A CHARACTERIZATION OF STRICTLY APF EXTENSIONS

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Abstract. Let K denote a finite extension of $\mathbf{Q}_p$. We give necessary and sufficient conditions for an infinite totally wildly ramified extension L/K to be strictly APF in the sense of Fontaine-Wintenberger. Our conditions are phrased in terms of the existence of a certain tower of intermediate subfields. These conditions are well-suited to producing examples of strictly APF extensions, and in particular, our main theorem proves that the φ-*iterate* extensions previously considered by the first two authors are strictly APF.

1. Introduction

Let p be a prime and K a finite extension of $\mathbf{Q}_p$ with residue field k and valuation v_K normalized so that $v_K(K^\times) = \mathbf{Z}$. Fix an algebraic closure $\overline{K}$ of K, and for any subfield E of $\overline{K}$ containing K write $G_E := \mathrm{Gal}(\overline{K}/E)$. Recall [13] that an infinite, totally wildly ramified extension L/K is said to be *arithmetically profinite* (APF) if the upper numbering ramification groups $G_K^u G_L$ are *open* in G_K for all $u \geq 0$. The field of norms machinery of Fontaine–Wintenberger [13] functorially associates to any such APF extension L/K a complete, discretely valued field $X_K(L)$ of equicharacteristic p and residue field k with the amazing property that the étale sites of L and $X_K(L)$ are equivalent; in particular, one has a canonical isomorphism of topological groups $G_L \simeq \mathrm{Gal}(X_K(L)^{\mathrm{sep}}/X_K(L))$ that is compatible with the upper numbering ramification filtrations. In certain special cases, this isomorphism plays a foundational role in Fontaine's theory of (φ, Γ)-modules [6] and in the integral p-adic Hodge theory of Faltings [5], Breuil [2, 3], and Kisin [7], and in general provides a key ingredient of Scholze's recent theory of perfectoid spaces and tilting [10].

A famous theorem of Sen [11] guarantees that any infinite, totally wildly ramified *Galois* extension L/K with $\mathrm{Gal}(L/K)$ a p-adic Lie group is strictly[1] APF; however, there are many other interesting and important cases in which one is given an infinite and totally wildly ramified extension L/K, and one would like to decide whether or not L/K is strictly APF. Such examples occur naturally in the theory of p-adic analytic dynamics as follows: Choosing a uniformizer π_1 of K, let $\varphi \in \mathscr{O}_K[\![x]\!]$ be a power series which reduces modulo π_1 to some power of the Frobenius endomorphism of $k[\![x]\!]$ and which fixes zero, and let $\{\pi_n\}_{n\geq 1}$ be a compatible system (i.e., $\varphi(\pi_n) = \pi_{n-1}$) of choices of roots of $\varphi^{(n)} - \pi_1$. The arithmetic of the rising union $L := \cup_{n\geq 1} K(\pi_n)$ is of serious interest (e.g., [9]). For example, if G is a Lubin–Tate formal group over (the valuation ring of) a subfield F of K and φ is the power series giving multiplication by a uniformizer of F, then one may choose $\{\pi_n\}_{n\geq 1}$ so that L/K is the Lubin–Tate extension generated by the p-power torsion points of G in $\overline{K}$. While it is true

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[1] The meaning of the *strictness* condition, whose definition (Definition 2.12) is somewhat technical, is that the norm field $X_K(L)$ of L/K admits a canonical embedding into the fraction field $\widetilde{\mathbf{E}}$ of Fontaine's ring $\widetilde{\mathbf{E}}^+ := \varprojlim_{x\mapsto x^p} \mathscr{O}_{\mathbf{C}_K}/(p)$; see also Remark 2.13 for a geometric interpretation of strictness.

that L/K is strictly APF if its Galois closure L^{gal}/K is [13, Proposition 1.2.3(iii)], it is often very difficult or impossible in practice to describe $\mathrm{Gal}(L^{\mathrm{gal}}/K)$, and so Sen's theorem is of limited use in these cases.

In this note, we establish the following elementary and explicit characterization of strictly APF extensions:

Theorem 1.1. *Let L/K be an infinite, totally wildly ramified extension. Then L/K is strictly APF if and only if there exists a tower of finite extensions $\{E_n\}_{n\geq 2}$ of $E_1 := K$ inside L with $L = \cup E_n$ and a norm compatible sequence $\{\pi_n\}_{n\geq 1}$ with π_n a uniformizer of E_n such that:*

(1) The degrees $q_n := [E_{n+1} : E_n]$ are bounded above.
(2) If $f_n(x) = x^{q_n} + a_{n,q_n-1}x^{q_n-1} + \cdots + a_{n,1}x + (-1)^p\pi_n \in E_n[x]$ is the minimal polynomial of π_{n+1} over E_n, then the non-constant and non-leading coefficients $a_{n,i}$ of f_n satisfy $v_K(a_{n,i}) > \epsilon$ for some $\epsilon > 0$, independent of n and i.

