r_1^2 + 4r_2^2 - 4r_3^2, \\
\dot{r}_1 &= -2a_1r_1, \\
\dot{r}_2 &= 2(a_1 - b_1)r_2, \\
\dot{r}_3 &= 2(a_1 + 2b_1)r_3,
\end{align*}
\]

\[
\begin{align*}
\dot{\beta}_1 &= 2a_2, \\
\dot{\beta}_2 &= -2a_2 + 2b_2, \\
\dot{\beta}_3 &= -2a_2 - 4b_2.
\end{align*}
\]

So we may assume that \( a_2 = b_2 = \beta_1 = \beta_2 = \beta_3 = 0 \), i.e.,

\[
a_2 = b_2 = c_2 = d_2 = e_2 = 0.
\]

Substitute these conditions to the matrix formulas for \( g^{-1}Ag \) and \( g^{-1}gt \) to get

\[
P := g^{-1}Ag = \begin{pmatrix}
0 & -2c_1 \\
2c_1 & 0 & -2a_1 & -2d_1 \\
2a_1 & 0 & -2d_1 \\
2d_1 & 0 & -2b_1 & -2e_1 & c_1 \\
2d_1 & 0 & -2b_1 & -2e_1 & c_1 \\
2e_1 & c_1 & 0 & -2a_1 - 2b_1 \\
-c_1 & -2e_1 & 2a_1 + 2b_1 & 0
\end{pmatrix},
\]

\[
Q := g^{-1}gt = \begin{pmatrix}
0 & -2c_1 \\
0 & 0 & 2d_1 \\
2c_1 & 0 & -c_1 & 2e_1 \\
-2d_1 & 0 & -c_1 & 2e_1 \\
c_1 & -2e_1 & 0 \\
-2e_1 & c_1 & 0
\end{pmatrix}
\]
Since $\psi^{-1}\psi_\omega = u_0 + u_{-1} \in \mathfrak{h}_0 + \mathfrak{h}_{-1}$, $P = u_0 + \bar{u}_0 + u_{-1} + \bar{u}_{-1}$ and $Q = -i(u_0 - \bar{u}_0 + u_{-1} - \bar{u}_{-1})$. By assumption that $a, b, \ldots, e$ are real, so $u_0 = \bar{u}_0$, and
\[
P = 2u_0 + u_{-1} + \bar{u}_{-1}, \quad Q = i(u_{-1} - \bar{u}_{-1}), \tag{5.2}
\]
where
\[
\begin{aligned}
    \begin{cases}
        u_0 = a_1 Y_3 + b_1 Z_5 \in \mathfrak{h}_0 \cap \mathfrak{g}_2, \\
        u_{-1} = c_1 (X_2 - \frac{Z_7}{2}) + i(X_3 + \frac{Z_6}{2}) + d_1 (Y_4 + iY_5) + e_1 (Z_6 - iZ_7) \in \mathfrak{h}_{-1}.
    \end{cases}
\end{aligned}
\]
Thus we have

**Proposition 5.1.** Suppose $(u_0, u_{-1}) : \mathbb{R} \to (\mathfrak{h}_0 \cap \mathfrak{g}_2) \times \mathfrak{h}_{-1}$ satisfies
\[
(2u_0 + u_{-1} + \bar{u}_{-1})_t = [2u_0 + u_{-1} + \bar{u}_{-1}, i(u_{-1} - \bar{u}_{-1})], \tag{5.3}
\]
and there exist a constant $A \in (\mathfrak{h}_0 \cap \mathfrak{g}_2) + \mathfrak{h}_{-1}$ and $g : \mathbb{R} \to G_2$ such that
\[
\begin{cases}
    g^{-1} Ag = 2u_0 + u_{-1} + \bar{u}_{-1}, \\
    g^{-1} g_t = u_{-1} - \bar{u}_{-1}.
\end{cases} \tag{5.4}
\]
Then $f(s, t) = e^{Ag(t)}$ is an almost complex curve in $S^6$. Moreover, $f$ is $S^1$-symmetric if and only if $A$ is rational, and is doubly periodic if and only if $A$ is rational and $g$ is periodic.

