CANONICAL COHEN RINGS FOR NORM FIELDS

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ABSTRACT. Fix $K/\mathbf{Q}_p$ a finite extension and let L/K be an infinite, strictly APF extension in the sense of Fontaine–Wintenberger [21]. Let $X_K(L)$ denote its associated *norm field*. The goal of this paper is to associate to L/K, in a canonical and functorial way, a p-adically complete subring $\mathbf{A}^+_{L/K} \subset \widetilde{\mathbf{A}}^+$ whose reduction modulo p is contained in the valuation ring of $X_K(L)$. When the extension L/K is of a special form, which we call a *φ-iterate extension*, we prove that $X_K(L)$ is (at worst) a finite purely inseparable extension of $\mathrm{Frac}(\mathbf{A}^+_{L/K}/p\mathbf{A}^+_{L/K})$. The class of φ-iterate extensions includes all Lubin–Tate extensions, as well as many other extensions such as the non-Galois "Kummer" extension occurring in work of Faltings, Breuil, and Kisin. In particular, our work provides a canonical and functorial construction of every characteristic zero lift of the norm fields that have thus far played a foundational role in (integral) p-adic Hodge theory, as well as many other cases which have yet to be studied.

1. INTRODUCTION

Let K be a finite extension of $\mathbf{Q}_p$ with residue field k_K, fix an algebraic closure $\overline{K}$ of K, and let $G_K := \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of K. A natural approach to the classification of p-adic representations of G_K is to first study their restrictions to subgroups G_L of G_K for L/K an infinite extension. When L/K is *arithmetically profinite* (APF) in the sense that the higher ramification subgroups $G_K^v G_L$ are *open* in G_K, a miracle occurs: by the field of norms machinery of Fontaine–Wintenberger [21], there is a canonical isomorphism of Galois groups

$$(1.1) \qquad \mathrm{Gal}(\overline{K}/L) \simeq \mathrm{Gal}(X_K(L)^{\mathrm{sep}}/X_K(L)),$$

where $X_K(L)$ is the *field of norms* of L/K, which is a discretely valued field of equicharacteristic p, noncanonically isomorphic to $k_L((u))$, where k_L is the residue field of L. As a consequence of (1.1), Fontaine proved [13] that G_L-representations on $\mathbf{F}_p$-vector spaces are classified by étale φ-modules over $X_K(L)$. In order to lift this to a classification of G_L-representations on $\mathbf{Z}_p$-modules, it is necessary to lift the norm field $X_K(L)$ together with its Frobenius endomorphism to characteristic zero. Provided that this can be done, one then aims to recover the entire G_K-representation via appropriate descent data from G_L to G_K. This strategy has successfully been carried out in three important cases:

(1) For $L = K(\mu_{p^\infty})$, Fontaine's work [13] provides the necessary lift of $X_K(L)$, and results in the theory of (φ, Γ)-modules which has played a pivotal role in the development and major applications of p-adic Hodge theory. This lift of the norm field is also essential in Berger's theory of Wach modules, which provides a classification of (certain) crystalline representations of G_K when K is an abelian extension of an absolutely unramified extension of $\mathbf{Q}_p$ [1], [3].

(2) More recently, the natural generalization of Fontaine's theory in which the cyclotomic extension of K is replaced by an arbitrary Lubin–Tate extension determined by a Lubin–Tate formal

group over a subfield F of K has been studied by several authors [14], [8], [17]. As in (1) (which is the special case $F = W(k_K)$ with formal group $\widehat{\mathbf{G}}_m$), the underlying formal group plays an essential role in constructing a characteristic zero lift of the norm field $X_K(L)$ that has enough structure to encode the desired descent data from G_L to G_K.

(3) For $L = K(\pi_0^{1/p^\infty})$ the (non-Galois) "Kummer" extension obtained by adjoining a compatible system of p-power roots of a uniformizer π_0 in K, the resulting theory of φ-modules (due to Fontaine) has been extensively exploited by Faltings [12], Breuil [5] [6], and Kisin [16] in their development of integral p-adic Hodge theory. In this case, the ramification structure of L/K is such that for *crystalline* G_K-representations, there is a *unique* descent datum from G_L to G_K so that one obtains a classification of these representations by φ-modules alone (without any Γ!).

In each of these special cases, the required lift of the norm field to characteristic zero is constructed in a rather *ad hoc* manner, and it is not in general clear that it is independent of the choices made to construct it.

In this paper, we propose a candidate for a *canonical* and *functorial* lift to characteristic zero of the valuation ring $A_K(L)$ of the norm field $X_K(L)$. In order to encompass the aforementioned special cases, we work in slightly greater generality: let F be a subfield of K with residue field $k_F \subseteq k_K$ of cardinality $q := |k_F|$, choose a uniformizer π of F, and for any $W(k_F)$-algebra A, write $A_F := A \otimes_{W(k_F)} \mathscr{O}_F$.

Theorem 1.2. *Let L/K be an arbitrary strictly[1] APF extension and let k_L denote the residue field of L. There is a canonical $W(k_L)_F$-subalgebra $\mathbf{A}_{L/K}^+$ of $\widetilde{\mathbf{A}}_F^+ := W(\widetilde{\mathbf{E}}^+)_F$, depending only on L/K and the choices F, π, such that:*

(1) $\mathbf{A}_{L/K}^+$ *is π-adically complete and separated, and closed for the weak topology on $\widetilde{\mathbf{A}}_F^+$.*

(2) $\mathbf{A}_{L/K}^+$ *is stable under the q-power Frobenius automorphism of $\widetilde{\mathbf{A}}_F^+$.*

(3) *Via the inclusion $\mathbf{A}_{L/K}^+ \hookrightarrow \widetilde{\mathbf{A}}_F^+$, the residue ring $\mathbf{A}_{L/K}^+/\pi\mathbf{A}_{L/K}^+$ of $\mathbf{A}_{L/K}^+$ is naturally a k_L-subalgebra of $A_K(L)$.*

(4) *The ring $\mathbf{A}_{L/K}^+$ is functorial in L/K: for any strictly APF extension L'/K and any K-embedding $\tau : L \hookrightarrow L'$ with $L'/\tau(L)$ finite and of wild ramification degree a power[2] of $q = |k_F|$, there is associated a canonical embedding $\mathbf{A}_{L/K}^+ \hookrightarrow \mathbf{A}_{L'/K}^+$ whose reduction modulo π is the restriction of the map on norm fields induced by τ [21, 3.1.1]; in particular, $\mathbf{A}_{L/K}^+$ is naturally equipped with an action of $\mathrm{Aut}(L/K)$.*

For general strictly APF extensions L/K, the canonical homomorphism $W(k_L)_F \to \mathbf{A}_{L/K}^+$ is often an *isomorphism*, and our candidate ring $\mathbf{A}_{L/K}^+$ is too small; following recent work of Berger [2], we prove that this happens, for example, whenever L/K is Galois with group that is a p-adic Lie group admitting no abelian quotient by a finite subgroup (see Remark 5.15). Nevertheless, we isolate a large class of (not necessarily Galois) strictly APF extensions L/K—containing each of the extensions (1)–(3) above as special cases—for which we are able to prove that $\mathbf{A}_{L/K}^+$ provides the desired functorial lift of the valuation ring in the norm field of L/K:

Definition 1.3. With notation above, let π_0 be any choice of uniformizer of K, and let $\varphi(x) \in \mathscr{O}_F[\![x]\!]$ be any power series satisfying $\varphi(0) = 0$ and $\varphi(x) \equiv x^q \mod \pi\mathscr{O}_F$. Beginning with $\pi_0 \in \mathscr{O}_K$, recursively

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[1]The strictness hypothesis guarantees that $A_K(L)$ is naturally a subring of Fontaine's ring $\widetilde{\mathbf{E}}^+ := \varprojlim_{x \mapsto x^p} \mathscr{O}_{\overline{K}}/(p)$.

[2]This restriction is vacuous if $q = p$, *i.e.* if $F/\mathbf{Q}_p$ is totally ramified

choose $\pi_{i+1} \in \mathscr{O}_{\overline{K}}$ a root of $\varphi(x) - \pi_i$, and let $L := \cup_i K(\pi_i)$. We call a strictly APF extension L/K of this form[3] a φ-*iterate extension*.

For any φ-iterate extension L/K, we prove that our candidate ring $\mathbf{A}_{L/K}^+$ lifts the valuation ring $A_K(L)$ of the norm field of L/K, up to possibly a finite, purely inseparable extension (which for applications to p-adic Hodge theory is sufficient; see *e.g.* [13, 3.4.4 (a)]):

Theorem 1.4. *Let L/K be a φ-iterate extension. There exists $u \in \mathbf{A}_{L/K}^+$ whose reduction modulo π is the q^d-th power of a uniformizer in $A_K(L)$, for some integer $d \geq 0$. In particular, $\mathbf{A}_{L/K}^+ \simeq W(k_L)_F[\![u]\!]$, and the norm field $X_K(L)$ is a finite, purely inseparable extension of $\mathrm{Frac}(\mathbf{A}_{L/K}^+/\pi\mathbf{A}_{L/K}^+)$.*

Our work thus provides a *unified* and *canonical* construction of every characteristic zero lift of the norm fields that have thus far played a foundational role in (integral) p-adic Hodge theory, and raises the possibility of a general theory for arbitrary strictly APF extensions.

Throughout this article, we keep the following conventions and notations:

Notation 1.5. We fix a finite extension F of $\mathbf{Q}_p$ and a choice π of uniformizer of $\mathscr{O}_F$, and we denote by $\mathfrak{m}_F$ and k_F the maximal ideal and residue field of $\mathscr{O}_F$, respectively. We write $p^a = q := |k_F|$, and for any $W(k_F)$-algebra A we set $A_F := A \otimes_{W(k_F)} \mathscr{O}_F$. When we consider a strictly APF extension L/K, the bottom field K is assumed to contain F. We let K_0 and K_1 denote, respectively, the maximal unramified and tamely ramified subextensions of L/K. We let k denote the residue field of L (or equivalently, of K_0). Our construction of $\mathbf{A}_{L/K}^+$ will depend implicitly on our choices F and π, though we suppress this in our notation.

2. Generalized Witt vectors

Let $F, q, \pi, \varphi(x)$ be defined as in Notation 1.5 and Definition 1.3. Our construction of $\mathbf{A}_{L/K}^+$ is inspired by the alternative description $\widetilde{\mathbf{A}}^+ \cong \varprojlim_{\mathrm{Fr}} W(\mathscr{O}_{\mathbf{C}_F})$ due to the second author and Kedlaya [10] (see also Proposition 3.4 below), in which the inverse limit is taken along the Witt vector Frobenius map. To account for ramification and to account for q-powers instead of p-powers, we use generalized Witt vectors.

The power series φ determines a unique continuous $\mathscr{O}_F$-algebra homomorphism

$$\mathscr{O}_F[\![x]\!] \to \mathscr{O}_F[\![x]\!], \qquad x \mapsto \varphi(x).$$

In the special case that $F = \mathbf{Q}_p$, meaning $q = p$ and we can take $\pi = p$, then such a ring homomorphism determines a unique ring homomorphism

$$\lambda_\varphi : \mathscr{O}_F[\![x]\!] \to W(\mathscr{O}_F[\![x]\!])$$

which is a section to the projection map $W(\mathscr{O}_F[\![x]\!]) \to \mathscr{O}_F[\![x]\!]$ and which satisfies

$$\lambda_\varphi \circ \varphi = \mathrm{Fr} \circ \lambda_\varphi,$$

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[3] Together with Jonathan Lubin, the authors expect to show that the strictly APF condition is automatically satisfied by every extension of this type.

where Fr is the Witt vector Frobenius. In fact, a completely analogous result holds for the general fields $F/\mathbf{Q}_p$ considered in this paper; see Proposition 2.13. For this result, we will require a generalization of the classical p-typical Witt vectors. These generalized Witt vectors have appeared in the work of Drinfeld [11, §1] and Hazewinkel [15, (18.6.13)]. Our exposition follows that of Borger[4] in [4]. The goal of this section is to recall the properties of these generalized Witt vectors.

Define functors $\bullet^{\mathbf{N}}$ from sets to sets, from $\mathscr{O}_F$-algebras to sets, and from $\mathscr{O}_F$-algebras to $\mathscr{O}_F$-algebras in the obvious way.

Definition 2.1. Let $W_\pi(\bullet)$ denote the unique functor from $\mathscr{O}_F$-algebras to $\mathscr{O}_F$-algebras satisfying the following two properties:

- The induced functor from $\mathscr{O}_F$-algebras to sets agrees with $\bullet^{\mathbf{N}}$.
- The ghost map $W_\pi(\bullet) \to \bullet^{\mathbf{N}}$ given by

$$(a_0, a_1, a_2, \ldots) \mapsto (a_0, a_0^q + \pi a_1, a_0^{q^2} + \pi a_1^q + \pi^2 a_2, \ldots)$$

is a natural transformation of functors.

Remark 2.2. Perhaps the notation $W_{\pi,q}$ would be more precise. We hope the notation W_π is sufficient to remind the reader that these are not the usual p-typical Witt vectors.

This defines the ring operations on $W_\pi(R)$ for any $\mathscr{O}_F$-algebra R in the usual way: If R is π-torsion free, then the ghost map is injective, and so the ring operations are uniquely determined by the requirement that the ghost map be a ring homomorphism. In the general case, we take a surjection from a π-torsion free ring S to R; we then use the ring operations on S to define the ring operations on R.

Proposition 2.3. *The functor $W_\pi(\bullet)$ defined above exists. In particular, for any $\mathscr{O}_F$-algebra R, $W_\pi(R)$ is an $\mathscr{O}_F$-algebra.*

Proof. The functor $W_\pi(\bullet)$ is constructed in a more abstract way in [4]. For the agreement with our definition, see [4, §3.1]. See also page 235 of *loc. cit.* for a more direct formulation. ∎

Lemma 2.4. *Let R denote an $\mathscr{O}_F$-algebra, and let $a, b \in R$ satisfy $a \equiv b \mod \pi^i$. Then $a^q \equiv b^q \mod \pi^{i+1}$. In particular, if $a \equiv b \mod \pi$, then $a^{q^j} \equiv b^{q^j} \mod \pi^{j+1}$.*

Proof. The second statement follows from the first by induction. To prove the first statement, we write $b = a + \pi^i x$ and compute

$$b^q = (a + \pi^i x)^q \equiv a^q + \pi^{qi} x^q \mod p\pi^i$$

by the binomial theorem. Using that $p \in (\pi)$, this completes the proof. ∎

Proposition 2.5 (Generalized Dwork Lemma). *Let R denote a π-torsion free $\mathscr{O}_F$-algebra, and let ϕ denote an $\mathscr{O}_F$-algebra homomorphism such that*

$$\phi(x) \equiv x^q \mod \pi.$$

Let (y_j) denote a sequence of elements such that

$$\phi(y_{i-1}) \equiv y_i \mod \pi^i.$$

Then the sequence (y_j) is in the image of the ghost map.