Moreover, if L/K is strictly APF, one may take $\{E_n\}$ to be the tower of elementary subextensions (see Definition 2.8) and $\{\pi_n\}$ to be any norm-compatible sequence of uniformizers.

As a consequence of our work, we are able to produce many concrete examples of (typically non-Galois) strictly APF extensions as follows: let π_1 be a uniformizer of $E_1 := K$; for $n \geq 1$ and given E_n and $\pi_n \in E_n$ a uniformizer, choose a monic polynomial $\varphi_n(x) \in \mathscr{O}_{E_n}[x]$ satisfying $\varphi_n(0) = 0$ and $\varphi_n(x) \equiv x^{q_n} \bmod \pi_n\mathscr{O}_{E_n}$ for $q_n > 1$ a power of p, and let π_n be a choice of root of $f_n(x) := \varphi_n(x) - \pi_{n-1} = 0$. If the degrees q_n are bounded above and the non-leading and non-constant coefficients of the f_n have v_K-valuation bounded below, then it follows at once from Theorem 1.1 that $L := \cup_n E_n$ is a strictly APF extension of K. In particular, the "φ-iterate" extensions described above are always strictly APF.

In §4, we provide several examples of infinite, totally ramified extensions L/K which are not APF, or which are APF but not strictly APF, to illustrate the subtlety of these conditions.

As any strictly APF extension L/K has norm field $X_K(L)$ that is canonically identified with a subfield of Fontaine's field $\widetilde{\mathbf{E}}$, one can try to find a canonical and functorial lift of $X_K(L)$ to a subring of $\widetilde{\mathbf{A}} := W(\widetilde{\mathbf{E}})$. Such lifts play a crucial role in Fontaine's classification [6] of p-adic representations of G_L by étale φ-modules. The question of functorially lifting $X_K(L)$ inside $\widetilde{\mathbf{A}}$ is studied in [4] and [1], and the main theorem of the present paper provides essential input for [4].

Remark 1.2. Much of Theorem 1.1 continues to hold if we allow K to be an *equicharacteristic* local field. In particular, for $\{E_n\}$ satisfying Conditions (1) and (2), the field $L := \cup E_n$ is a strictly APF extension of K. Conversely, for L/K infinite totally wildly ramified and strictly APF and for $\{E_n\}$ the tower of elementary subextensions and $\{\pi_n\}$ a norm compatible sequence of uniformizers, Condition (2) continues to hold. (The proofs given below in the mixed characteristic case work in the equal characteristic case as well.) However, Example 4.4 below shows we cannot expect Condition (1) to hold in general.

Remark 1.3. The proofs below produce an explicit lower bound for the constant $c(L/K)$ appearing in the definition of strictly APF (Definition 2.12). The lower bound is given in terms of $\max q_n$ and ϵ as in Theorem 1.1.

2. Transition functions and ramification

Following [8, §2], we briefly review the definition and properties of the Herbrand transition functions, and recall the definitions of APF and strictly APF as in [13, 1.2].

Let L/K be a finite, totally ramified extension contained in $\overline{K}$, and π_L a uniformizer of L. Write v_L for the valuation of $\overline{K}$, normalized so that $v_L(\pi_L) = 1$. Denote by G the Galois set of all K-embeddings of L into $\overline{K}$, and for real $t \geq 0$ set

$$G_t := \{\sigma \in G \ : \ v_L(\sigma(\pi_L) - \pi_L) \geq t\}.$$

We define the transition function

$$\phi_{L/K}(u) := \frac{1}{[L:K]} \int_0^u |G_t|\, dt;$$

it is a continuous, piecewise linear and increasing bijection on $[0, \infty)$, so we may define $\psi_{L/K} := \phi_{L/K}^{-1}$, which is again continuous, piecewise linear and increasing. For L'/L any finite, totally ramified extension contained in $\overline{K}$, one has the transitivity relations

$$\phi_{L'/K} = \phi_{L/K} \circ \phi_{L'/L} \qquad \text{and} \qquad \psi_{L'/K} = \psi_{L'/L} \circ \psi_{L/K}. \tag{2.1}$$

In practice, we may compute $\phi_{L'/L}$ as follows:

Proposition 2.2 ([8, Lemma 1])**.** *Let L'/L be a finite, totally ramified extension of subfields of $\overline{K}$ containing K. Choose a uniformizer $\pi_{L'}$ of L' and let $f(x) \in L[x]$ be the minimal polynomial of $\pi_{L'}$ over L. Set $g(x) := f(x+\pi_{L'}) \in L'[x]$, and let Ψ_g be the function whose graph is the boundary of the Newton copolygon of $g(x) = \sum_{n\geq 1} b_n x^n$ formed by the intersection of the half-planes $\{y \leq ix + v_K(b_i)\}_{i\geq 1}$. Then*

$$\phi_{L'/L}(x) = e_{L/K} \Psi_g(x/e_{L'/K}). \tag{2.3}$$

If L/K is finite Galois, then the G_t are the usual lower-numbering ramification subgroups of G, and we define the ramification subgroups in the *upper-numbering* to be $G^t := G_{\psi_{L/K}(t)}$. Unlike the lower-numbering groups, the G^t are well-behaved with respect to quotients: if K' is a finite Galois extension of K contained in L then for $H := \mathrm{Gal}(L/K') \trianglelefteq G$ one has $(G/H)^t = G^t H/H$ for all real $t \geq 0$. It follows that by taking projective limits, we may define the upper numbering filtration $\{G^t\}_{t\geq 0}$ for *any* Galois extension L/K, finite or infinite, contained in $\overline{K}$; this is a separated and exhaustive decreasing filtration of G by closed normal subgroups.