Define $v_1, v_2, v_3$ by
\[
\begin{aligned}
    \begin{cases}
        e^{2v_1} = c_1^2, \\
        e^{2(v_2 - v_1)} = d_1^2, \\
        e^{2(v_3 - v_2)} = e_1^2.
    \end{cases}
\end{aligned}
\]
Then $a_1, b_1, v_1, v_2, v_3$ satisfy
\[
\begin{aligned}
    &\begin{cases}
        \dot{a}_1 = 2e^{2v_1} - 4e^{2(v_2 - v_1)}, \\
        \dot{b}_1 = -e^{2v_1} + 4e^{2(v_2 - v_1)} - 4e^{2(v_3 - v_2)}, \\
        \dot{v}_1 = -2a_1, \\
        \dot{v}_2 = -2b_1, \\
        \dot{v}_3 = 2(a_1 + b_1).
    \end{cases} \tag{5.5}
\end{aligned}
\]
Clearly, $(v_1 + v_2 + v_3)_t = 0$. Moreover, $v_1, v_2, v_3$ satisfy
\[
\begin{aligned}
    &\begin{cases}
        \ddot{v}_1 = -4e^{2v_1} + 8e^{2(v_2 - v_1)}, \\
        \ddot{v}_2 = 2e^{2v_1} - 8e^{2(v_2 - v_1)} + 8e^{2(v_3 - v_2)}, \\
        \ddot{v}_3 = 2e^{2v_1} - 8e^{2(v_3 - v_2)}.
    \end{cases}
\end{aligned}
\]
These are equivalent to the periodic Toda lattice equations of $G_2$-type.

If $a_1 + b_1 = 0$, i.e., the type (iii) case, then $\dot{a}_1 + \dot{b}_1 = e^{2v_1} - 4e^{2(v_3 - v_2)} = 0$, $\dot{v}_1 + \dot{v}_2 = \dot{v}_3 = 0$, so there is a positive constant $C_1$ such that:
\[
e^{2(v_1 + v_2)} = 4e^{2v_3} = C_1.
\]
Then $v_1$ satisfies
\[
\ddot{v}_1 + 4 e^{2v_1} - 8C_1 e^{-4v_1} = 0
\]
Multiply $\dot{v}_1$ to both sides and integrating once to get
\[(\dot{v}_1)^2 + 4e^{2v_1} + 4C_1e^{-4v_1} = 4C_2,\]
where $C_2$ is a positive constant. Let $y = e^{2v_1} = r_1^2$. Then the above equation becomes
\[(y)^2 = -16y^2 + 16C_2y^2 - 16C_1.\]
One can verify easily that $4C_2^2 \geq 27C_1$. Therefore this equation has three real constant solutions $\Gamma_1, \Gamma_2, \Gamma_3$. Let us label these solutions so that $\Gamma_1 < 0 < \Gamma_2 \leq \Gamma_3$. Then we can rewrite the previous equation as
\[(\dot{y})^2 = -16(y - \Gamma_1)(y - \Gamma_2)(y - \Gamma_3)\]
Haskins ([12]) showed that this equation has the following solution:
\[y = \Gamma_3 - (\Gamma_3 - \Gamma_2) \text{sn}^2(B_1t + B_2, B_3)\]
where $B_2$ is a constant determined by the initial condition of $y$,
\[B_2^2 = 4(\Gamma_3 - \Gamma_1), \quad B_3^2 = \frac{\Gamma_3 - \Gamma_2}{\Gamma_3 - \Gamma_1}\]
and $\text{sn}$ is the Jacobi elliptic sn-noidal function. Recall that $\text{sn}(t, k)$ is defined to be the unique solution of the equation
\[\dot{z}^2 = (1 - z^2)(1 - k^2z^2)\]
with $z(0) = 0, \dot{z}(0) = 1$, where $0 \leq k \leq 1$. It is straightforward to see from this definition that $\text{sn}(t, 0) = \sin t$ and $\text{sn}(t, 1) = \tanh t$. The period of $\text{sn}(t, k)$ is given by
\[\int_0^{2\pi} \frac{dx}{\sqrt{1 - k^2\sin^2 x}}\]
Thus $y$ is a periodic function, so are $a_1, b_1, v_1, v_2$. They all have same period denoted by $T$.