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[4]Borger has indicated to the authors that the constructions and arguments of his paper work equally well under the more general hypotheses $\varphi(x) \equiv x^{q^j} \mod \pi$ for any $j \geq 1$ and $\varphi(0) = 0$; we stick to the present assumptions for simplicity.

Proof. We will construct (x_j), the pre-image of (y_j), inductively. Constructing x_0 is trivial. Now assume we have constructed $(x_0, \ldots, x_{n-1})$. Then we wish to choose x_n such that

$$\sum_{j=0}^{n} \pi^j x_j^{q^{n-j}} = y_n.$$

We know by our inductive hypothesis that

$$\sum_{j=0}^{n-1} \pi^j x_j^{q^{n-1-j}} = y_{n-1}.$$

Applying ϕ to both sides

$$\sum_{j=0}^{n-1} \pi^j \phi\left(x_j\right)^{q^{n-1-j}} = \phi(y_{n-1}).$$

The right side is congruent to y_n modulo π^n. On the other hand, $\phi(x_j) \equiv x_j^q \bmod \pi$, and so by Lemma 2.4, $\phi(x_j)^{q^{n-1-j}} \equiv x_j^{q^{n-j}} \bmod \pi^{n-j}$. This shows

$$\sum_{j=0}^{n-1} \pi^j \phi\left(x_j\right)^{q^{n-1-j}} \equiv \sum_{j=0}^{n-1} \pi^j x_j^{q^{n-j}} \bmod \pi^n.$$

Hence we can find x_n such that

$$\sum_{j=0}^{n} \pi^j x_j^{q^{n-j}} = y_n,$$

which completes the construction. ∎

Definition 2.6 ([4, §1.7])**.** The *generalized Witt vector Frobenius* is the unique natural transformation $\mathrm{Fr} : W_\pi(\bullet) \to W_\pi(\bullet)$ which has the following effect on ghost components:

$$(a_0, a_1, a_2, \ldots) \mapsto (a_1, a_2, \ldots).$$

In particular, for any fixed $\mathscr{O}_F$-algebra A, the map $\mathrm{Fr} : W_\pi(A) \to W_\pi(A)$ is an $\mathscr{O}_F$-algebra homomorphism.

Definition 2.7 ([4, §3.7])**.** The *generalized Witt vector Verschiebung* V is defined on Witt vector components by

$$(a_0, a_1, a_2, \ldots) \mapsto (0, a_0, a_1, \ldots).$$

It is $\mathscr{O}_F$-linear but not multiplicative.

Lemma 2.8. *The Witt vector Frobenius has the following description in terms of Witt components:*

$$(a_0, a_1, \ldots) \mapsto (a_0^q + \pi f_0(a_0, a_1), a_1^q + \pi f_1(a_0, a_1, a_2), \ldots).$$

In other words, for arbitrary i, the i-th Witt component of $\mathrm{Fr}(\underline{a})$ has the form $a_i^q + \pi f_i$, where f_i is some universal polynomial over $\mathscr{O}_K$ in the variables $a_0, \ldots, a_{i+1}$. Moreover, f_i is homogeneous of degree q^{i+1} under the weighting in which a_j has weight q^j.

Proof. The proof of the analogous result in the p-typical case [9, Lemma 1.4] easily adapts to this case. The idea for proving homogeneity is to use induction and the fact that the ghost components have this property. See also [4, Lemma 3.2] for a proof of everything except homogeneity. ∎

Lemma 2.9. *If R is any $\mathscr{O}_F$-algebra, we have $\mathrm{Fr}\,V = \pi$, where π on the right side refers to the map $W_\pi(R) \to W_\pi(R)$ given by multiplication by π. If R is a $k_F = \mathscr{O}_F/\pi$-algebra, then we furthermore have $V\,\mathrm{Fr} = \mathrm{Fr}\,V = \pi$.*

Proof. The fact that $\mathrm{Fr}\,V = \pi$ is clear in terms of ghost components. Using Lemma 2.8, it is clear in terms of Witt components that when $\pi = 0$ in R, then $\mathrm{Fr}\,V = V\,\mathrm{Fr}$. ∎

Corollary 2.10. *If the $\mathscr{O}_F$-algebra R is a $k_F = \mathscr{O}_F/\pi$-algebra, then multiplication by π on $W_\pi(R)$ has the following effect on Witt coordinates:*

$$\pi : (a_0, a_1, a_2, \ldots) \mapsto (0, a_0^q, a_1^q, \ldots).$$

Proposition 2.11. *If the $\mathscr{O}_F$-algebra R is a perfect k_F-algebra, then $W_\pi(R)$ is the unique π-adically complete and separated π-torsion free $\mathscr{O}_F$-algebra which satisfies $W_\pi(R)/\pi W_\pi(R) \cong R$.*

Proof. Corollary 2.10 makes it clear that when R is a k_F-algebra for which the q-power map is injective, then $W_\pi(R)$ is π-torsion free, π-adically complete, and π-adically separated. This corollary, together with surjectivity of the q-power map, also makes it clear that $\pi W_\pi(R) = V(W_\pi(R))$, and so the residue ring is isomorphic to R.

It remains to prove that $W_\pi(R)$ is the unique such ring. The proof is similar to the usual proof for Witt vectors; see for example [20, §II.5]. The idea is to first note that if $a \equiv b \bmod \pi$, then $a^{q^i} \equiv b^{q^i} \bmod \pi^i$ as in Lemma 2.4. Using this fact one constructs a unique family of Teichmüller lifts, and then notes that elements in a ring satisfying the properties of the proposition can be expressed uniquely as π-adic combinations of Teichmüller lifts. ∎

Let $\widetilde{\mathbf{E}}^+ := \varprojlim_{x \mapsto x^p} \mathscr{O}_{\mathbf{C}_F}/p\mathscr{O}_{\mathbf{C}_F}$; this is a perfect ring of characteristic p and its fraction field $\widetilde{\mathbf{E}}$ is algebraically closed. By [21, §4], the norm fields which we seek to lift can be embedded naturally into $\widetilde{\mathbf{E}}$. Note that the usual p-typical Witt vectors appear on the right side of the following corollary.

Corollary 2.12. $W_\pi(\widetilde{\mathbf{E}}^+) \cong \widetilde{\mathbf{A}}_F^+ := W(\widetilde{\mathbf{E}}^+) \otimes_{W(k_F)} \mathscr{O}_F$.

Proof. This follows from the uniqueness assertion of Proposition 2.11. ∎

Borger's approach to Witt vectors emphasizes the importance of the following result. It is also of central importance to our work, in that we use it to lift a (power of a) uniformizer for $\mathbf{E}_{L/K}^+$ to our Cohen ring $\mathbf{A}_{L/K}^+$ in the case that L/K is a φ-iterate extension. This existence of a lift is the only part of our construction which fails to work for arbitrary strictly APF extensions L/K.

Proposition 2.13. *Let R denote a π-torsion free $\mathscr{O}_F$-algebra, and let $\varphi : R \to R$ denote an $\mathscr{O}_F$-algebra homomorphism that induces the q-power Frobenius modulo π. Then there is a unique $\mathscr{O}_F$-algebra homomorphism*

$$\lambda_\varphi : R \to W_\pi(R)$$

which is a section to the projection $W_\pi(R) \to R$, and such that

$$\lambda_\varphi \circ \varphi = \mathrm{Fr} \circ \lambda_\varphi.$$

Proof. See [4, Proposition 1.9(c)]. The π-torsion free requirement in our statement of the proposition corresponds to the "E-flatness" condition in Borger's paper. ∎

Corollary 2.14. *Keep notation as in Proposition 2.13. For any $r \in R$, the ghost components of $\lambda_\varphi(r)$ are $(r, \varphi(r), \varphi^2(r), \ldots)$.*

Proof. Notice that for any $n \geq 1$,

$$\lambda_\varphi \circ \varphi^n(r) = \mathrm{Fr}^n \circ \lambda_\varphi(r).$$

Consider the first Witt component (which is also the first ghost component) of both sides. Consider the left-hand side first. Because λ_φ is a section to the first Witt component, we have that this first Witt component is $\varphi^n(r)$. Now consider the right-hand side. By the definition of the Witt vector Frobenius (Definition 2.6), the first ghost component of $\mathrm{Fr}^n \circ \lambda_\varphi(r)$ is the $(n+1)$-st ghost component of $\lambda_\varphi(r)$. This completes the proof. ∎

The preceding results, especially Proposition 2.13, are the fundamental properties we will require of $W_\pi(R)$. We conclude this section with several lemmas of a more technical nature which we will require in our proofs.

Lemma 2.15. *If $\underline{x} = (x_0, x_1, x_2, \ldots) \in W_\pi(R)$ is divisible by π^i, then $x_j \in \pi^{i-j}R$ for all $0 \leq j \leq i$.*

Proof. This follows easily using induction and ghost components. ∎

Lemma 2.16. *If the $\mathscr{O}_F$-algebra R is π-torsion free, then $W_\pi(R)$ is also π-torsion free.*

Proof. If $\pi\underline{x} = 0$ for $\underline{x} \in W_\pi(R)$, then at least the component x_0 is zero (because R is assumed π-torsion free). But using the fact that Verschiebung is injective and $\mathscr{O}_F$-linear, we reduce to this case. ∎

Lemma 2.17. *If the $\mathscr{O}_F$-algebra R is π-adically separated, then $W_\pi(R)$ is π-adically separated.*

Proof. Assume $\underline{x} \in W_\pi(R)$ is non-zero, and say for example that $x_j \neq 0$. Then by the assumption that R is π-adically separated, we know that $x_j \notin \pi^N R$ for some suitably large N. But then by Lemma 2.15, we know that $\underline{x} \notin \pi^{j+N} W_\pi(R)$. ∎

Remark 2.18. The authors suspect that when R is π-adically complete and π-adically separated, then $W_\pi(R)$ is also π-adically complete. It is not hard to find a candidate limit by working componentwise, but proving it is actually the limit seems difficult, because π-divisibility of elements in $W_\pi(R)$ seems difficult to detect when R is π-torsion free.

Lemma 2.19. *Let $(R, \mathfrak{m})$ denote a complete local $\mathscr{O}_F$-algebra such that $\pi \in \mathfrak{m}$. Let $\underline{x}_0, \underline{x}_1, \ldots \in W_\pi(R)$ denote a sequence of Witt vectors such that the ghost components*

$$(x_{i0}, x_{i1}, \ldots) \in R^{\mathbf{N}}$$

converge termwise to some sequence $(x_{\infty 0}, x_{\infty 1}, \ldots)$. Then there is a Witt vector $\underline{x}_\infty \in W(R)$ whose ghost components are equal to $(x_{\infty 0}, x_{\infty 1}, \ldots)$. In fact, the Witt vector components of $\underline{x}_0, \underline{x}_1, \ldots$ converge termwise to $\underline{x}_\infty$.

Proof. We work momentarily in $R_\pi := R\left[\frac{1}{\pi}\right]$. In this setting, the ghost map $W_\pi(R_\pi) \to R_\pi^{\mathbf{N}}$ is an isomorphism, and the inverse is given in terms of sequences of polynomials. In particular, the components of the inverse map are $\mathfrak{m}$-adically continuous functions. The element $\underline{x}_\infty$ exists in $W_\pi(R_\pi)$, and its components are $\mathfrak{m}$-adic limits of elements in R. Hence its components are themselves elements of R, and the claim follows. ∎

3. Witt vector constructions

As in Notation 1.5, let $F/\mathbf{Q}_p$ be a finite extension, $q := |k_F|$ its residue cardinality, and π a uniformizer of $\mathscr{O}_F$. Let $W_\pi(R)$ denote generalized Witt vectors with coefficients in R, as in Definition 2.1. Consider the inverse system $\varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F})$, where the transition maps are the Witt vector Frobenius as in Definition 2.6. We first note that this Witt vector inverse limit enjoys the same functoriality properties as the Witt vectors.

Remark 3.1. Let $\gamma : \mathscr{O}_{\mathbf{C}_F} \to \mathscr{O}_{\mathbf{C}_F}$ denote any $\mathscr{O}_F$-algebra homomorphism. This extends to a ring homomorphism $W_\pi(\gamma) : W_\pi(\mathscr{O}_{\mathbf{C}_F}) \to W_\pi(\mathscr{O}_{\mathbf{C}_F})$ by Witt vector functoriality, and because the components of the Witt vector Frobenius are defined by polynomials with coefficients in $\mathscr{O}_F$ (Lemma 2.8), we have that $\mathrm{Fr} \circ W_\pi(\gamma) = W_\pi(\gamma) \circ \mathrm{Fr}$. Hence γ induces a ring homomorphism

$$\varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F}) \to \varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F}), \qquad (\underline{x}_1, \underline{x}_q, \ldots) \mapsto (W_\pi(\gamma)(\underline{x}_1), W_\pi(\gamma)(\underline{x}_q), \ldots).$$

There is also a natural action of Fr on $\varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F})$, defined as follows.

Definition 3.2. Define the Frobenius ring homomorphism on $\varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F})$ by the formula

$$\mathrm{Fr} : \varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F}) \to \varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F}), \qquad (\underline{x}_1, \underline{x}_q, \underline{x}_{q^2}, \ldots) \mapsto (\mathrm{Fr}(\underline{x}_1), \underline{x}_1, \underline{x}_q, \ldots).$$

This definition is equivalent to applying Fr to each entry in the inverse system.

Despite the fact that the Witt vector Frobenius $\mathrm{Fr} : W_\pi(\mathscr{O}_{\mathbf{C}_F}) \to W_\pi(\mathscr{O}_{\mathbf{C}_F})$ is neither injective nor surjective, the above map $\mathrm{Fr} : \varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F}) \to \varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F})$ is an automorphism.