Remark 2.4. Because of our desire to have the simple description of $\phi_{L'/L}$ given in Proposition 2.2, our transition functions differ from the ones considered by Serre [12] and Wintenberger [13] by a shift. Indeed, following [8, §2], if ${}_S\phi_{L'/L}$ and ${}_S\psi_{L'/L}$ denote the transition functions defined by Serre [12, IV §3], then one has the relations

$$\phi_{L'/L}(x) = 1 + {}_S\phi_{L'/L}(x-1) \qquad \text{and} \qquad \psi_{L'/L}(x) = 1 + {}_S\psi_{L'/L}(x-1).$$

Correspondingly, the relation between our ramification groups G_t and G^t and those defined by Serre ${}_SG_t$, ${}_SG^t$ is a through shift of one: $G_t = {}_SG_{t-1}$ and $G^t = {}_SG^{t-1}$.

For any extension E of K contained in $\overline{K}$, we define

$$i(E/K) := \sup_{t\geq 0}\{t \ : \ G_K^t G_E = G_K\}. \tag{2.5}$$

Definition 2.6. Let L/K be an arbitrary (possibly infinite) totally ramified extension of K contained in $\overline{K}$. We say that L/K is *arithmetically profinite* (APF) if $G_K^u G_L$ is open in G_K for all $u \geq 0$. If L/K is APF, we define

$$\psi_{L/K}(u) := \int_0^u [G_K : G_K^v G_L]\, dv, \tag{2.7}$$

which is a continuous and piecewise linear increasing bijection on $[0,\infty)$, and we write $\phi_{L/K} := \psi_{L/K}^{-1}$.

Observe that any finite totally ramified extension L/K is APF, and the functions $\phi_{L/K}$ and $\psi_{L/K}$ of Definition 2.6 coincide with the previously defined transition functions of the same name. It follows from the definition that if L/K is an infinite APF extension, then the set of ramification breaks $\{b \in \mathbf{R}_{\geq 0} \ :\ G_K^{b+\varepsilon}G_L \neq G_K^b G_L \ \forall\, \varepsilon > 0\}$ is *discrete* and unbounded, so we may enumerate these real numbers as $b_1 < b_2 < \ldots$.

Definition 2.8. The *n-th elementary subextension of* L/K is the subfield K_n of $\overline{K}$ fixed by $G_K^{b_n}G_L$.

We note that each K_n is a finite extension of K contained in L, that L is the rising union of the K_n, and that K_{n+1}/K_n is *elementary of level* i_n for $i_n := i(K_{n+1}/K_n) = i(L/K_n)$ in the sense that there is a unique break at $u = i_n$ in the filtration $\{G_{K_n}^u G_{K_{n+1}}\}_{u\geq 0}$ of G_{K_n}. Equivalently, the transition function ϕ_{K_{n+1}/K_n} is the boundary function of the intersection of the two half-planes $y \leq x$ and $y \leq [K_{n+1} : K_n]^{-1}(x - i_n) + i_n$, and has a single vertex at (i_n, i_n). As in [13, 1.4.1], it follows that $\{i_n\}_{n\geq 1}$ is an increasing and unbounded sequence, and that one has

$$b_n = i_1 + \frac{i_2 - i_1}{[K_2 : K_1]} + \frac{i_3 - i_2}{[K_3 : K_1]} + \cdots + \frac{i_n - i_{n-1}}{[K_n : K_1]}. \tag{2.9}$$

with $\{b_n\}_{n\geq 1}$ increasing and unbounded. It follows easily from definitions that the vertices of the function $\phi_{L/K}$ of Definition 2.6 are $\{(i_n, b_n)\}_{n\geq 1}$, and the slope of the segment immediately to the right of (i_n, b_n) is $[K_{n+1} : K]^{-1}$.

We will make use of the following characterization:

Proposition 2.10. *Let $\{E_n\}_{n\geq 2}$ be a tower of finite extensions of $E_1 := K$ and let $L = \cup_{n\geq 1} E_n$ be their rising union. Set $\Phi_n := \phi_{E_n/K}$ and define $\alpha_n := \sup\{x \ :\ \Phi_{n+1}(x) = \Phi_n(x)\}$. Then L/K is APF if and only if the following two conditions hold:*

1. *We have $\lim_{n\to\infty} \alpha_n = \infty$. In particular, the pointwise limit $\Phi(x) := \lim_{n\to\infty} \Phi_n(x)$ exists, and moreover, for fixed x_1, we have $\Phi(x) = \Phi_n(x)$ for all $x \leq x_1$ and all n sufficiently large.*
2. *The function $\Phi(x)$ of (1) is piecewise linear and continuous, with vertices $\{(i_n, b_n)\}_{n\geq 1}$ where $\{i_n\}$ and $\{b_n\}$ increasing and unbounded sequences.*

If L/K is APF, then $\Phi(x) = \phi_{L/K}$ for $\phi_{L/K}$ as in Definition (2.6).