In fact, Haskins proved in [12] that not only (5.3) has a periodic solution but he also proved that the solution $g$ of (5.4) is also periodic for some rational $A$. So he proved the existence of infinitely many $S^1$-symmetric type (iii) almost complex curves (hence infinitely many special Lagrangian cones in $\mathbb{C}^3$).

6. $S^1$-SYMMETRIC SOLUTIONS AND LOOP GROUP FACTORIZATION

The first equation of (5.4) implies that the solution $2u_0(t) + u_{-1}(t) + \bar{u}_{-1}(t)$ must lie the same conjugate class for all $t$, and there is $g$ solves (5.4). Although these conditions seem to be extra conditions for solutions of (5.3), we will see below that (5.3) has a Lax pair and is a Toda type equation, and hence the AKS theory implies that if $(u_0, u_{-1})$ is a solution of (5.3) then there exists $g$ satisfies (5.4) automatically.

Set $P = 2u_0 + u_{-1} + \bar{u}_{-1}$ and $Q = i(u_{-1} - \bar{u}_{-1})$ as in (5.2). Then (5.4) is $P_t = [P, Q]$, or equivalently, $iP_t = [iP, Q]$, i.e.,
\[(v_0 + v_{-1} - \bar{v}_{-1})_t = [v_0 + v_{-1} - \bar{v}_{-1}, v_{-1} + \bar{v}_{-1}], \quad (6.1)\]
Theorem 6.1. (Iwasawa loop group factorization Theorem [17, 10])

Iwasawa decomposition of simple Lie groups for loop groups: the identity of $G$ that can be extended to a holomorphic maps inside $G$, where $v_0 \in \mathfrak{h}_0 \cap i\mathfrak{g}_2$ and $v_{-1} \in \mathfrak{h}_{-1}$.

Equation (6.1) has a Lax pair

A simple calculation shows that $(v_0, v_1)$ satisfies (6.1) if and only if

$$(v_0 + v_{-1}\lambda^{-1} - \bar{v}_{-1}\lambda t) = [v_0 + v_{-1}\lambda^{-1} - \bar{v}_{-1}\lambda, v_{-1}\lambda^{-1} + \bar{v}_{-1}\lambda]$$

holds for all $\lambda \in \mathbb{C} \setminus \{0\}$. Here $v_0 \in \mathfrak{h}_0$ is pure imaginary and $v_{-1} \in \mathfrak{h}_{-1}$.

Results from the Adler-Kostant-Symes (AKS) Theory (cf. [1, 6, 2])

Let $G$ be a group, $G_+, G_-$ subgroups of $G$ such that the multiplication map $G_+ \times G_- \to G$ defined by $(g_+, g_-) \to g_+g_-$ is a bijection. So $G = G_+ + G_-$ as direct sum of vector subspaces. Suppose $G$ admits a non-degenerate, ad-invariant bilinear form $(\cdot, \cdot)$. Let

$$G^+_+ = \{y \in G \mid (y, x) = 0 \ \forall \ x \in G_+\},$$

and $\pi_+$ denote the projection of $G$ onto $G_+$ with respect to the decomposition $G = G_+ + G_-$. Suppose $M \subset G^+_+$ is invariant under the flow

$$\frac{dx}{dt} = [x(t), \pi_+(x(t))].$$

Given $x_0 \in M$, consider the following ODE:

$$\begin{cases}
\frac{dx}{dt} = [x(t), \pi_+(x(t))], \\
x(0) = x_0.
\end{cases}$$

The AKS theory gives a method to solve the initial value problem (6.4) via factorizations as follows:

(i) Find the one-parameter subgroup $f(t)$ for $x_0$, i.e., solve $f^{-1}f_t = x_0$ with $f(0) = e$.