Lemma 3.3. *The ring homomorphism* Fr *of Definition 3.2 is an automorphism.*

Proof. It's clear that the left-shift map

$$(\underline{x}_1, \underline{x}_q, \ldots) \mapsto (\underline{x}_q, \underline{x}_{q^2}, \ldots)$$

is a two-sided inverse to Fr. Hence the ring homomorphism Fr is bijective, hence an isomorphism. ∎

We will now use this Witt vector inverse limit construction to describe a familiar ring from p-adic Hodge theory. Recall the perfect characteristic p ring $\widetilde{\mathbf{E}}^+$ described above Corollary 2.12.

Proposition 3.4 ([10, Proposition 4.5]). *There are canonical isomorphisms*

$$\varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F}) \xrightarrow{\varpi} \varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F}/(\pi)) \xrightarrow{\alpha} W_\pi(\varprojlim \mathscr{O}_{\mathbf{C}_F}/(\pi)) \xrightarrow{\sim} W(\widetilde{\mathbf{E}}^+) \otimes_{W(k_F)} \mathscr{O}_F =: \widetilde{\mathbf{A}}_F^+.$$

Here the transition maps in $\varprojlim \mathscr{O}_{\mathbf{C}_F}/(\pi)$ *are the q-power Frobenius, and only the Witt vectors in the right-most term are the classical p-typical Witt vectors; all other Witt vectors are the generalized Witt vectors as in Definition 2.1.*

Note that the statement that these isomorphisms are *canonical* includes equivariance with respect to the action of the absolute Galois group of F on every term.

Proof. The ring $\varprojlim \mathscr{O}_{\mathbf{C}_F}/(\pi)$ is isomorphic to $\widetilde{\mathbf{E}}^+$ (*cf.* the proof of Proposition 4.2), which is a perfect k_F-algebra of characteristic p. Hence the final isomorphism follows from Proposition 2.11. Thus we concentrate on the first two maps.

The map ϖ is induced by functoriality of Witt vectors. We check that it is injective. Write an element $\underline{x} \in \varprojlim W_\pi(\mathscr{O}_{\mathbf{C}_F})$ as $(\underline{x}_{q^i})$, where $\underline{x}_{q^i} \in W_\pi(\mathscr{O}_{\mathbf{C}_F})$ and where $\mathrm{Fr}^n(\underline{x}_{q^{i+n}}) = \underline{x}_{q^i}$. Using the definition of the Witt vector Frobenius, we obtain the equation

$$(3.5) \qquad x_{q^i 0} = \sum_{j=0}^{n} \pi^j x_{q^{i+n} j}^{q^{n-j}} \qquad (i, n \geq 0).$$

Suppose now that $\varpi(\underline{x}) = 0$. This implies that $v(x_{q^i j}) \geq 1$ for all $i, j \geq 0$, where we write v for the π-adic valuation on $\mathscr{O}_{\mathbf{C}_F}$. By (3.5), for all $i, n \geq 0$, we have $v(x_{q^i 0}) \geq n$ because $j + q^{n-j} \geq n$ for $0 \leq j \leq n$. Hence $x_{q^i 0} = 0$ for all $i \geq 0$. If for some n we have $x_{q^i j} = 0$ for all i and all $j < n$, then from (3.5) we immediately obtain $x_{q^i n} = 0$ for all $i \geq 0$. We thus conclude that $x_{q^i j} = 0$ for all $i, j \geq 0$, so ϖ is injective.

To see that ϖ is surjective, we construct a preimage of $\underline{x} \in \varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F}/(\pi))$. For each $i, j \geq 0$, choose any lift $y_{q^i j} \in \mathscr{O}_{\mathbf{C}_F}$ of $x_{q^i j} \in \mathscr{O}_{\mathbf{C}_F}/(\pi)$, and put $\underline{y}_{q^i} = (y_{q^i 0}, y_{q^i 1}, \dots) \in W_\pi(\mathscr{O}_{\mathbf{C}_F})$. Using the polynomials expressing Frobenius in terms of Witt components (Lemma 2.8), one can check that for each $i, j \geq 0$, as $k \to \infty$, the j-th Witt component of $\mathrm{Fr}^k(\underline{y}_{q^{i+k}})$ converges π-adically to some limit $z_{q^i j}$. These define an element $\underline{z} \in \varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F})$ with $\varpi(\underline{z}) = \underline{x}$.

Having proved that ϖ is an isomorphism, we now consider the map α, which we must first define. For $\underline{x} \in \varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F}/(\pi))$, the sequence $\underline{y}_i = (x_{1i}, x_{qi}, x_{q^2 i}, \dots)$ defines an element of $\varprojlim \mathscr{O}_{\mathbf{C}_F}/(\pi)$ because the Witt vector Frobenius on $W(\mathscr{O}_{\mathbf{C}_F}/(\pi))$ is the map which in each coordinate sends $x \mapsto x^q$. Using the polynomials defining the ring operations on generalized Witt vectors [4, §1.19], we can check that setting $\alpha(\underline{x}) = (\underline{y}_0, \underline{y}_1, \dots)$ in $W(\varprojlim \mathscr{O}_{\mathbf{C}_F}/(\pi))$ defines a ring homomorphism. Finally, the map α is clearly injective and surjective, as it simply involves permuting certain indices. This completes the proof. ∎

Remark 3.6. Recall that the *weak topology* on $W(\widetilde{\mathbf{E}}^+)$ is obtained by identifying $W(\widetilde{\mathbf{E}}^+)$ with $\prod \widetilde{\mathbf{E}}^+$ (as a set), and equipping the latter with the product topology, where each factor is given its valuation topology. In terms of $\varprojlim W_\pi(\mathscr{O}_{\mathbf{C}_F})$, this corresponds to the coarsest topology such that the maps

$$\varprojlim_{\mathrm{Fr}} W(\mathscr{O}_{\mathbf{C}_F}) \to \varprojlim_{x \mapsto x^q} \mathscr{O}_{\mathbf{C}_F}/(\pi), \qquad (\underline{x}_1, \underline{x}_q, \underline{x}_{q^2}, \dots) \mapsto (\overline{x}_{1j}, \overline{x}_{qj}, \overline{x}_{q^2 j}, \dots)$$

(which are not ring maps if $j > 0$) are continuous for all $j \geq 0$.

Recall that we write $q = p^a$. Using the fact that modulo π, the Witt vector Frobenius is the same as raising each coordinate to the q-th power, one can check that our map Fr defined in Definition 3.2 agrees with the map $\mathrm{Fr}^a \otimes 1$ on $W(\widetilde{\mathbf{E}}^+) \otimes_{W(k_F)} \mathscr{O}_F$, via the isomorphism in Proposition 3.4.

Definition 3.7. For an element $x_{q^i j} \in \mathscr{O}_{\mathbf{C}_F}$, write $\overline{x}_{q^i j}$ for its image in $\mathscr{O}_{\mathbf{C}_F}/(\pi)$. Let β denote the ring homomorphism

$$\beta : \varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F}) \to \varprojlim_{x \mapsto x^q} \mathscr{O}_{\mathbf{C}_F}/(\pi), \qquad \underline{x} \mapsto (\overline{x}_{11}, \overline{x}_{q1}, \overline{x}_{q^2 1}, \dots).$$

Remark 3.8. Under the isomorphism

$$\varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F}) \cong W(\widetilde{\mathbf{E}}^+) \otimes_{W(k_F)} \mathscr{O}_F =: \widetilde{\mathbf{A}}_F^+.$$

from Proposition 3.4, the ring homomorphism β corresponds to reduction modulo the ideal generated by $1 \otimes \pi$. In terms of $W_\pi(\widetilde{\mathbf{E}}^+)$, the map β corresponds to projection onto the first Witt component (or equivalently, onto the first ghost component).

Proposition 3.4 allows us to relate $\varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F})$ to $\widetilde{\mathbf{A}}^+ = W(\widetilde{\mathbf{E}}^+)$. One pleasant feature of this comparison is that it provides a simple description of the θ map from p-adic Hodge theory. We first recall this map.

Definition 3.9. Define the θ map

$$\theta : W\left(\varprojlim_{x\mapsto x^p} \mathscr{O}_{\mathbf{C}_F}/(\pi)\right) \to \mathscr{O}_{\mathbf{C}_F}, \qquad \sum p^i[a_i] \mapsto p^i \lim_{j\to\infty} \widetilde{a_{ij}}^{p^j},$$

where a_{ij} is the j-th term in the inverse system corresponding to a_i, and where $\widetilde{a_{ij}}$ is any lift from $\mathscr{O}_{\mathbf{C}_F}/(\pi)$ to $\mathscr{O}_{\mathbf{C}_F}$. If we replace the transition maps $x \mapsto x^p$ by $x \mapsto x^q$, the map becomes

$$\theta : \sum p^i[a_i] \mapsto p^i \lim_{j\to\infty} \widetilde{a_{ij}}^{q^j}.$$

For ramified Witt vectors, this becomes

$$\theta : W\left(\varprojlim_{x\mapsto x^q} \mathscr{O}_{\mathbf{C}_F}/(\pi)\right) \otimes_{W(k_F)} \mathscr{O}_F \to \mathscr{O}_{\mathbf{C}_F}, \qquad \sum \pi^i[a_i] \mapsto \sum \pi^i \lim_{j\to\infty} \widetilde{a_{ij}}^{q^j}.$$

Proposition 3.10. *For an element $\underline{x} \in \varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F})$, write $(\underline{x}_1, \underline{x}_q, \ldots, \underline{x}_{q^i}, \ldots)$ for the Witt vectors in this inverse system and write $x_{q^i j}$ for the j-th Witt component of the Witt vector $\underline{x}_{q^i}$. Define θ to be the ring homomorphism*

$$\theta : \varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F}) \to \mathscr{O}_{\mathbf{C}_F}, \qquad \underline{x} \mapsto x_{q^0 0}.$$

Under the isomorphisms of Proposition 3.4, this induces the same map θ as in Definition 3.9.

Proof. We adapt the argument [10, Proposition 5.8] which is due to Ruochuan Liu. Assume

$$\underline{a} = \sum \pi^i[a_i] \in W(\widetilde{\mathbf{E}}^+) \otimes_{W(k_F)} \mathscr{O}_F$$

corresponds to an element $\underline{x} \in \varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F})$ under the isomorphisms of Proposition 3.4. Recall that we let $\widetilde{a_{ij}}$ denote an arbitrary lift of $a_{ij} \in \mathscr{O}_{\mathbf{C}_F}/(\pi)$ to $\mathscr{O}_{\mathbf{C}_F}$. To prove the proposition, we will describe

$$\theta(\underline{a}) = \sum \pi^i \lim_{j\to\infty} \widetilde{a_{ij}}^{q^j}$$

in terms of the coordinates of $\underline{x} \in \varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F})$. If we view $\underline{a}$ as an element of $W_\pi\left(\varprojlim_{x\mapsto x^q} \mathscr{O}_{\mathbf{C}_F}\right)$, then by Corollary 2.10 it has coordinates $(a_0, a_1^q, a_2^{q^2}, \ldots)$. By definition of the maps ϖ, α in Proposition 3.4, we find that $x_{q^j i}$ is a lift of $a_{ij}^{q^i}$. Then we can take $x_{q^{j+i} i}$ for $\widetilde{a_{ij}}$, our lift of a_{ij}. We then compute

$$\sum_{i=0}^\infty \pi^i \lim_{j\to\infty} \widetilde{a_{ij}}^{q^j} = \lim_{j\to\infty} \sum_{i=0}^j \pi^i \widetilde{a_{ij}}^{q^j}$$

$$= \lim_{j\to\infty} \sum_{i=0}^j \pi^i x_{q^{j+i} i}^{q^j}$$

$$= \lim_{j\to\infty} \sum_{i=0}^j \pi^i x_{q^j i}^{q^{j-i}}.$$

The term $\sum_{i=0}^{j} \pi^i x_{q^j i}^{q^{j-i}} \in \mathscr{O}_{\mathbf{C}_F}$ is the j-th ghost component of the Witt vector $\underline{x}_{q^j}$. If we write w_j for this ghost component, then the above limit becomes

$$= \lim_{j \to \infty} w_j(\underline{x}_{q^j}).$$

Because $\underline{x} \in \varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F})$, we have $\mathrm{Fr}(\underline{x}_{q^j}) = \underline{x}_{q^{j-1}}$. By the definition of the Frobenius map (Definition 2.6), we find that $w_j(\underline{x}_{q^j})$ is independent of j. Taking $j = 0$ completes the proof. ∎

4. NORM RINGS

Let L/K denote an arbitrary strictly APF extension. Recall [21, 1.4] that canonically attached to L/K is the sequence of ramification breaks $\{b_m\}_{m>0}$: this is the increasing sequence of real numbers b for which $G_K^{b+\varepsilon} G_L \neq G_K^b G_L$ for all $\varepsilon > 0$. Let K_0 be the maximal unramified extension of K contained in L, and for $m > 0$ denote by K_m the fixed field of $G_K^{b_m} G_L$ acting on $\overline{K}$; it is a subfield of L of finite degree over K with the property that K_{n+1}/K_n is *elementary of level* $i_n := \Psi_{L/K}(b_n)$, where $\Psi_{L/K}$ is the transition function of Herbrand as in [21, 1.2.1]. Following [21, 1.4], we call $\{K_m\}_{m\geq 0}$ the *tower of elementary extensions* associated to L/K, and for $m \geq 1$ we write r_m for the unique positive integer determined by $p^{r_m} = [K_m : K_1]$.

Lemma 4.1. *Each K_m is preserved by $\mathrm{Aut}(L/K)$. In particular, when L/K is Galois, so is K_m/K.*

Proof. Fix $\sigma \in \mathrm{Aut}(L/K)$, and let $\widetilde{\sigma} \in G_K$ be any extension of σ to an automorphism of $\overline{K}$. Then

$$\widetilde{\sigma}^{-1} G_K^{b_m} G_L \widetilde{\sigma} = (\widetilde{\sigma}^{-1} G_K^{b_m} \widetilde{\sigma})(\widetilde{\sigma}^{-1} G_L \widetilde{\sigma}) = G_K^{b_m} G_L$$

as $G_K^{b_m}$ is normal in G_K and G_L is stable under conjugation by any automorphism of $\overline{K}$ that preserves L. The result follows. ∎

Let us recall the definition of the *perfect norm field* associated to L/K:

Proposition 4.2. *Let $\mathfrak{a} \subseteq \mathscr{O}_L$ be any ideal with the property that the $\mathfrak{a}$-adic and p-adic topologies on $\mathscr{O}_L$ coincide. Then reduction modulo $\mathfrak{a}$ induces a multiplicative bijection*

$$(4.3) \qquad \varprojlim_{x \mapsto x^p} \mathscr{O}_{\widehat{L}} \xrightarrow{\simeq} \varprojlim_{x \mapsto x^p} \mathscr{O}_L/\mathfrak{a}.$$

In particular, $\widetilde{\mathbf{E}}_{L/K}^+ := \varprojlim_{x \mapsto x^p} \mathscr{O}_L/\mathfrak{a}$ is naturally a subring of $\widetilde{\mathbf{E}}^+$ that is independent of $\mathfrak{a}$.