Proof. Assume first that the two numbered conditions hold. From the assumption that the $\{b_n\}$ sequence is unbounded, we know the inverse function $\Phi^{-1}(x)$ is defined for all $x \geq 0$ and is the pointwise limit of $\Phi_n^{-1}(x)$ (for any x, we have $\Phi^{-1}(x) = \Phi_n^{-1}(x)$ for all n suitably large). By definition, $\Phi_n^{-1}(x) = \phi_{E_n/K}^{-1}(x) = \psi_{E_n/K}(x)$. Thus, the convergence condition (and the definition of ψ) implies that for any u we have $[G_K : G_K^u G_{E_n}] = [G_K : G_K^u G_{E_{n+1}}]$ for all n suitably large. Writing momentarily K' for the fixed field of G_K^u acting on $\overline{K}$, it follows that $K' \cap E_n = K' \cap E_{n+1}$ for all n sufficiently large. Hence this intersection is also equal to $K' \cap L$ and so, for fixed u, we find $[G_K : G_K^u G_{E_n}] = [G_K : G_K^u G_L]$ for n suitably large. In particular, $G_L G_K^u$ is of finite index—and hence open—in G_K for every u, and L/K is APF.

Now assume L/K is APF, and let $\{K_n\}$ be the associated tower of elementary extensions as in Definition 2.8. By [13, 1.4.1], we have $\lim_{n\to\infty} i(L/K_n) = \infty$. This implies that for any fixed u, there exists $n_0 := n_0(u)$ with $[G_{K_n} : G_{K_n}^u G_L] = 1$ and hence $\psi_{L/K_n}(u) = u$ for all $n \geq n_0$. As $L = \cup E_m$, for any u there exists $m_0 = m_0(u)$ with $E_m \supseteq K_{n_0(u)}$ whenever $m \geq m_0(u)$. We then have $\alpha_{m+1} \geq u$ for all $m \geq m_0(u)$; as u was arbitrary, this implies (1). It follows that $\Phi := \lim_{n\to\infty} \Phi_n$ is well-defined, piecewise linear and continuous, and is the unique such function with $\Phi'(u) = [G_K : G_K^u G_L]^{-1}$ whenever u is not the x-coordinate of a vertex. In particular, $\Phi = \phi_{L/K}$ for $\phi_{L/K}$ as in Definition 2.6; since L/K is APF we conclude that (2) holds. ■

Corollary 2.11 ([13, 1.4.2])**.** *Set $E_1 := K$ and for $n \geq 1$, assume that E_{n+1}/E_n is elementary of level i_n with $\{i_n\}$ strictly increasing and unbounded, and let $\{b_n\}$ be given by (2.9). Then $L := \cup_n E_n$ is an APF extension of K if and only if $\{b_n\}$ is unbounded. Moreover, if L/K is APF, then E_n is the n-th elementary subextension of L/K as in Definition 2.8.*

Definition 2.12 ([13, 1.4.1])**.** Let L/K be an infinite APF extension with associated elementary tower $\{K_n\}$, and recall the function $i(\cdot)$ of (2.5). We define

$$c(L/K) := \inf_{u \geq i(L/K)} \frac{\psi_{L/K}(u)}{[G_K : G_K^u G_L]} = \inf \frac{i_n}{[K_{n+1} : K]}$$

for $i_n := i(K_{n+1}/K_n) = i(L/K_n)$. We say that L/K is *strictly APF* if $c(L/K) > 0$.

Remark 2.13. If L/K is an infinite APF extension, it follows immediately from Definition 2.12 and the discussion preceding Proposition 2.10 that the constant $c(L/K)$ is equal to $\inf v_n m_n$ where v_n is the x-coordinate of the n-th vertex of $\phi_{L/K}$ and m_n is the slope of the segment of $\phi_{L/K}$ immediately to the right of v_n. Thus, L/K is strictly APF if and only if the sequence $\{v_n m_n\}$ is bounded below by a constant $c > 0$. More geometrically, the strictness condition is equivalent to $[G_K : G_K^u G_L]^{-1} \geq c/u$ for $u \geq i(L/K)$, which, upon integrating, is equivalent to the bound

$$\phi_{L/K}(x) \geq c \log(x) + d \qquad \text{for} \qquad d := i(L/K) - c\log(i(L/K))$$

for all $x \geq i(L/K)$.