(ii) Factor $f(t) = f_+(t)f_-(t)$ with $f_{\pm}(t) \in G_{\pm}$.

(iii) Set $x(t) = f_+(t)^{-1}x_0f_+(t)$. Then $x(t)$ is the solution for the initial value problem (6.4). Moreover, $f_{\pm}^{-1}f_t = \pi_+(x(t))$. If $G = SL(n, \mathbb{R})$, $G_+ = SO(n)$, $G_-$ is the subgroup of upper triangular matrices, and $M$ is the space of all tri-diagonal matrices in $sl(n, \mathbb{R})$, then ODE (6.4) is the standard Toda lattice. So we call a system obtained from a factorization a Toda type equation.

Equation (6.1) is of Toda type

Let $L(G^+_2)$ denote the group of smooth loops from $S^1$ to $G^+_2$ satisfying the reality condition $g(\lambda^{-1}) = g(\lambda)$, $L_+(G^+_2)$ the subgroup of $g \in L(G^+_2)$ with $g(\lambda) \in G_+$ for all $\lambda \in S^1$, and $L_-(G^+_2)$ denote the subgroups of $f \in L(G^+_2)$ that can be extended to a holomorphic maps inside $S^1$ such that $f(0) = e$ the identity of $G$. Pressley and Segal proved in [17] an analogue of the Iwasawa decomposition of simple Lie groups for loop groups:

Theorem 6.1. (Iwasawa loop group factorization Theorem [17, 10])
The multiplication map \( L_+ (G_2^C) \times L_- (G_2^C) \to L(G_2^C) \) is a diffeomorphism. In particular, given \( g \in L(G_2^C) \), we can factor \( g = g_+ g_- \) uniquely with \( g_\pm \in L_\pm (G_2^C) \).

Note that
\[
\hat{\sigma}(g)(\lambda) = \sigma(g(e^{-\frac{\pi i}{2}} \lambda))
\]
defines an automorphism of \( L(G_2^C) \). Let \( L_\sigma^\alpha (G_2^C) \) and \( L_\sigma^\pm (G_2^C) \) denote the subgroups fixed by \( \hat{\sigma} \) of \( L(G_2^C) \) and \( L_\pm (G_2^C) \) respectively. Then we have

**Corollary 6.2.** If \( g \in L_\sigma^\alpha (G_2^C) \) and \( g = g_+ g_- \) with \( g_\pm \in L_\pm (G_2^C) \), then \( g_\pm \in L_\sigma^\alpha (G_2^C) \).

Let \( B \) denote the Borel subgroup of \( G_2^C \) such that the Iwasawa decomposition is \( G_2^C = G_2 B \), and \( g_2^C = g_2 + b \) at the Lie algebra level. It is easier to write down the factorization at the Lie algebra level:
\[
L_\sigma^\alpha (g_2^C) = L_+^\sigma (g_2^C) + L_-^\sigma (g_2^C),
\]
where
\[
L_+^\sigma (g_2^C) = \{ \xi = \sum_{j \in \mathbb{Z}} \xi_j \lambda^j \mid \xi_j \in g_2^C, \xi_j \in h_j \},
\]
\[
L_+^\sigma (g_2^C) = \{ \xi = \sum_{j \in \mathbb{Z}} \xi_j \lambda^j \in L_\sigma^\alpha (g_2^C) \mid \xi_{-j} = \bar{\xi}_j \},
\]
\[
L_-^\sigma (g_2^C) = \{ \xi = \sum_{j \geq 0} \xi_j \lambda^j \in L_\sigma^\alpha (g_2^C) \mid \xi_0 \in b \}.
\]