Proof. This is standard (*e.g.* [7, Proposition 4.3.1]): if $(x_n)_n \in \widetilde{\mathbf{E}}_{L/K}^+$, then for any choices of lifts $\widehat{x}_n \in \mathscr{O}_L$ of x_n, our hypotheses on $\mathfrak{a}$ ensure that the $\mathfrak{a}$-adic and p-adic topologies on $\mathscr{O}_{\widehat{L}}$ coincide, so for each fixed n, the sequence $\{\widehat{x}_{n+m}^{p^m}\}_m$ converges to an element y_n of $\mathscr{O}_{\widehat{L}}$ that is independent of these choices with the resulting sequence $\{y_n\}_n$ compatible under the p-power map. The association $x_n \rightsquigarrow y_n$ gives the desired inverse mapping to (4.3). The rest of the Proposition follows easily. ∎

The *strictness* hypothesis on the APF extension L/K amounts to the assumption that the quantities $i_n/[K_{n+1} : K_n]$ are bounded below by a positive constant, and we define the constants [21, 1.2.1, 1.4.1]

$$(4.4) \qquad c(L/K_m) := \inf_{n \geq m} \frac{i_n}{[K_{n+1} : K_m]}.$$

Denote by $\mathfrak{b}$ the ideal of $\mathscr{O}_L$ defined by

$$(4.5) \qquad \mathfrak{b} := \{\alpha \in \mathscr{O}_L \ : \ v_{K_1}(\alpha) \geq c(L/K_1)\}.$$

If $\mathfrak{a}$ is any ideal of $\mathscr{O}_L$, then for any extension E of K_1 contained in L we write $\mathfrak{a}_E := \mathfrak{a} \cap \mathscr{O}_E$ for the induced ideal of $\mathscr{O}_E$, which is visibly the kernel of the canonical map $\mathscr{O}_E \hookrightarrow \mathscr{O}_L \twoheadrightarrow \mathscr{O}_L/\mathfrak{a}$. For any subfields $E' \supseteq E$ of L containing K, we may (and do) therefore view $\mathscr{O}_E/\mathfrak{a}_E$ as a subring of $\mathscr{O}_{E'}/\mathfrak{a}_{E'}$.

Definition 4.6. Let L/K be an infinite and strictly APF extension, and let $\mathfrak{a} \subseteq \mathscr{O}_L$ be any ideal such that $\mathfrak{a}^N \subseteq \mathfrak{b} \subseteq \mathfrak{a}$ for some integer $N \geq 1$. We define

$$\mathbf{E}_{L/K}^+(\mathfrak{a}) := \left\{ (x_n) \in \varprojlim_{x \mapsto x^p} \mathscr{O}_L/\mathfrak{a} \ : \ x_{r_m} \in \mathrm{im}\,(\mathscr{O}_{K_m}/\mathfrak{a}_{K_m} \hookrightarrow \mathscr{O}_L/\mathfrak{a}) \text{ for all } m \right\}$$

For ease of notation, we set $\mathbf{E}_{L/K}^+ := \mathbf{E}_{L/K}^+(\mathfrak{b})$ and $\mathbf{E}_{L/K} := \mathrm{Frac}(\mathbf{E}_{L/K}^+)$.

By definition, $\mathbf{E}_{L/K}^+$ is a subring of $\widetilde{\mathbf{E}}_{L/K}^+$—depending only on L/K—which, thanks to Lemma 4.1, is stable under the natural coordinate-wise action of $\mathrm{Aut}(L/K)$ on $\widetilde{\mathbf{E}}_{L/K}^+$.

Let $X_K(L)$ be the imperfect norm field attached to L/K as in [21, §2.1]. For each finite intermediate extension E/K_1, define

$$(4.7) \qquad r(E) := \left\lceil \frac{(p-1)}{p} i(L/E) \right\rceil \qquad \text{where} \qquad i(L/E) := \sup\{i \geq -1 \ : \ G_E^i G_L = G_E\}.$$

Due to [21, 2.2.3.1], the integers $r(E)$ are nondecreasing and tend to infinity with respect to the directed set $\mathscr{E}_{L/K_1}$ of intermediate extensions $L \supseteq E \supseteq K_1$ that are finite over K_1. Let $s := \{s(E)\}_{E \in \mathscr{E}_{L/K_1}}$ be any fixed choice of positive integers $s(E) \leq r(E)$ with $s(E)$ nondecreasing and tending to infinity (with respect to $\mathscr{E}_{L/K_1}$). Thanks to [21, 2.2.1, 2.2.3.3], the norm maps induce *ring homomorphisms* $\mathrm{Nm}_{E'/E} : \mathscr{O}_{E'}/\mathfrak{m}_{E'}^{s(E')} \to \mathscr{O}_E/\mathfrak{m}_E^{s(E)}$ by and we put

$$(4.8) \qquad\qquad\qquad A_K(L)_s := \varprojlim_{E \in \mathscr{E}_{L/K_1}} \mathscr{O}_E/\mathfrak{m}_E^{s(E)}.$$

Lemma 4.9. *Let* $r := \{r(E)\}_{E \in \mathscr{E}_{L/K_1}}$ *and let* $s := \{s(E)\}_{E \in \mathscr{E}_{L/K_1}}$ *be any choice of positive integers as above. Then the natural reduction map* $A_K(L)_r \to A_K(L)_s$ *is an isomorphism of rings; in particular, the ring* $A_K(L) := A_K(L)_s$ *is independent of the choice of* s. *Moreover,* $A_K(L)$ *is canonically identified with the valuation ring of the norm field* $X_K(L)$.

Proof. Given a norm compatible sequence $(x_E)_E \in \varprojlim \mathscr{O}_E/\mathfrak{m}_E^{s(E)}$, we choose for each n an arbitrary lift $\widehat{x}_E \in \mathscr{O}_E$ of x_E, and we set $y_E := \varinjlim_{E' \in \mathscr{E}_{L/E}} \mathrm{Nm}_{E'/E}(\widehat{x}_{E'})$; this limit exists and is independent of the choices of lifts $\widehat{x}_{E'}$ by the proof 2.3.2 of [21, Proposition 2.3.1]. By construction, $(y_E \bmod \mathfrak{m}_E^{r(E)})_E$ is then a norm compatible sequence in $\varprojlim_{\mathrm{Nm}} \mathscr{O}_E/\mathfrak{m}_E^{r(E)}$ lifting the given sequence $(x_E)_E$, and the association $(x_E)_E \mapsto (y_E)_E$ provides the desired inverse to the reduction map $A_K(L)_r \to A_K(L)_s$; see [21, §2.3] for further details. That $A_K(L)$ is canonically identified with the valuation ring of $X_K(L)$ is [21, Proposition 2.3.1]. ∎

By 4.4, we have

$$i(L/K_n) = i(K_{n+1}/K_n) = [K_{n+1} : K_1]\frac{i(K_{n+1}/K_n)}{[K_{n+1} : K_1]} \geq [K_{n+1} : K_1]c(L/K_1),$$

and it follows that $r(K_n) \geq \lceil (p-1)[K_n : K_1]c(L/K_1) \rceil$. In particular, the sequence of positive integers $s(K_n) := \lceil [K_n : K_1]c(L/K_1) \rceil$ has $s(K_n) \leq r(K_n)$, is nondecreasing and tends to infinity with n.

Furthermore, as the value group of v_{K_n} is $\mathbf{Z}$, we have by definition (4.5) of $\mathfrak{b}$

$$\mathfrak{b}_{K_n} = \{x \in \mathscr{O}_{K_n} : v_{K_n}(x) \geq [K_n : K_1]c(L/K_1)\} = \mathfrak{m}_{K_n}^{s(K_n)}.$$

If $\mathfrak{a} \subseteq \mathscr{O}_L$ is any ideal with $\mathfrak{a}^N \subseteq \mathfrak{b} \subseteq \mathfrak{a}$, then for any $E \in \mathscr{E}_{L/K_1}$ we have $\mathfrak{a}_E^N \subseteq (\mathfrak{a}^N)_E \subseteq \mathfrak{b}_E \subseteq \mathfrak{a}_E$, so defining integers $s_\mathfrak{a}(E)$ by $\mathfrak{a}_E = \mathfrak{m}_E^{s_\mathfrak{a}(E)}$ we have $s_\mathfrak{a}(K_n) \leq s(K_n) \leq Ns_\mathfrak{a}(K_n)$ and the sequence $\{s_\mathfrak{a}(K_n)\}_n$ tends to infinity with n and has $s_\mathfrak{a}(K_n) \leq r(K_n)$. We claim that the sequence $s_\mathfrak{a}(K_n)$ is nondecreasing. To see this, let E'/E denote any two fields in $\mathscr{E}_{L/K_1}$. We then have

$$\mathfrak{m}_E^{s_\mathfrak{a}(E)} = \mathfrak{a}_E = \mathfrak{a}_{E'} \cap \mathscr{O}_E = \mathfrak{m}_{E'}^{s_\mathfrak{a}(E')} \cap \mathscr{O}_E \supseteq (\mathfrak{m}_{E'} \cap \mathscr{O}_E)^{s_\mathfrak{a}(E')} = \mathfrak{m}_E^{s_\mathfrak{a}(E')};$$

comparing the left and right sides shows that $s_\mathfrak{a}(E) \leq s_\mathfrak{a}(E')$, as desired.

We thus obtain natural maps of rings

$$(4.10) \quad \varprojlim_{E \in \mathscr{E}_{L/K}, \mathrm{Nm}} \mathscr{O}_E/\mathfrak{m}_E^{r(E)} \xrightarrow{\text{forget}} \varprojlim_{n, \mathrm{Nm}} \mathscr{O}_{K_n}/\mathfrak{m}_{K_n}^{r(K_n)} \xrightarrow{\text{mod } \mathfrak{m}_\star^{s_\mathfrak{a}(\star)}} \varprojlim_{n, \mathrm{Nm}} \mathscr{O}_{K_n}/\mathfrak{m}_{K_n}^{s_\mathfrak{a}(K_n)} = \varprojlim_{n, \mathrm{Nm}} \mathscr{O}_{K_n}/\mathfrak{a}_{K_n}$$

Lemma 4.11. *The maps* (4.10) *are ring isomorphisms.*

Proof. The first is an isomorphism as $\{K_n\}$ is cofinal in the directed system $\mathscr{E}_{L/K_1}$ of intermediate extensions $L/E/K$. That the second is an isomorphism follows from the proof of Lemma 4.9. ∎

Corollary 4.12. *For any ideal $\mathfrak{a} \subseteq \mathscr{O}_L$ with $\mathfrak{a}^N \subseteq \mathfrak{b} \subseteq \mathfrak{a}$ for some integer N, there is a canonical isomorphism of rings $A_K(L) \simeq \mathbf{E}_{L/K}^+(\mathfrak{a})$. In particular, $\mathbf{E}_{L/K}^+(\mathfrak{a}) \simeq \mathbf{E}_{L/K}^+$ and these rings all correspond to the same subring of $\widetilde{\mathbf{E}}^+$.*

Proof. As the sequence $\{r_m\}$ is cofinal in $\mathbf{N}$, the forgetful map induces an isomorphism of rings

$$(4.13) \qquad \mathbf{E}_{L/K}^+(\mathfrak{a}) \simeq \{(x_n) : x_n \in \mathscr{O}_{K_n}/\mathfrak{a}_{K_n}, \text{ and } x_n^{[K_n:K_{n-1}]} = x_{n-1} \text{ inside } \mathscr{O}_L/\mathfrak{a}\}.$$

On the other hand, we claim that $\mathrm{Nm}_{K_{n+1}/K_n}(\alpha) - \alpha^{[K_{n+1}:K_n]} \in \mathfrak{b} \subseteq \mathfrak{a}$ for all $\alpha \in \mathscr{O}_{K_{n+1}}$, so that the right sides of (4.10) and (4.13) are equal. In other words, we claim that

$$v_{K_1}\left(\mathrm{Nm}_{K_{n+1}/K_n}(\alpha) - \alpha^{[K_{n+1}:K_n]}\right) \geq c(L/K_1).$$

This follows from

$$v_{K_n}\left(\mathrm{Nm}_{K_{n+1}/K_n}(\alpha) - \alpha^{[K_{n+1}:K_n]}\right) \geq c(K_{n+1}/K_n) \geq c(L/K_n) \geq c(L/K_1)[K_n : K_1],$$

with the first two inequalities given by 4.2.2.1 and 1.2.3 (iv) of [21], respectively, and the final inequality following immediately from the definition (4.4) of $c(L/K_m)$. This completes the proof of the claim.