Lemma 2.14. *Let $\{E_n\}_{n\geq 2}$ be a tower of finite extensions of $E_1 := K$ and $L := \cup_n E_n$. Suppose that L/K is APF, and let Φ and Φ_n be the transition functions of Proposition 2.10. Let V_n be the set of x-coordinates of vertices of Φ_n, and for $v \in V_n$ let m_v be the slope of the segment of Φ_n immediately to the right of v. Then*

$$c(L/K) \geq \liminf_{n\to\infty} \left(\min_{v \in V_n} v m_v \right).$$

Proof. Writing V for the set of x-coordinates of vertices of Φ, we have $c(L/K) = \inf_{v\in V} v m_v$ by Remark 2.13. This means that for any $\epsilon > 0$, we can find $v \in V$ such that $v m_v < c(L/K) + \epsilon$. It follows from Proposition 2.10(1) that any vertex v of Φ is a vertex of Φ_n for all n sufficiently large, and the slopes of the segments on Φ and Φ_n to the immediate right of v agree. Thus $\min_{v\in V_n} v m_v < c(L/K) + \epsilon$ for all n sufficiently large, which completes the proof. ■

3. Proof of Theorem 1.1

From now until the end of Proposition 3.4, fix an infinite totally wildly ramified extension L/K with a tower of subextensions $\{E_n\}$ satisfying Conditions (1) and (2) from Theorem 1.1. We will show that such an extension L/K is strictly APF, thus proving one direction of Theorem 1.1.

Lemma 3.1. *Let $f_n(x)$ and π_n be as in Theorem 1.1(2). Write*

$$f_n(x) = x^{q_n} + a_{n,q_n-1}x^{q_n-1} + \cdots + a_{n,1}x + (-1)^p \pi_n,$$

so

$$g_n(x) := f_n(x+\pi_{n+1}) = \sum_{i=1}^{q_n} b_{n,i}x^i, \quad \textit{for} \quad b_{n,i} := \sum_{j\geq i} a_{n,j}\binom{j}{i}\pi_{n+1}^{j-i}. \tag{3.2}$$

Let $1 > \epsilon > 0$ *be such that* $v_K(a_{n,i}) > \epsilon$ *for all* $0 < i < q_n$*. If* $0 < i < q_n$*, then* $v_K(b_{n,i}) > \epsilon$*.*

Proof. If $j \neq q_n$, then $v_K(a_{n,j}) > \epsilon$ by hypothesis and so $v_K\left(a_{n,j}\binom{j}{i}\pi_{n+1}^{j-i}\right) > \epsilon$. If $j = q_n$ and $0 < i < q_n$, then $v_K\binom{j}{i} \geq v_K(p) \geq 1$. ∎

Proposition 3.3. *The extension* L/K *is APF.*

Proof. We prove this by verifying Conditions (1) and (2) of Proposition 2.10. We begin with Condition (1). Because $\Phi_{n+1}(x) = \Phi_n(\phi_{E_{n+1}/E_n}(x))$, we know that $\Phi_{n+1}(x) = \Phi_n(x)$ for all $x \leq v$, where v is the x-coordinate of the first vertex of $\phi_{E_{n+1}/E_n}(x)$. Let $q := \max(q_n)$, let ϵ be as in Lemma 3.1, and set $x_0 := \frac{\epsilon}{q}$. We claim that $v \geq e_{E_{n+1}/K}x_0$, which will complete the verification of Condition (1). By Proposition 2.2, it suffices to show that the first vertex of $\Psi_{g_n}(x)$ has x-coordinate at least x_0, where as usual $g_n(x) := f_n(x + \pi_{n+1})$ and $f_n(x)$ is the minimal polynomial of π_{n+1} over E_n. From Lemma 3.1, the only contribution to the Newton copolygon of $g_n(x)$ with y-intercept 0 occurs with slope q_n. All other contributions to the Newton copolygon have positive slope and y-intercept at least ϵ. The line $y = q_n x$ crosses the line $y = \epsilon$ at $x = \epsilon/q_n \geq \epsilon/q$, as required.

We now verify that Condition (2) of Proposition 2.10 holds. We have seen that $\Phi(x) = \Phi_n(x)$ for all $x \leq e_{E_{n+1}/K}x_0$. If $\max(q_n) = p^s$, then $\Phi_n(x)$ has at most ns vertices and so $i_{ns+1} \geq e_{E_{n+1}/K}x_0$, and in particular, the sequence $\{i_n\}$ is unbounded. It remains to check that the $\{b_n\}$ sequence is unbounded. Because $\Phi(x)$ is monotone increasing, it suffices to show that $\lim_{x\to\infty}\Phi(x) = \infty$. This will follow from the claim that for any $x \geq e_{E_{n+1}/K}x_0$, we have $\Phi(x) \geq q_1x_0 + (q_2-1)x_0 + \cdots + (q_n-1)x_0$. To see this, notice that between $x = e_{E_i/K}x_0$ and $x = e_{E_{i+1}/K}x_0$, the slope of $\Phi(x)$ is at least $\frac{1}{e_{E_i/K}} = \frac{1}{q_1\cdots q_{i-1}}$. We then compute that for $x \geq e_{E_{n+1}/K}x_0$, we have