Let \( \pi_{g_2} \) and \( \pi_b \) denote the projections of \( g_2^C \) onto \( g_2 \) and \( b \) respectively, and \( \pi_\pm \) the projections of \( L_\sigma^\alpha (g_2^C) \) onto \( L_\pm^\alpha (g_2^C) \) with respect to the decomposition (6.5). Then for \( \xi = \sum_j \xi_j \lambda^j \),
\[
\pi_+ (\xi) = \pi_{g_2} (\xi_0) + \sum_{j > 0} \xi_{-j} \lambda^{-j} + \bar{\xi}_{-j} \lambda^j,
\]
\[
\pi_- (\xi) = \pi_b (\xi_0) + \sum_{j > 0} (\xi_j - \bar{\xi}_{-j}) \lambda^j.
\]

Let \( (\ , ) \) be the Killing form on \( G_2^C \). Then
\[
(\xi, \eta) = \sum_{i+j=0} (\xi_i, \eta_j)
\]
is an ad-invariant bilinear form on \( L(G) \). So
\[
L_+ (G)^\perp = \{ \xi = \sum_j \xi_j \lambda^j \mid \xi_{-j} = -\bar{\xi}_j \}.
\]

Let \( M = \{ \xi = \xi_0 + \xi_{-1} \lambda^{-1} - \bar{\xi}_{-1} \lambda \mid \xi_0 \in h_0 \cap (iG_2), \xi_{-1} \in h_{-1} \} \). Note that
\[
\pi_+ (\xi_0 + \xi_{-1} \lambda^{-1} - \bar{\xi}_{-1} \lambda) = \xi_{-1} \lambda + \bar{\xi}_{-1} \lambda.
\]
It is easy to check that $[\xi, \pi_+(\xi)] \in M$ if $\xi \in M$, so $M$ is invariant under the flow $\xi_t = [\xi, \pi_+(\xi)]$. So we can use the Iwasawa loop group factorization to construct solution of (6.2) as described in the AKS theory and get

**Theorem 6.3.** Let $A = 2h_0 + h_{-1} + \bar{h}_{-1}$ be a constant with $h_0 \in \mathfrak{h}_0 \cap \mathfrak{g}_2$ and $h_{-1} \in \mathfrak{h}_{-1}$. Then the solution of (5.3) with initial value $A$ can be obtained as follows:

1. Set $\xi_0(\lambda) = 2i h_0 + i h_{-1} \lambda^{-1} + i \bar{h}_{-1} \lambda$, and construct $g(t, \lambda)$ such that
   \[
   \begin{cases}
   g^{-1} g_t = \xi_0(\lambda), \\
   g(0, \lambda) = I,
   \end{cases}
   \]
   i.e., $g(t, \cdot)$ is the one-parameter subgroup of $\xi_0$ in $L^\sigma(G_2^\mathbb{C})$.
2. Factor $g(t, \lambda) = g_+(t, \lambda)g_-(t, \lambda)$ such that $g_\pm(t, \cdot) \in L^\sigma_\pm(G_2^\mathbb{C})$.
3. Set $\xi(t, \lambda) = g_+(t, \lambda)^{-1} \xi_0(\lambda)g_+(t, \lambda)$. Then
   \[
   \xi(t, \lambda) = v_0(t) + v_{-1}(t) \lambda^{-1} + \bar{v}_{-1}(t) \lambda
   \]
   for some $v_0(t) \in \mathfrak{h}_0 \cap (i\mathfrak{g}_2)$ and $v_{-1}(t) \in \mathfrak{h}_{-1}$.
4. Set $u_0 = -iv_0$, $u_{-1} = -iv_{-1}$, and $g(t) = g_+(t, 1)$. Then $g(t) \in G_2$ and $u_0, u_{-1}, g$ satisfy (5.3) and (5.4).

Moreover, $f(s, t) = e^{As}g(t)$ is almost complex in $S^6$.

**References**


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