We have subrings

$$\mathbf{E}_{L/K}^+(\mathfrak{b}) \subseteq \mathbf{E}_{L/K}^+(\mathfrak{a}) \subseteq \widetilde{\mathbf{E}}^+,$$

with the first inclusion given by reduction modulo $\mathfrak{a}$ in each coordinate. On the other hand, as observed in the previous paragraph, these first two rings are equal to the rings $A_K(L)_{s_\mathfrak{b}}$ and $A_K(L)_{s_\mathfrak{a}}$, respectively. We then in fact have

$$\mathbf{E}_{L/K}^+(\mathfrak{b}) = \mathbf{E}_{L/K}^+(\mathfrak{a}) \subseteq \widetilde{\mathbf{E}}^+,$$

by Lemma 4.11. ∎

If $K' \in \mathscr{E}_{L/K}$ if any finite extension of K contained in L, then as the finite subextensions of L/K' and L/K are co-final among each other, there is a canonical isomorphism of abstract rings $A_K(L) \simeq A_{K'}(L)$ cf. [21, 2.1.4]. We warn the reader, however, that the embedding of $A_K(L)$ into $\widetilde{\mathbf{E}}^+$ described in [21, 4.2]—whose image is exactly $\mathbf{E}^+_{L/K}$—is sensitive to K. More precisely:

Proposition 4.14. *Let K' be a finite extension of K contained in L. Let K_1 (resp., K_1') be the maximal tamely ramified subextension of L/K (resp., L/K'). Considered as subfields of $\widetilde{\mathbf{E}}$ via the embedding described in [21, 4.2], we have that $X_{K'}(L)$ is a purely inseparable extension of $X_K(L)$ of degree $[K_1' : K_1]$. In particular, the group $\mathrm{Aut}(L) = \mathrm{Aut}(L/\mathbf{Q}_p)$ acts naturally on $X_{K'}(L)$.*

Proof. As abstract rings, both $X_K(L)$ and $X_{K'}(L)$ are isomorphic to $k((x))$ by [21, Théorème 2.1.3]. Let $(\pi_E)_E$ denote a uniformizer for either $X_K(L)$ or $X_{K'}(L)$; we may choose the same elements π_E in both cases, although the index sets differ. Then the map in [21, 4.2] sends this element to the p-power compatible sequence with n-th term

$$\lim_{E,\,[E:K_1]\gg 0} \pi_E^{p^{-n}[E:K_1]} \qquad \text{or} \qquad \lim_{E,\,[E:K_1']\gg 0} \pi_E^{p^{-n}[E:K_1']}.$$

Thus we have subfields $k((x)) \subseteq k((x')) \subseteq \widetilde{\mathbf{E}}$, and clearly $(x')^{[K_1':K_1]} = x$. Because K_1' is a subextension of the totally wildly ramified extension L/K_1, we have that $[K_1' : K_1]$ is a power of p. The final assertion follows by taking $K = \mathbf{Q}_p$ and L/K' arbitrary strictly APF and the fact that $\mathrm{Aut}(L)$ acts naturally on $\mathbf{E}_{L/\mathbf{Q}_p}$ by the preceding constructions and discussion. This completes the proof. ∎

5. The ring $\mathbf{A}^+_{L/K}$

In Section 3, we described the ring $\widetilde{\mathbf{A}}^+_F$ using inverse limits of Witt vectors. In this section, we will use this formulation to construct canonical rings $\mathbf{A}^+_{L/K}$ sitting inside of $\widetilde{\mathbf{A}}^+_F$ for any infinite strictly APF extension L/K with $K \supseteq F$. (We remark that our ring $\mathbf{A}^+_{L/K}$ depends on F, π as in Notation 1.5; this dependence is suppressed from our notation.) We show in this section that the ring $\mathbf{A}^+_{L/K}$ is always π-adically complete and separated, and always has residue ring contained within $\mathbf{E}^+_{L/K}$. In the special case that L/K is a φ-iterate extension (Definition 1.3), we prove in Section 7 that $\mathbf{E}_{L/K}$ is a finite purely inseparable extension of $\mathrm{Frac}\left(\mathbf{A}^+_{L/K}/\pi\mathbf{A}^+_{L/K}\right)$.

The ring $\widetilde{\mathbf{E}}^+_{L/K}$ of Proposition 4.2 is a perfect ring of characteristic p, so the classical theory of Witt vectors suffices to lift it to a Cohen ring. In the next proposition, we put this classical description in our "Witt inverse limit" framework.

Proposition 5.1. *Let L/K be an infinite strictly APF extension, Then there is a canonical identification*

$$(5.2) \qquad \widetilde{\mathbf{A}}^+_{L/K} := W_\pi(\widetilde{\mathbf{E}}^+_{L/K}) = \varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\widehat{L}}).$$

Proof. It is straightforward to adapt the proof of Proposition 3.4. Note that it is important to work with $\mathscr{O}_{\widehat{L}}$ instead of $\mathscr{O}_L$, because we require π-adic completeness. ∎

Remark 5.3. Note that $\widetilde{\mathbf{E}}^+_{L/K}$ and $\widetilde{\mathbf{A}}^+_{L/K}$ are independent of K in the sense that for any finite extension K'/K contained in L we have $\widetilde{\mathbf{E}}^+_{L/K'} = \widetilde{\mathbf{E}}^+_{L/K}$ and $\widetilde{\mathbf{A}}^+_{L/K'} = \widetilde{\mathbf{A}}^+_{L/K}$.

The following is the key definition of this paper.

Definition 5.4. Let F, π, q be as in Notation 1.5. Let K/F denote a finite extension and let L/K be a strictly APF extension. Let $\{K_m\}_{m \geq 0}$ be the tower of elementary extensions of L/K defined in Section 4. We define

$$\mathbf{A}^+_{L/K} := \left\{ (\underline{x}_{q^i})_i \in \varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\widehat{L}}) \; : \; \underline{x}_{q^j} \in W_\pi(\mathscr{O}_{K_m}) \text{ whenever } q^j \mid [K_m : K_1] \right\},$$

viewed as a subring of $\widetilde{\mathbf{A}}^+_{L/K} \subseteq \widetilde{\mathbf{A}}^+_F$ via Proposition 5.1.

Remark 5.5. As observed in Lemma 4.1, the elementary fields K_m are stable under the action of $\mathrm{Aut}(L/K)$ on L, so the ring $\mathbf{A}^+_{L/K}$ carries a natural action of $\mathrm{Aut}(L/K)$. In particular, if L/K is Galois, then $\mathbf{A}^+_{L/K}$ is a G_K-stable subring of $\widetilde{\mathbf{A}}^+_F$ that is fixed element-wise by G_L.

Because $W_\pi(\mathscr{O}_E)$ is always an $\mathscr{O}_F$-algebra for any intermediate field $K \subseteq E \subseteq L$ and because the Witt vector Frobenius Fr is an $\mathscr{O}_F$-algebra homomorphism, we find that $\mathbf{A}^+_{L/K}$ is an $\mathscr{O}_F$-algebra.

Recall that the weak topology on $\widetilde{\mathbf{A}}^+_F \cong \varprojlim W_\pi(\mathscr{O}_{\mathbf{C}_F})$ was described in Remark 3.6.

Proposition 5.6. *The $\mathscr{O}_F$-subalgebra $\mathbf{A}^+_{L/K}$ of $\widetilde{\mathbf{A}}^+_F$ is closed for the weak topology.*

Proof. We will show that the complement is open. Let $\underline{x} = (\underline{x}_{q^i})$ denote an element in the complement. Then there exist some indices i, j such that $q^i \mid [K_m : K_1]$ and $x_{q^i j} \notin \mathscr{O}_{K_m}$. Let n be such that there is no element of $\mathscr{O}_{K_m}$ congruent to $x_{q^i j}$ modulo $\pi^n \mathscr{O}_{\mathbf{C}_F}$. The collection of all elements $\underline{y} \in \varprojlim W_\pi(\mathscr{O}_{\mathbf{C}_F})$ such that $v_\pi(y_{q^i j} - x_{q^i j}) > n$ is an open set containing $\underline{x}$ and contained in the complement of $\mathbf{A}^+_{L/K}$. This completes the proof. ∎

Proposition 5.7. *Recall that $\beta : \varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F}) \to \widetilde{\mathbf{E}}^+$ was defined in Definition 3.7. The induced map $\beta : \mathbf{A}^+_{L/K} \to \widetilde{\mathbf{E}}^+$ has image contained in $\mathbf{E}^+_{L/K}$.*

Proof. Let $\mathfrak{a} := (\mathfrak{b}, \pi)$. This ideal satisfies the hypotheses of Corollary 4.12. By that lemma (and using its notation), it suffices to show that for $\underline{x} \in \mathbf{A}^+_{L/K}$, we have $\beta(\underline{x}) \in \mathbf{E}^+_{L/K}(\mathfrak{a})$. We will show $\beta(\underline{x})_{r_n} \in \mathscr{O}_{K_n}/(\pi)$ for all n. Write $q = p^a$ and let s_n be the smallest index such that $q^{s_n} \mid [K_n : K_1]$. Then by the definition of $\mathbf{A}^+_{L/K}$, we know that $x_{q^{s_n} 0} \in \mathscr{O}_{K_n}$. This shows that $\beta(\underline{x})_{p^{a s_n}} \in \mathscr{O}_{K_n}/(\pi)$. Because $r_n = a s_n - d$ for some $d \geq 0$, we have that $\beta(\underline{x})_{r_n}$ is the p^d-th power of some element in $\mathscr{O}_{K_n}/(\pi)$, and hence is itself in $\mathscr{O}_{K_n}/(\pi)$, which completes the proof. ∎

Our next goal is to study the effect of multiplication by π on an element in $\mathbf{A}^+_{L/K}$. We will use these results to show that $\mathbf{A}^+_{L/K}$ is π-adically complete and separated. Recall that Lemma 2.15 showed that for a Witt vector $\underline{x} \in W_\pi(R)$ which is divisible by π^i, then x_j is divisible by π^{i-j} for any $0 \leq j \leq i$. The converse to this lemma is not true; for example, the first Witt component being divisible by π does not imply that the Witt vector is divisible by π. For example, $V([1]) \in W_\pi(\mathscr{O}_K)$ is not divisible by π, even though its first component is zero. However, we do have the following result (whose proof was suggested to us by Abhinav Kumar), which states that if we take a Witt vector whose first component is divisible by π, and apply Frobenius to it, then the result is a Witt vector which is divisible by π. This result will be very useful with respect to our inverse systems under Frobenius maps.

Lemma 5.8. *Let R denote any $\mathscr{O}_F$-algebra, and let $\underline{y} \in W_\pi(R)$ with $y_0 \in \pi R$. Then $\mathrm{Fr}(\underline{y}) \in \pi W(R)$.*

Proof. Write $\underline{y} = \pi[y'] + V(\widetilde{y})$, for some $y' \in R$ and $\widetilde{y} \in W_\pi(R)$. Applying Fr to both sides, and using the facts that Fr is an $\mathscr{O}_F$-algebra homomorphism (Definition 2.6) and that the composition Fr V is equal to multiplication by π (Lemma 2.9), immediately yields the result. ∎

The following proposition shows that the ring $\mathbf{A}^+_{L/K}$ satisfies many of the desired properties which were enumerated in Theorem 1.2.

Proposition 5.9. *Recall that k denotes the residue field of $\mathscr{O}_L$ (or, equivalently, of $\mathscr{O}_{K_0}$) and that $\beta : \varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\mathbf{C}_F}) \to \widetilde{\mathbf{E}}^+$ was defined in Definition 3.7. The ring $\mathbf{A}^+_{L/K}$ is a π-torsion free, π-adically complete and separated $W_\pi(k)$-algebra. Furthermore, the map $\beta : \mathbf{A}^+_{L/K} \to \mathbf{E}^+_{L/K}$ has kernel equal to $\pi\mathbf{A}^+_{L/K}$.*

Proof. We first show that $\mathbf{A}^+_{L/K}$ is a $W_\pi(k)$-algebra; this is slightly harder than showing it is a $W_\pi(k_F)$-algebra. The Witt vector Frobenius Fr : $W_\pi(k) \to W_\pi(k)$ is an $\mathscr{O}_F$-algebra homomorphism which induces the q-power map modulo π. Hence using Proposition 2.13, we have a natural ring homomorphism $\lambda_{\mathrm{Fr}} : W_\pi(k) \to W_\pi(W_\pi(k))$, and the family of ring homomorphisms $\lambda_{\mathrm{Fr}} \circ \mathrm{Fr}^{-n} : W_\pi(k) \to W_\pi(W_\pi(k))$ are Frobenius compatible, so we have an induced map to $\varprojlim_{\mathrm{Fr}} W_\pi(W_\pi(k))$. From the explicit description of $W_\pi(k)$ given in Proposition 2.11, we know that we have a natural inclusion $W_\pi(k) \hookrightarrow \mathscr{O}_{K_0}$, so we have a natural map $W_\pi(k) \to \varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{K_0})$. Because $K_0 \subseteq K_n$ for all n, this completes the proof that $\mathbf{A}^+_{L/K}$ is a $W_\pi(k)$-algebra.

Lemma 2.16 shows that $\mathbf{A}^+_{L/K}$ is π-torsion free and Lemma 2.17 shows that $\mathbf{A}^+_{L/K}$ is π-adically separated. To check that $\mathbf{A}^+_{L/K}$ is π-adically complete, let $\underline{x}^{(1)}, \underline{x}^{(2)}, \dots$ denote a π-adic Cauchy sequence of elements in $\mathbf{A}^+_{L/K}$. Lemma 2.15 shows that for any i, j, the coordinates $x^{(1)}_{q^i j}, x^{(2)}_{q^i j}, \dots \in \mathscr{O}_{\widehat{L}}$ form a π-adic Cauchy sequence. The resulting limits form a Frobenius-compatible system of Witt vectors in $W_\pi(\mathscr{O}_{\widehat{L}})$, because the definition of Fr in terms of coordinates involves polynomials over $\mathscr{O}_F$ by Lemma 2.8, and such polynomials are in particular π-adically continuous. We thus have a well-defined candidate for the limit $\underline{x} \in \widetilde{\mathbf{A}}^+_{L/K}$. On the other hand, we know that there does exist a π-adic limit in $\widetilde{\mathbf{A}}^+_{L/K}$, because $\widetilde{\mathbf{A}}^+_{L/K} \cong W_\pi(\widetilde{\mathbf{E}}^+_{L/K})$, and the latter is π-adically complete. Each Witt vector $\underline{x}_{q^i}$ in the inverse system corresponding to $\underline{x}$ has at least the same first Witt coordinate as the actual limit (because addition and multiplication are defined in the obvious way on the first Witt coordinate). But Lemma 5.8 and the fact that $\widetilde{\mathbf{A}}^+_{L/K}$ is π-adically separated shows that an element in $\widetilde{\mathbf{A}}^+_{L/K}$ is uniquely determined by its sequence of first Witt components. We have shown that our π-adic Cauchy sequence has a limit in $\widetilde{\mathbf{A}}^+_{L/K}$. To show that the limit is in fact in $\mathbf{A}^+_{L/K}$, we note that each elementary subfield K_m is a finite extension of K, and hence is π-adically complete.