$$\Phi(x) \geq 1\cdot q_1x_0 + \frac{1}{q_1}(q_1q_2 - q_1)x_0 + \cdots + \frac{1}{q_1\cdots q_{n-1}}(q_1\cdots q_n - q_1\cdots q_{n-1})x_0,$$

which completes the proof. ∎

Proposition 3.4. *The extension* L/K *is strictly APF.*

Proof. By Proposition 3.3, we know that L/K is APF; let $\Phi_n(x)$ and $\Phi(x)$ be the functions of Proposition 2.10 and let V_n be the set of x-coordinates of vertices of Φ_n. For $x_0 = \epsilon/q$ as in the proof of Proposition 3.3, we will prove that

$$\min_{v\in V_n} vm_v \geq x_0; \tag{3.5}$$

it will then follow from Lemma 2.14 that L/K is strictly APF.

We will prove (3.5) using induction on n. In the proof of Proposition 3.3, we showed that any $v \in V_2$ satisfies $v \geq q_1x_0$; on the other hand, the slopes of $\Phi_2(x)$ are all at least $1/q_1$. This settles the base case $n = 2$. For the inductive step, let $v \in V_{n+1}$ and consider the following two cases:

(1) Assume $v < e_{E_{n+1}/K}x_0$. In this range, $\Phi_{n+1}(x) = \Phi_n(x)$ and we are finished by the inductive hypothesis.
(2) Assume $v \geq e_{E_{n+1}/K}x_0$. Then $vm_v \geq e_{E_{n+1}/K}x_0m_v \geq e_{E_{n+1}/K}x_0 \cdot e_{E_{n+1}/K}^{-1} = x_0$.

■

Proposition 3.4 concludes the proof that L/K is strictly APF, giving one direction of Theorem 1.1. The remainder of this section is devoted to proving the converse.

We now fix an infinite and totally wildly ramified strictly APF extension L/K, and let $\{K_n\}_{n\geq 1}$ be the associated tower of elementary extensions as in Definition 2.8, so that $K_1 = K$ and K_{n+1}/K_n is elementary of level i_n; we set $q_n := [K_{n+1} : K_n]$, so that $[K_{n+1} : K] = q_1 q_2 \cdots q_n$. Let $\pi_n \in K_n$ be any choice of a norm-compatible family of uniformizers.[2]

Proposition 3.6. *Let*

$$f_n(x) = x^{q_n} + a_{n,q_n-1}x^{q_n-1} + \cdots + a_{n,1}x + (-1)^p \pi_n$$

denote the minimal polynomial of π_{n+1} over K_n. Then the valuations of the coefficients $v_K(a_{n,i})$ for $0 < i < q_n$ are bounded below by a positive constant (independent of n and i).

Proof. We prove this by contradiction. As L/K is strictly APF, there exists $c > 0$ such that

$$\inf_n \frac{i_n}{q_1 \cdots q_n} \geq c. \tag{3.7}$$

Suppose that

$$v_K(a_{n,i}) < c \tag{3.8}$$

for some n and i. From (3.7) and (3.8) we will reach a contradiction.

Because K_{n+1}/K_n is elementary, from the discussion following Definition 2.8 we know that the transition function $\phi_{K_{n+1}/K_n}(x)$ has a unique vertex (i_n, i_n). By Proposition 2.2, this means that for $g_n(x) := f_n(x + \pi_{n+1})$, the copolygon boundary function $\Psi_{g_n}(x)$ has a unique vertex with x-coordinate $i_n/(q_1 \cdots q_n)$. By the correspondence between Newton polygons and copolygons (see for example [8, §1]), we know that the Newton polygon of g_n has exactly one segment of slope

$$\frac{-i_n}{q_1 \cdots q_n} \leq -c, \tag{3.9}$$

where the inequality follows from (3.7). On the other hand, writing $g_n(x) = \sum_{j\geq 1} b_{n,i}x^i$ we have

$$v_K(b_{n,i}) = v_K\left(\sum_{j\geq i} a_{n,j}\binom{j}{i}\pi_{n+1}^{j-i}\right) = \min_{j\geq i} v_K\left(a_{n,j}\binom{j}{i}\pi_{n+1}^{j-i}\right) \tag{3.10}$$

as the valuations of the nonzero terms in the sum are all distinct: in fact, they are all distinct modulo $1/(q_1 \cdots q_{n-1})$. Now, using (3.8), we have $v_K(b_{n,i}) \leq v_K\left(a_{n,i}\binom{i}{i}\pi_{n+1}^0\right) < c$.