Next we check that the kernel of β is equal to $\pi\mathbf{A}^+_{L/K}$. The fact that every element divisible by π is contained in the kernel of β follows immediately from the definitions (or from the fact that the target is characteristic p). So to prove the reverse inclusion, let $\underline{x}$ denote an element in the kernel of β. From the fact that $\beta(\underline{x}) = 0$, we have that $x_{q^i 1}$ is divisible by π for every i. Using Lemma 5.8, this shows that each Witt vector $\underline{x}_{q^i}$ comprising $\underline{x}$ is divisible by π. Using that $W(\mathscr{O}_{\widehat{L}})$ is π-torsion free by Lemma 2.16, we deduce that the Witt vectors $\underline{x}_{q^i}/\pi$ are Frobenius-compatible, and hence $\underline{x}$ is divisible by π, as required. ∎

Our hope is that the ring $\mathbf{A}^+_{L/K}$ provides a functorial lift of the valuation ring in the norm field of L/K in many situations. In §7, we will prove that this is indeed the case (up to a finite, purely

inseparable extension) for a large class of strictly APF extensions. However, in many cases of interest (*e.g.* when L/K is Galois with $\mathrm{Gal}(L/K)$ admitting no abelian quotient by a finite subgroup) our candidate lift $\mathbf{A}_{L/K}^+$ is too small. The remainder of this section is devoted to making this statement precise.

Corollary 5.10. *Write* $\mathbf{E}^+$ *and* $\mathbf{E}$ *for* $\mathbf{A}_{L/K}^+/\pi\mathbf{A}_{L/K}^+$ *and its fraction field, respectively. If the residue ring* $\mathbf{E}^+$ *is strictly larger than* k, *then* $\mathbf{E}_{L/K}$ *is a finite extension of* $\mathbf{E}$. *In particular, there is a non-canonical isomorphism of rings* $\mathbf{A}_{L/K}^+ \simeq W_\pi(k)[\![y]\!]$.

Proof. By [21, Théorème 2.1.3] and our results from Section 4, we know that $\mathbf{E}_{L/K}^+ \cong k[\![x]\!]$. Let $y \in (x) \subseteq k[\![x]\!]$ denote an element in the image of $\mathbf{E}^+$ such that $\mathrm{ord}_x(y)$ is minimal. We can find a lift $\widetilde{y} \in \mathbf{A}_{L/K}^+$ of y of the form $[y] + \pi z$, where $z \in \widetilde{\mathbf{A}}_F^+$. (Note that we do not claim $z \in \mathbf{A}_{L/K}^+$.) We have an inclusion of rings $W_\pi(k)[\widetilde{y}] \subseteq \mathbf{A}_{L/K}^+$. Because $\mathbf{A}_{L/K}^+ \subseteq \widetilde{\mathbf{A}}_F^+$ is weakly closed (Proposition 5.6), using our explicit description of $\widetilde{y}$, we see that in fact $W_\pi(k)[\![\widetilde{y}]\!] \subseteq \mathbf{A}_{L/K}^+$, which implies $k[\![y]\!] \subseteq \mathbf{E}^+$. As $k(\!(y)\!) \subseteq k(\!(x)\!)$ is a finite extension of degree $\mathrm{ord}_x(y)$ (see for example [19, Theorem 3]), the first assertion follows.

We next claim that $k(\!(y)\!) = \mathbf{E}$ as subrings of $\mathbf{E}_{L/K}$. Consider more generally any field $k(\!(y)\!) \subseteq l \subseteq k(\!(x)\!)$. Any element of $l \cap k[\![x]\!]$ can be written uniquely in the form

$$a_0 + a_1 x + \cdots + a_{n-1}x^{n-1},$$

where $n := \mathrm{ord}_x(y)$ and where $a_0, \ldots, a_{n-1} \in k[\![y]\!]$. Unless $a_1 = \cdots = a_{n-1} = 0$, one may (after subtracting off a_0) divide by y^m, where $m = \min(\mathrm{ord}_y(a_1), \ldots, \mathrm{ord}_y(a_{n-1}))$ to produce an element g in l such that $0 < \mathrm{ord}_x(g) < n$. In the specific case $l = \mathbf{E}$, this contradicts the minimality of $\mathrm{ord}_x(y)$. This proves the claim. The embedding of rings $W_\pi(k)[\![\widetilde{y}]\!] \to \mathbf{A}_{L/K}^+$ is surjective modulo π, and hence is an isomorphism because the target is π-adically separated. ∎

Lemma 5.11. *The group* $\mathrm{Aut}(L)$ *acts faithfully on* $X_K(L)$.

Proof. One adapts the proof of [21, Lemme 3.1.3.1]: if τ is an automorphism acting trivially on $X_K(L)$, then in particular τ fixes the residue field k of L and also fixes uniformizers for subfields $E \subseteq L$ which are finite over K. ∎

Lemma 5.12. *Continue to write* $\mathbf{E} := \mathrm{Frac}\left(\mathbf{A}_{L/K}^+/\pi\mathbf{A}_{L/K}^+\right)$ *and write* $\mathbf{E}^{\mathrm{sep}}$ *for the separable closure of* $\mathbf{E}$ *in* $\mathbf{E}_{L/K}$. *The kernel of the action of* $\mathrm{Aut}(L/K)$ *on* $\mathbf{A}_{L/K}^+$ *has order at most* $\sigma_{L/K} := [\mathbf{E}^{\mathrm{sep}} : \mathbf{E}]$.

Proof. Let H denote the kernel of the action of $\mathrm{Aut}(L/K)$ on $\mathbf{A}_{L/K}^+$. We have natural maps

$$H \to \mathrm{Aut}(\mathbf{E}_{L/K}/\mathbf{E}) \to \mathrm{Aut}(\mathbf{E}^{\mathrm{sep}}/\mathbf{E}) \to \mathrm{Aut}(\mathbf{E}).$$

The first two maps are injective (by Lemma 5.11 and the fact that $\mathbf{E}_{L/K}/\mathbf{E}^{\mathrm{sep}}$ is purely inseparable, respectively). The composite map $H \to \mathrm{Aut}(\mathbf{E})$ is clearly trivial, by our definitions of H and $\mathbf{E}$. It thus suffices to bound the size of the kernel of the third map, and the result follows by Galois theory. ∎

The reduction map $\mathscr{O}_L \to k$ induces a ring homomorphism

$$\rho : \mathbf{A}_{L/K}^+ \to W_\pi(\mathscr{O}_L) \to W_\pi(k),$$

where the first map is projection onto the first Witt vector in the inverse system.

Lemma 5.13. *Assume that the residue ring* $\mathbf{E}^+ := \mathbf{A}^+_{L/K}/\pi\mathbf{A}^+_{L/K}$ *is strictly larger than* k, *so that by Corollary 5.10 we have a noncanonical isomorphism*

$$\mathbf{A}^+_{L/K} \cong W_\pi(k)[\![y]\!].$$

Then $\mathrm{Fr}(y) \in \ker\rho$ *and for any unit* $u \in W_\pi(k)[\![y]\!]$ *we have*

$$v_\pi\left(\rho\left(\frac{\mathrm{Fr}(y)}{y}\right)\right) = v_\pi\left(\rho\left(\frac{\mathrm{Fr}(uy)}{uy}\right)\right).$$

Proof. The first result follows from

$$\rho\left(\mathrm{Fr}(y)\right) = \mathrm{Fr}\left(\rho(y)\right) = \mathrm{Fr}(0) = 0.$$

The second result follows from the fact that u is a unit, and hence so are $\mathrm{Fr}(u)$ and $\rho(\mathrm{Fr}(u)/u)$. ∎

It follows from Lemma 5.13 that if the residue ring $\mathbf{E}^+$ strictly contains k, then we may choose an isomorphism $\mathbf{A}^+_{L/K} \simeq W_\pi(k)[\![y]\!]$, and writing $\mathrm{Fr}(y) = a_1 y + \cdots$, the absolute value $|\mathrm{Fr}'(0)| := |a_1|$ is well-defined (*i.e.* is independent of any choices and depends only on $\mathbf{A}^+_{L/K}$).

Proposition 5.14. *Let* L/K *be a strictly APF extension and suppose that the residue ring* $\mathbf{E}^+$ *strictly contains* k, *so that* $\mathbf{E}_{L/K}$ *is a finite extension of* $\mathbf{E}$. *Let* $\sigma_{L/K}$ *be the separable degree of* $\mathbf{E}_{L/K}/\mathbf{E}$, *and suppose that* $|\mathrm{Fr}'(0)| \neq 0$. *There is a natural homomorphism of groups* $\mathrm{Aut}(L/K) \to W_\pi(k)^\times$ *with finite kernel of order at most* $\sigma_{L/K}$. *In particular, if* $\mathbf{E}_{L/K}$ *is a purely inseparable extension of* $\mathbf{E}$, *then* $\mathrm{Aut}(L/K)$ *is abelian.*

Proof. Lemma 5.12 shows that the kernel H of the action of $\mathrm{Aut}(L/K)$ on $\mathbf{A}^+_{L/K}$ has order at most $\sigma_{L/K}$. By Corollary 5.10 and our hypotheses, we thus have a power series ring $W_\pi(k)[\![y]\!]$ equipped with an endomorphism Fr determined by $\mathrm{Fr}(y) = f_1 y + \cdots$ with $f_1 \neq 0$ and a commuting, faithful action of the group $G := \mathrm{Aut}(L/K)/H$. Put $E := \mathrm{Frac}(W_\pi(k))$. Under these circumstances, we prove that G is isomorphic to a subgroup of $W_\pi(k)^\times$ and is in particular abelian. For each $g \in G$, we have $gu = a_1(g)u + \cdots$ with $a_1(g) \neq 0$ as G acts by automorphisms. As in the proof of [2, Theorem 4.1], there is a unique power series $A(u) := a_1 u + \cdots \in E[\![u]\!]$ with $a_1 = 1$ which satisfies $A(\mathrm{Fr}(u)) = f_1 A(u)$ (*cf.* [18, Proposition 1.2]). As the action of G commutes with Fr, we have

$$a_1(g)^{-1}A(g\,\mathrm{Fr}(u)) = a_1(g)^{-1}A(\mathrm{Fr}(gu)) = f_1\left(a_1(g)^{-1}A(gu)\right)$$

for all $g \in G$, and it follows from $a_1(g)^{-1}A(gu) = u + \cdots$ and the uniqueness of A that

$$A(gu) = a_1(g)A(u).$$

From the construction of A one sees that $A(gu) = A(u)$ if and only if $gu = u$, which happens if and only if $g = 1$ since G acts faithfully. We conclude that the mapping $G \to W_\pi(k)^\times$ given by $g \mapsto a_1(g)$ is an injective homomorphism, whence G is isomorphic to a subgroup of $W_\pi(k)^\times$ and in particular is abelian. ∎

Remark 5.15. If L/K is Galois with group that is a p-adic Lie group, then one has $|\mathrm{Fr}'(0)| \neq 0$ thanks to [2, Lemma 4.5]. It follows that if the residue ring $\mathbf{A}^+_{L/K}/\pi\mathbf{A}^+_{L/K}$ is strictly larger than k, then $\mathrm{Gal}(L/K)$ admits an abelian quotient by a finite subgroup.

The above results complete the proof of Theorem 1.2 (1)-(3). The proof of Theorem 1.2 (4), concerning functoriality of our construction, is completed in Section 6. In Section 7, we prove Theorem 1.4, which says that in the case that L/K is a φ-iterate extension, then the norm field $\mathbf{E}_{L/K}$ is a finite, purely inseparable extension of $\mathbf{E}$.

6. Functoriality

The goal of this section is to prove Theorem 6.9, which describes a sense in which the association

$$L/K \rightsquigarrow \mathbf{A}_{L/K}^{+}$$

is functorial in L. This functoriality is compatible with the functoriality of $\mathbf{E}_{L/K}^{+}$. In order to prove functoriality, we are restricted to working with finite extensions L'/L which have wild ramification degree equal to a power of q.

Lemma 6.1. *Let L/K denote a strictly APF extension and let L'/L denote a finite extension of degree N. Let K_n (respectively, K_n'), denote the associated elementary subfields of L/K (respectively, L'/K). For every n, there exists an $r(n)$ such that $K_n' \cap L = K_{r(n)}$.*

Proof. There exists some u such that

$$K_n' = \mathrm{Fix}(G_K^u G_{L'}).$$

Because $G_{L'} \leq G_L$, we have (for the same u),

$$K_n' \cap L = \mathrm{Fix}(G_K^u G_{L'} G_L) = \mathrm{Fix}(G_K^u G_L).$$

Every such fixed field occurs in the tower of elementary extensions of L/K, which completes the proof. ∎

Lemma 6.2. *If $LK_n' = L'$, then the natural map*

$$(6.3) \qquad\qquad L \otimes_{L \cap K_n'} K_n' \to L'$$

is an isomorphism.

Proof. Surjectivity of the map is clear, so we concentrate on injectivity. It suffices to show that $L \otimes_{L \cap K_n'} K_n'$ is a field, which we prove by showing it is an integral domain, and in particular is not a direct sum of multiple fields.

Let u be such that $K_n' = \mathrm{Fix}(G_K^u G_{L'})$. Set $K_n'' := \mathrm{Fix}(G_K^u)$. Then K_n'' is a Galois extension of K, and hence is a Galois extension of any intermediate field between itself and K. In particular, K_n'' is a Galois extension of $L \cap K_n''$. This implies that the natural map

$$(6.4) \qquad\qquad L \otimes_{L \cap K_n''} K_n'' \to LK_n''$$

is injective. Because $L \subseteq L'$, we have

$$L \cap K_n'' = L \cap (K_n'' \cap L') = L \cap K_n'.$$

Using this result and flatness, we have an injective map

$$L \otimes_{L \cap K_n'} K_n' \to L \otimes_{L \cap K_n'} K_n'' = L \otimes_{L \cap K_n''} K_n''.$$

By (6.4), the right-hand side injects into a field, and hence the left-hand side $L \otimes_{L \cap K_n'} K_n'$ is an integral domain. ∎

Corollary 6.5. *For n sufficiently large, the degree of K_n' over $K_n' \cap L$ is independent of n.*

Proof. Indeed, the previous lemma shows that if n is sufficiently large that $LK_n' = L'$, then this degree is always the degree of L'/L. ∎

Corollary 6.6. *For any n, let $r(n)$ be such that $K'_n \cap L = K_{r(n)}$; this is possible by Lemma* 6.1. *For n sufficiently large, there exists a constant a independent of n such that*

$$[K'_n : K'_1] = a[K_{r(n)} : K_1].$$

Proof. A calculation shows that we may take $a = [K'_n : K_{r(n)}]/[K'_1 : K_1]$ and we have already seen that the numerator of this is independent of n for n sufficiently large. ∎

The formula stated in the previous proof, together with the descriptions of K_1 and K'_1 as the maximal tamely ramified subextensions, shows the following.

Corollary 6.7. *The constant a of Corollary* 6.6 *is equal to the wild ramification degree of L'/L.*

Corollary 6.8. *Let a denote the same constant as in Corollary* 6.6. *Then for all $n \geq 1$*

$$[K'_n : K'_1] \leq a[K_{r(n)} : K_1].$$

Proof. The assumption that n be sufficiently large was used only only to ensure the surjectivity of (6.3); the proof given in Lemma 6.2 shows that this map is always injective. This shows that

$$[K'_n : K_{r(n)}] \leq [L' : L] = a[K'_1 : K_1]$$

for all n. Using this, we have

$$[K'_n : K'_1] = \frac{[K'_n : K_{r(n)}][K_{r(n)} : K_1]}{[K'_1 : K_1]} \leq a[K_{r(n)} : K_1],$$

as required. ∎

Theorem 6.9. *If L'/L has wild ramification degree a power of q, say q^b, then the map*

$$\mathbf{A}^+_{L/K} \to \mathbf{A}^+_{L'/K}$$

given by

$$(\underline{x}_{q^i})_i \mapsto (\underline{x}'_{q^j})_j, \qquad x'_{q^{j+b}} := x_{q^j}$$

lifts the norm field map $\mathbf{E}^+_{L/K} \to \mathbf{E}^+_{L'/K}$ defined in [21, 3.1.1].

Proof. By Corollary 6.7, we may write $a = q^b$, where a is the constant of Corollary 6.8 and the surrounding results. We first claim that this map does have image in $\mathbf{A}^+_{L'/K}$. In other words, we are given an element

$$(x_{q^j}) \in \varprojlim_{\mathrm{Fr}} W_\pi(\mathscr{O}_{\widehat{L}})$$

such that for all j, if $q^j \mid [K_m : K_1]$ then $x_{q^j} \in \mathscr{O}_{K_m}$, and we want to show that if $q^{j+b} \mid [K'_n : K'_1]$, then $x_{q^j} \in \mathscr{O}_{K'_n}$. By Corollary 6.8, the condition $q^{j+b} \mid [K'_n : K'_1]$ implies $q^j \mid [K_{r(n)} : K_1]$. Hence $x_{q^j} \in \mathscr{O}_{K_{r(n)}} = \mathscr{O}_{K'_n} \cap L \subseteq \mathscr{O}_{K'_n}$, as required.

We now show that the given shift map does lift the functorial map on norm fields. By Corollary 4.12, we may view the non-zero elements of $\mathbf{E}^+_{L/K}$ as norm compatible families of elements in $\mathscr{O}_{K_n}$ for n large enough that the statement of Corollary 6.6 holds, and similarly for L'/K. The description in [21, 3.1.1], together with Lemma 6.2, shows that the norm field map is given by

$$(z_{K_n}) \mapsto (z'_{K'_n}), \qquad z'_{K'_n} := z_{K_{r(n)}}.$$

To complete the proof, we must relate this norm-compatible description of the norm field to the q-power compatible description of the norm field, as in our definition of $\mathbf{E}^+_{L/K}$. If we begin with an

element (z_{K_n}) of the norm field, viewed as a norm-compatible family, we may find a corresponding q-power compatible family using the map in [21, Proposition 4.2.1]. In particular, the q^s-component in the q-power compatible system is equal to

$$\lim_{n\to\infty} z_{K_{r(n)}}^{q^{-s}[K_{r(n)}:K_1]} = \lim_{n\to\infty} z_{K_{r(n)}}^{q^{-s-b}[K'_n:K'_1]} = \lim_{n\to\infty} (z'_{K'_n})^{q^{-s-b}[K'_n:K'_1]}.$$

This is equal to the q^{s+b} component of the image of $(z_{K'_n})$. This shows that the norm field map, in terms of q-power compatible systems, is given again by the "b-shift" map. This is clearly compatible with our "b-shift" map on $\mathbf{A}^+_{L/K} \to \mathbf{A}^+_{L'/K}$, which completes the proof. ∎

7. Lifting a uniformizer to $\mathbf{A}^+_{L/K}$ for φ-iterate extensions

Throughout this section, we fix a φ-iterate extension L/K in the sense of Definition 1.3. This definition includes the (likely redundant) hypothesis that L/K is strictly APF, so there is a positive constant c such that

$$(7.1) \qquad c \le \frac{\psi_{L/K}(u)}{[G_K : G_K^u G_L]}$$

for all u. The goal of this section is to prove Theorem 1.4. This theorem says loosely that, in the case of a φ-iterate extension, the map $\beta : \mathbf{A}^+_{L/K} \to \mathbf{E}^+_{L/K}$ is nearly surjective. The ring $\mathbf{E}^+_{L/K}$ is noncanonically isomorphic to $k[\![x]\!]$, and our strategy is to construct a lift of some q^d-th power of the uniformizer x. From our assumption that L/K is a φ-iterate extension, we know explicit uniformizers π_i for a tower of intermediate fields $K(\pi_i)$. This enables us to construct an explicit uniformizer for $\mathbf{E}^+_{L/K}$. The difficulty then is to relate these elements π_i to the elementary extensions K_m which appear in the definition of $\mathbf{A}^+_{L/K}$.

Lemma 7.2. *Let L/K denote a φ-iterate extension. There exist positive constants A, B such that for any $i \ge 1$, and any non-trivial embedding σ of $K(\pi_i)$ into $\overline{L}$ fixing $K(\pi_{i-1})$, we have*

$$Aq^i \le \mathrm{ord}_{\pi_i}(\sigma(\pi_i) - \pi_i) \le Bq^i.$$

Proof. We first find an A such that

$$Aq^i \le \mathrm{ord}_{\pi_i}(\sigma(\pi_i) - \pi_i).$$

We know that $\varphi(\pi_i) = \varphi(\sigma(\pi_i))$, which shows that

$$\mathrm{ord}_{\pi_i}\left((\sigma(\pi_i))^q - \pi_i^q\right) \ge \mathrm{ord}_{\pi_i} \pi,$$

where π is as in the definition of a φ-iterate extension. On the other hand,

$$(\sigma(\pi_i) - \pi_i)^q \equiv \sigma(\pi_i)^q - \pi_i^q \bmod p.$$

Thus

$$\mathrm{ord}_{\pi_i}(\sigma(\pi_i) - \pi_i) \ge \frac{\mathrm{ord}_{\pi_i} \pi}{q}.$$

This shows that we may take $A = \frac{\mathrm{ord}_{\pi_0} \pi}{q}$.

We now find B such that for all i,

$$\mathrm{ord}_{\pi_i}(\sigma(\pi_i) - \pi_i) \le Bq^i.$$

Write $\varphi(x) = \sum a_j x^j$. Let j_0 be the smallest index at which the minimum $B_1 := \min_j \operatorname{ord}_{\pi_0}(ja_j)$ is attained. (Note that $B_1 > 0$, because a_j is a unit only for $j = q$.) We then claim that for $i \gg 0$, we have

$$\operatorname{ord}_{\pi_i} \varphi'(\pi_i) = \operatorname{ord}_{\pi_i}(j_0 a_{j_0} \pi_i^{j_0-1}) = j_0 - 1 + B_1 q^i.$$

It suffices to show that for all $j \neq j_0$ and $i \gg 0$, we have

$$j - 1 + \operatorname{ord}_{\pi_i}(ja_j) > j_0 - 1 + B_1 q^i.$$

If $j > j_0$, then this follows immediately from the definition of B_1. This leaves only the finitely remaining values $j < j_0$. We must show that for i sufficiently large, we have

$$\left(\operatorname{ord}_{\pi_0}(ja_j) - B_1\right) q^i > j_0 - j,$$

and this is clear. After choosing B to account for the finitely many values of i which are not sufficiently large, we find that for *every* $i \geq 1$, we have

$$\operatorname{ord}_{\pi_i} \varphi'(\pi_i) \leq Bq^i.$$

We now show how this implies the result. By Weierstrass preparation, we can write

$$\varphi(x) - \pi_{i-1} = w(x)h(x),$$

where $h(x)$ is a monic polynomial of degree q and where $w(x)$ is a unit in $\mathscr{O}_{K(\pi_{i-1})}[\![x]\!]$. Taking the derivative of both sides and evaluating at π_i yields

$$\varphi'(\pi_i) = w'(\pi_i)h(\pi_i) + w(\pi_i)h'(\pi_i) = w(\pi_i) \prod_{\sigma \neq \mathrm{id}} (\pi_i - \sigma(\pi_i)).$$

(Here the product is over the $q - 1$ embeddings of $K(\pi_i)$ into $\overline{L}$ which fix $K(\pi_{i-1})$.) Hence we have

$$\operatorname{ord}_{\pi_i}(\pi_i - \sigma(\pi_i)) \leq \operatorname{ord}_{\pi_i}(\varphi'(\pi_i)) \leq Bq^i,$$

as required. ∎

Lemma 7.3. *Let τ denote a non-trivial embedding of $K(\pi_i)$ into $\overline{L}$, fixing K. Then*

$$\operatorname{ord}_{\pi_i}(\tau(\pi_i) - \pi_i) \leq Bq^i,$$

where B is the same as in Lemma 7.2.

Proof. We already know the result if τ fixes $K(\pi_{i-1})$, so choose a τ which doesn't fix $K(\pi_{i-1})$. Assume towards contradiction that

$$\operatorname{ord}_{\pi_i}(\tau(\pi_i) - \pi_i) > Bq^i.$$

Considering the power series φ which relates π_i and π_{i-1}, we then have

$$\operatorname{ord}_{\pi_i}(\tau(\pi_{i-1}) - \pi_{i-1}) \geq \operatorname{ord}_{\pi_i}(\tau(\pi_i) - \pi_i) > Bq^i,$$

and so

$$\operatorname{ord}_{\pi_{i-1}}(\tau(\pi_{i-1}) - \pi_{i-1}) > Bq^{i-1}.$$

Using descent on i, we reach a contradiction. ∎

Lemma 7.4. *Let $\overline{K(\pi_i)}$ denote the Galois closure of $K(\pi_i)$ over K. Let ϖ_i denote a uniformizer of $\overline{K(\pi_i)}$. There exists a constant $\overline{B}$, independent of i, such that if $\tau \in \operatorname{Gal}(\overline{K(\pi_i)}/K)$ is nontrivial and acts trivially on the maximal unramified subextension. Then*

$$\operatorname{ord}_{\varpi_i}(\tau(\varpi_i) - \varpi_i) \leq \overline{B}q^i.$$

Proof. We have

$$\operatorname{ord}_{\varpi_i}(\tau(\varpi_i) - \varpi_i) = \operatorname{ord}_{\varpi_i}(\pi_i) \cdot \operatorname{ord}_{\pi_i}(\tau(\varpi_i) - \varpi_i)$$
$$\leq \operatorname{ord}_{\varpi_i}(\pi_i) \cdot \operatorname{ord}_{\pi_i}(\tau(\pi_i) - \pi_i)$$

(to obtain this bound, write π_i as a polynomial in ϖ_i with coefficients in the maximal unramified subextension)

$$\leq \operatorname{ord}_{\varpi_i}(\pi_i) \cdot Bq^i,$$

where the last inequality follows from Lemma 7.3. Because the degree of $\overline{K(\pi_i)}/K(\pi_i)$ is bounded by $(q-1)!$ (independent of i), taking $\overline{B} = (q-1)!B$ proves the inequality. ∎

Lemma 7.5. *For any i and any $x \leq Aq^i$, we have*

$$\psi_{K(\pi_i)/K(\pi_{i-1})}(x) = x.$$

In particular, for any i and any $x \leq Aq^{i+1}$

$$\psi_{L/K(\pi_i)}(x) = x.$$

Proof. For the first assertion, it suffices to show that $\phi_{K(\pi_i)/K(\pi_{i-1})}(x) = x$ for $x \leq Aq^i$. By Lemma 7.2, every embedding σ of $K(\pi_i)$ into $\overline{L}$ which fixes $K(\pi_{i-1})$ satisfies

$$\operatorname{ord}_{\pi_i}(\sigma(\pi_i) - \pi_i) \geq Aq^i.$$

Thus using the definition of [21, §1.1.1], we see that $\phi_{K(\pi_i)/K(\pi_{i-1})}(x) = x$ for $x \leq Aq^i$.

For the second assertion, we use the first assertion and the fact that for any $j \geq 1$

$$\psi_{L/K(\pi_i)} = \psi_{L/K(\pi_{i+j})} \circ \psi_{K(\pi_{i+j})/K(\pi_{i+j-1})} \circ \cdots \circ \psi_{K(\pi_{i+1})/K(\pi_i)}.$$

∎

Lemma 7.6. *There exists an increasing and unbounded sequence of real numbers (independent of i, j) $A_0, A_1, A_2, \ldots$ such that for all i, j we have*

$$\psi'_{K(\pi_{i+j})/K(\pi_i)}(x) \leq \begin{cases} 1 & \text{for} & x \leq A_0 q^{i+1} \\ q & \text{for} & A_0 q^{i+1} \leq x \leq A_1 q^{i+1} \\ \ldots & & \\ q^j & \text{for} & A_{j-1} q^{i+1} \leq x < \infty. \end{cases}$$

Proof. Note first that for any finite extension of fields E/F, we have that $\psi'_{E/F}(x) \leq [E : F]$. This fact will be used without further comment below.