We now compute the Newton polygon associated to g_n. It must pass through the point $(q_n, 0)$ and by the discussion in the previous paragraph, it must pass below the point (i, c). Such a Newton polygon has slope strictly greater than (i.e., negative and smaller in absolute value than) $\frac{-c}{q_n - i} \geq -c$. This contradicts (3.9). ■

[2] Such a choice exists as L/K is (strictly) APF. Indeed, the norm field of L/K is by definition $X_K(L) := \varprojlim_{E\in\mathscr{E}_{L/K}} E$, where $\mathscr{E}_{L/K}$ is the collection of finite extensions of K in L and the limit is taken with respect to the Norm mappings. For any nonzero $(\alpha_E)_E \in X_K(L)$, one defines $v(\alpha) := v_K(\alpha_K)$. By [13, 2.2.4, 2.3.1], one knows that $(X_K(L), v)$ is a complete, discretely valued field with residue field k, and any choice of uniformizer in $X_K(L)$ corresponds to a norm compatible sequence $(\pi_E)_E$ with π_E a uniformizer of E.

Proposition 3.11. *With notation as in Proposition 3.6, the degrees q_n are bounded above.*

Proof. The proof is similar to the proof of Proposition 3.6. As L/K is strictly APF, we can find a positive constant c such that for all n,

$$\frac{i_n}{q_1 \cdots q_n} \geq c.$$

Since K_{n+1}/K_n is elementary, the Newton polygon of $f_n(x + \pi_{n+1})$ consists of a single segment with slope having absolute value greater than or equal to c. In the notation of (3.10), this implies that

$$c \leq \frac{v_K(b_{n,1})}{q_n - 1} = \frac{v_K\left(\sum_{j \geq 1} a_{n,j} \binom{j}{1} \pi_{n+1}^{j-1}\right)}{q_n - 1} \leq \frac{v_K\left(q_n \pi_{n+1}^{q_n - 1}\right)}{q_n - 1} = \frac{v_K(p) \cdot \log_p(q_n) + \frac{q_n - 1}{q_1 \cdots q_n}}{q_n - 1}. \tag{3.12}$$

This implies $\{q_n\}_{n \geq 1}$ is bounded. ∎

Remark 3.13. Notice that in the equicharacteristic case, the term $v_K(p)$ appearing in (3.12) is $v_K(0)$, and so our argument fails. See also Example 4.4.

Proof of Theorem 1.1. The content of Theorem 1.1 is that, in order for L/K to be strictly APF, it is necessary and sufficient that there exist a tower of subfields satisfying Conditions (1) and (2). That an infinite totally wildly ramified extension containing such a tower of subextensions is strictly APF follows from Proposition 3.4. That the tower of elementary subextensions of a strictly APF extension, together with any norm compatible family of uniformizers, satisfies Conditions (1) and (2) follows from Proposition 3.6 and Proposition 3.11. ∎

4. Examples

We conclude with examples which illustrate the subtlety of the APF and strictly APF conditions.

Example 4.1. Fix a sequence of positive integers $\{r_n\}_{n \geq 1}$ and set $q_n := p^{r_n}$. Let K be a finite extension of $\mathbf{Q}_p$, choose a uniformizer π_1 of K, and for $n \geq 1$ recursively choose a root π_{n+1} of $f_n(x) := x^{q_n} + \pi_1 x + (-1)^p \pi_n = 0$. Set $E_1 := K$ and for $n \geq 2$ let $E_{n+1} := E_n(\pi_{n+1})$ and put $L = \cup_{n \geq 1} E_n$.

We first claim that E_{n+1}/E_n is elementary of level $i_n = q_1 q_2 \cdots q_n/(q_n - 1)$. As in the proof of Proposition 3.6, we would like to show that the Herbrand transition function $\phi_{E_{n+1}/E_n}(x)$ has exactly two segments: a segment of slope 1 from $x = 0$ to $x = i_n$, and a segment of slope $1/q_n$ for $x > i_n$. Equivalently, it suffices to show that the Newton polygon of $f_n(x + \pi_{n+1})$ has exactly one segment of slope $-i_n/e_{E_{n+1}/K}$. (As always, we use the v_K valuation for drawing Newton polygons.)

Using that q_n is a power of p, the binomial theorem shows that the Newton polygon of $f_n(x+\pi_{n+1})$ is the lower convex hull of the collection of vertices containing $(1, 1)$, $(q_n, 0)$, and other vertices with y-coordinate at least 1. Hence the Newton polygon consists of a single segment of slope $-1/(q_n - 1)$. Thus $i_n = q_1 q_2 \cdots q_n/(q_n - 1)$, as desired.

Notice that the $\{i_n\}_n$ is strictly increasing. We may thus use Corollary 2.11 to analyze the extension L/K. Define b_n as in (2.9). Substituting $i_n = q_1 q_2 \cdots q_n/(q_n - 1)$ into the definition of the terms b_n, we find

$$b_n = \frac{q_1}{q_1 - 1} + \sum_{k=2}^{n} \left(\frac{q_k}{q_k - 1} - \frac{1}{q_{k-1} - 1} \right),$$

and it follows from Corollary 2.11 that L/K is APF for every choice of q_n (i.e., for every choice of r_n). On the other hand, by Definition 2.12, L/K is strictly APF if and only if

$$\inf_{n>0} \frac{i_n}{[E_{n+1} : K]} = \inf_{n>0} \frac{1}{(q_n - 1)} > 0.$$

In other words, the extension L/K is strictly APF if and only if the degrees q_n are bounded above.

Example 4.2. Fix an increasing sequence $\{s_n\}_{n\geq 1}$ of positive integers and let K be a finite extension of $\mathbf{Q}_p$ with absolute ramification index e. Choose a uniformizer π_1 of K, set $E_1 := K$ and for $n \geq 2$ recursively choose π_{n+1} a root of $x^p + \pi_n^{s_n} x - \pi_n = 0$ and put $E_{n+1} := E_n(\pi_{n+1})$. Set $L = \cup_{n\geq 1} E_n$.

As in Example 4.1, if we assume that $s_n \leq p^{n-1}e$, we compute that E_{n+1}/E_n is elementary of level $i_n = ps_n/(p-1)$, and because we have chosen s_n to be an increasing sequence, we may again apply Corollary 2.11. With b_n as in (2.9), we compute

$$b_n = \frac{ps_1}{p-1} + \frac{p}{p-1} \sum_{k=2}^{n} \frac{s_k - s_{k-1}}{p^{k-1}}.$$

As the following examples illustrate, whether or not the extension L/K is APF, strictly APF, or neither, depends crucially on the choice of s_n:

(1) If one takes $s_n = n$, then the b_n terms are increasing but bounded. In this case, the extension L/K is not APF.
(2) Assume $p \geq 5$ and take $s_n = \lfloor p^{n-1}/n \rfloor$. Then $\{i_n\}_{n\geq 1}$ is strictly increasing (using the hypothesis $p \geq 5$). Moreover, the sequence $\{b_n\}_{n\geq 1}$ is increasing and unbounded and so L/K is APF, but
$$\inf_{n>0} \frac{i_n}{[E_{n+1} : K]} = \inf_{n>0} \frac{s_n}{p^{n-1}(p-1)} = 0,$$
and so L/K is APF but not strictly APF.
(3) If we take $s_n = p^{n-1}$, then $\{b_n\}_{n\geq 1}$ is increasing and unbounded, and
$$\inf_{n>0} \frac{s_n}{p^{n-1}(p-1)} = \frac{1}{p-1} > 0,$$
so L/K is strictly APF.

Remark 4.3. (1) Assume L/K is an infinite totally wildly ramified strictly APF extension. One cannot expect that Condition (1) of Theorem 1.1 hold for *every* tower of subextensions $\{E_n\}$. For example, for K a finite extension of $\mathbf{Q}_p$ and π_1 a uniformizer of K, consider the extension L/K formed by recursively extracting roots of the polynomials $f_n(x) = x^{p^n} - \pi_n$. These polynomials determine the same extension as the polynomials $f_n(x) = x^p - \pi_n$; however the former collection of polynomials has unbounded degrees, while the degrees in the latter collection are all equal to p.
(2) The authors do not know whether Condition (2) of Theorem 1.1 holds for *every* tower of subextensions and every norm-compatible choice of uniformizers.

Example 4.4. Here we give an example to show that the full strength of our theorem does not hold in characteristic p; see Remark 1.2 for positive results. Assume K is a local field of characteristic p, and let $\pi_1 \in K$ denote a uniformizer. Consider the polynomials

$$f_n(x) = x^{p^n} + \pi_1^{p^n} x - \pi_n,$$

and let π_{n+1} denote a root of $f_n(x)$. Set $E_{n+1} := E_n(\pi_{n+1})$ and $L := \cup E_n$. We claim that L/K is strictly APF, and that $\{E_n\}$ is the associated tower of elementary extensions. Because the degrees $\deg f_n = p^n$ are unbounded, this shows that Theorem 1.1 is not true for local fields of characteristic p.

We compute $f_n(x + \pi_{n+1}) = x^{p^n} + \pi_1^{p^n} x$ and so the Newton polygon is a single segment with slope

$$\frac{-p^n}{p^n - 1} = \frac{-i_n}{p \cdot p^2 \cdots p^n},$$

which implies

$$i_n = \frac{p \cdot p^2 \cdots p^n \cdot p^n}{p^n - 1}.$$

This is a strictly increasing sequence, so we can apply Corollary 2.11 as above. One checks that the sequence $\{b_n\}$ defined by (2.9) is increasing and unbounded and

$$\inf \frac{i_n}{[E_{n+1} : E_1]} > 0.$$

Corollary 2.11 then shows that L/K is strictly APF, as desired.

Remark 4.5. Theorem 1.1 is perhaps better suited to producing strictly APF extensions than to establishing whether a given extension L/K is strictly APF. For example, consider the extension $\mathbf{Q}_p(\mu_{p^\infty}, p^{1/p^\infty})/\mathbf{Q}_p$. This is a Galois extension with Galois group a p-adic Lie group, hence is strictly APF extension by Sen's theorem [11, §4]. However, the authors do not know how to verify this fact using Theorem 1.1, because we do not know how to select a tower $\{E_n\}_{n\geq 1}$ and a norm compatible family of uniformizers $\{\pi_n\}_{n\geq 1}$ which is amenable to explicitly computing the polynomials f_n as in the statement of Theorem 1.1.

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