Let A be defined as above. We claim that we may take $A_{j-1} = jA - \frac{(j-1)A}{q}$. (Because $A_{j-1} \geq \frac{j}{2}A$, these numbers do approach infinity.) We prove this using induction on j. The claim for $j = 1$ follows from Lemma 7.5, together with the fact that $[K(\pi_{i+1}) : K(\pi_i)] = q$. Now assume the claim for some fixed $j - 1$. Our strategy is to use the fact that

$$\psi'_{K(\pi_{i+j})/K(\pi_i)}(x) = \psi'_{K(\pi_{i+j})/K(\pi_{i+j-1})}\left(\psi_{K(\pi_{i+j-1})/K(\pi_i)}(x)\right) \cdot \psi'_{K(\pi_{i+j-1})/K(\pi_i)}(x),$$

together with Lemma 7.5 and the inductive hypothesis. Because

$$\psi'_{K(\pi_{i+j})/K(\pi_{i+j-1})}(y) \leq 1$$

for $y \leq Aq^{i+j}$ and

$$\psi'_{K(\pi_{i+j})/K(\pi_{i+j-1})}(y) \leq q$$

for all y, we are finished by induction if we can show that

$$\psi_{K(\pi_{i+j-1})/K(\pi_i)}(A_{j-1}q^{i+1}) \leq Aq^{i+j}.$$

Using our slope bounds and $\psi(0) = 0$, we know that for all $x \geq A_{j-2}q^{i+1}$, we have

$$\psi_{K(\pi_{i+j-1})/K(\pi_i)}(x) \leq A_0q^{i+1} + q(A_1 - A_0)q^{i+1} + \cdots + q^{j-1}(x - A_{j-2}q^{i+1}).$$

Evaluating at $x = A_{j-1}q^{i+1}$ and using our formula for $A_0, \ldots, A_{j-1}$ in terms of A, we complete the proof. ∎

Corollary 7.7. *Let $\overline{B}$ denote some fixed constant. There exists a constant D, depending on $\overline{B}$ but independent of i, such that*

$$Dq^i \geq \psi_{L/K(\pi_i)}(\overline{B}q^i).$$

Proof. In the notation of Lemma 7.6, choose a j such that $\overline{B}q^i \leq A_jq^{i+1}$; note that this does not depend on i. Then as a coarse bound, we have $\psi_{L/K(\pi_i)}(\overline{B}q^i) \leq q^j\overline{B}q^i$. Hence for D we may take $q^j\overline{B}$. Note that this is independent of i. ∎

Lemma 7.8. *Let $K \subseteq K_1 \subseteq K_2$ be two Galois extensions of K, with Galois groups G_1, G_2. Then*

$$\mathrm{Fix}(G_2^u) \cap K_1 = \mathrm{Fix}(G_1^u).$$

Proof. Let $H_1 \trianglelefteq G_2$ denote the subgroup corresponding to the field K_1; we thus have $G_2/H_1 \cong G_1$. By properties of ramification groups in the upper numbering, we have

$$(G_2^u H_1/H_1) = (G_2/H_1)^u \cong G_1^u.$$

The claim follows immediately. ∎

Proposition 7.9. *There exists a constant D, independent of i, such that $K(\pi_i)$ is contained in the fixed field corresponding to $G_K^{\phi_{L/K}(Dq^i)}G_L$.*

Proof. Write $\overline{K(\pi_{i+j})}$ for the Galois closure of $K(\pi_{i+j})$ over K. It suffices to show that there exists a constant D, independent of i and j, such that

$$K(\pi_i) \subseteq \mathrm{Fix}\left(\mathrm{Gal}(\overline{K(\pi_{i+j})}/K)_{\psi_{\overline{K(\pi_{i+j})}/K} \circ \phi_{L/K}(Dq^i)}\right).$$

By Lemma 7.8, it suffices to consider the case $j = 0$. We want to show that if $\sigma(\pi_i) \neq \pi_i$, then

$$\sigma \notin \mathrm{Gal}(\overline{K(\pi_i)}/K)_{\psi_{\overline{K(\pi_i)}/K} \circ \phi_{L/K}(Dq^i)} = \mathrm{Gal}(\overline{K(\pi_i)}/K)_{\psi_{\overline{K(\pi_i)}/K(\pi_i)} \circ \phi_{L/K(\pi_i)}(Dq^i)}$$

Because $\psi(x) \geq x$, it suffices to prove that there exists a D, independent of i, such that

$$\sigma \notin \mathrm{Gal}(\overline{K(\pi_i)}/K)_{\phi_{L/K(\pi_i)}(Dq^i)}.$$

By Lemma 7.4, we know

$$\sigma \notin \mathrm{Gal}(\overline{K(\pi_i)}/K)_{\overline{B}q^i}.$$

Because $\psi_{L/K(\pi_i)}$ is an increasing function, it suffices to show that there exists a constant D, independent of i, such that

$$Dq^i \geq \psi_{L/K(\pi_i)}(\overline{B}q^i).$$

We are finished by Corollary 7.7. ∎

Lemma 7.10. *There exists a constant N, independent of i, such that*

$$[G_K : G_K^{\phi_{L/K}(Dq^i)}G_L] \leq Nq^i$$

for all i.

Proof. By (7.1), using the fact that $\psi_{L/K}$ is the inverse to $\phi_{L/K}$, we have

$$[G_K : G_K^{\phi_{L/K}(Dq^i)}G_L] \leq \frac{Dq^i}{c}.$$

Thus we may take $N = D/c$. ∎

Let D be as in Proposition 7.9. Let $v_i := \phi_{L/K}(Dq^i)$ and let F_i denote the fixed field corresponding to $G_K^{v_i}G_L$.

Lemma 7.11. *There does not exist an infinite sequence $0 = i_0 < i_1 < i_2 < \cdots$ such that*

$$[F_{i_{j+1}} : F_{i_j}] > q^{i_{j+1}-i_j}$$

for all j.

Proof. By Proposition 7.9, we know that $K(\pi_i) \subseteq F_i$. We also know that $[K(\pi_i) : K] = q^i$ for all i. By these observations and Lemma 7.10, we have that

$$q^i \mid [F_i : K] \qquad \text{and} \qquad [F_i : K] \leq Nq^i$$

for all i.

Now assume towards contradiction that there does exist a sequence as in the statement of the lemma. We have $[F_{i_1} : F_0] > q^{i_1}$, and also $q^{i_1} \mid [F_{i_1} : F_0]$, so $[F_{i_1} : F_0] \geq 2q^{i_1}$. We also have $[F_{i_2} : F_{i_1}] > q^{i_2-i_1}$, so $[F_{i_2} : F_0] > 2q^{i_2}$, and also $q^{i_2} \mid [F_{i_2} : F_0]$, so $[F_{i_2} : F_0] \geq 3q^{i_2}$. Continuing in this way, we eventually pass Nq^i, and thus reach a contradiction. ∎

Corollary 7.12. *There exists an $i_0 \gg 0$ such that $[F_{i_0+j} : F_{i_0}] \leq q^j$ for all j. In particular, there exists a field K_i in the tower of elementary extensions such that*

$$q^m \mid [K_{i+j} : K_i]$$

implies $\pi_{i_0+m} \in K_{i+j}$.

Proof. The first assertion follows immediately from Lemma 7.11. We now prove the second assertion. Notice that each of the fields F_j, being the fixed field of $G_K^{v_j}G_L$, occurs in the tower of elementary extensions, since the tower of elementary extensions contains the fixed field of $G_K^v G_L$ for every v. Take $K_i = F_{i_0}$. We know that $\pi_{i_0+j} \in F_{i_0+j}$ by Proposition 7.9, and we know that F_{i_0+j} occurs somewhere in the tower of elementary extensions. Because $[F_{i_0+m} : K_i] = [F_{i_0+m} : F_{i_0}] \leq q^m$, we deduce that if $q^m \mid [K_{i+j} : K_i]$, then $\pi_{i_0+m} \in F_{i_0+m} \subseteq K_{i+j}$, as required. ∎

The preceding results provide a connection between the elementary fields K_n and the fields $K(\pi_i)$. The fields K_n are important because of their appearance in the definition of $\mathbf{A}_{L/K}^+$ (and because they are canonically associated to the extension L/K). The fields $K(\pi_i)$ are important because in each such field we know an explicit uniformizer π_i, and from these uniformizers we can produce a uniformizer for $\mathbf{E}_{L/K}^+$.

Lemma 7.13. *Let $\mathfrak{b}$ be as in (4.5) and let $\mathfrak{a} = (\pi, \mathfrak{b})$. Let $q = p^a$. A uniformizer for $\mathbf{E}_{L/K}^+$ is given by the p-power compatible system in*

$$\varprojlim_{x \mapsto x^p} \mathscr{O}_L/\mathfrak{a}$$

which has ai component equal to π_i mod $\mathfrak{a}$.

Proof. Using that the system of fields $K(\pi_i)$ is cofinal among the system of all intermediate fields between L and K, we may express the valuation ring inside of $X_K(L)$ as $\varprojlim_{\mathrm{Nm}} \mathscr{O}_{K(\pi_i)}$. Using Weierstrass preparation, write $\varphi(x) = w(x)h(x)$, where $w(x) \in \mathscr{O}_F[\![x]\!]$ is a unit and where $h(x)$ is a monic degree q polynomial with no constant term. Because $\varphi(x) \equiv x^q \mod \pi$, we must have $w(x) \equiv 1 \mod \pi$. Because $\varphi(\pi_i) = \pi_{i-1}$, we must have that $h(\pi_i) \equiv \pi_{i-1} \mod \pi$.

Let $f_i(x) \equiv x^q \mod \pi_{i-1} \in \mathscr{O}_{K(\pi_{i-1})}[x]$ denote the minimal polynomial for π_i over $K(\pi_{i-1})$. We compare this polynomial to $h(x) - \pi_{i-1}$. Write

$$f_i(x) - (h(x) - \pi_{i-1}) = a_1 x^{q-1} + \cdots + a_q \in \mathscr{O}_{K(\pi_{i-1})}[x].$$

Evaluating this polynomial at π_i yields a sum of q elements of $\mathscr{O}_{K(\pi_i)}$ which have distinct π_i-valuations. On the other hand, evaluating this polynomial at π_i yields an element divisible by π. This shows that each of the q terms, and in particular the constant term, must be divisible by π. We deduce that the constant term of $f_i(x)$ differs from $-\pi_{i-1}$ by a multiple of π. In particular, $\mathrm{Nm}_{K(\pi_i)/K(\pi_{i-1})}(\pi_i) \equiv \pi_{i-1} \mod \pi$. (This holds for both $p = 2$ and for p odd.) Thus, for the ideal $\mathfrak{a}$ as in the statement of the lemma, we have that the elements $(\pi_i \mod \mathfrak{a})$ are norm-compatible. As in the proof of Lemma 4.9, this sequence lifts to a norm-compatible family of uniformizers, which in turn is a uniformizer for $\mathbf{E}_{L/K}^+$. ∎

Recall that the map β was defined in Definition 3.7 (see also Proposition 5.7). The following result shows that the image of β contains the uniformizer of Lemma 7.13 raised to some power of q.

Proposition 7.14. *For $i > 0$, set $\pi_{-i} := \varphi^i(\pi_0)$. There exists some $i_1 \geq 0$ and some $x \in \mathbf{A}_{L/K}^+$ such that $\beta(x) = (\pi_{n-i_1} \mod \mathfrak{b})_n$.*

Proof. We first show that there exists a Frobenius-compatible family of elements $\underline{x}_n \in W_\pi(\mathscr{O}_{K(\pi_n)})$ such that their first components satisfy

$$x_{n0} = \pi_n.$$

Consider the unique continuous $\mathscr{O}_F$-algebra homomorphism $\varphi : \mathscr{O}_F[\![x]\!] \to \mathscr{O}_F[\![x]\!]$ which sends x to $\varphi(x)$. By Proposition 2.13, there is a unique ring homomorphism

$$\lambda_\varphi : \mathscr{O}_F[\![x]\!] \to W_\pi(\mathscr{O}_F[\![x]\!])$$

which is a section to the projection $W_\pi(\mathscr{O}_F[\![x]\!]) \to \mathscr{O}_F[\![x]\!]$ and which satisfies $\mathrm{Fr} \circ \lambda_\varphi = \lambda_\varphi \circ \varphi$. By Corollary 2.14, the element $\lambda_\varphi(x)$ has ghost components $(x, \varphi(x), \varphi^2(x), \ldots)$. Now for any n, consider the $\mathscr{O}_F$-algebra homomorphism $\mathscr{O}_F[\![x]\!] \to \mathscr{O}_{K(\pi_n)}$ determined by $x \mapsto \pi_n$. This determines a map $W_\pi(\mathscr{O}_F[\![x]\!]) \to W_\pi(\mathscr{O}_{K(\pi_n)})$. Because the ghost components are defined by polynomials with coefficients in $\mathscr{O}_F$, we see that the composite

$$\mathscr{O}_F[\![x]\!] \to W_\pi(\mathscr{O}_F[\![x]\!]) \to W_\pi(\mathscr{O}_{K(\pi_n)})$$

sends x to a Witt vector with ghost components $(\pi_n, \varphi(\pi_n), \varphi^2(\pi_n), \ldots) = (\pi_n, \pi_{n-1}, \ldots, \pi_0, \varphi(\pi_0), \ldots)$. We define $\underline{x}_n$ to be the image of x under this composite. In particular, $x_{n0} = \pi_n$, and $\mathrm{Fr}(\underline{x}_n) = \underline{x}_{n-1}$. This completes our construction of the elements $\underline{x}_n \in W_\pi(\mathscr{O}_{K(\pi_n)})$.

To construct the desired element in $\mathbf{A}_{L/K}^+$, we need to work with the elementary subfields K_m, rather than the fields $K(\pi_n)$. Let K_i be is as in Corollary 7.12, and let i_1 be such that $[K_i : K_1] \mid q^{i_1}$. Using notation of the previous paragraph and the statement of the proposition, we claim that $(\underline{x}_{n-i_1})_n$, which we know is an element of $\widetilde{\mathbf{A}}^+$, is in fact an element of $\mathbf{A}_{L/K}^+$. Recalling the definition of $\mathbf{A}_{L/K}^+$, we need to show that if $q^n \mid [K_m : K_1]$, then $\pi_{n-i_1} \in K_m$. We deduce this as follows:

$$q^n \mid [K_m : K_1]$$
$$q^n \mid [K_m : K_i][K_i : K_1]$$
$$q^{n-i_1} \mid [K_m : K_i]$$

and by Corollary 7.12, this implies

$$\pi_{i_0+n-i_1} \in K_m,$$

and in particular $\pi_{n-i_1} \in \mathscr{O}_{K_m}$. This completes the proof, at least for $n - i_1 \geq 0$, but if $n - i_1 < 0$, then $\pi_{n-i_1} = \varphi^{i_1-n}(\pi_0) \in \mathscr{O}_K \subseteq \mathscr{O}_{K_m}$ for all m. ∎

Proposition 7.15. *As subrings of $\widetilde{\mathbf{E}}^+$, we have $\mathbf{A}_{L/K}^+/\pi\mathbf{A}_{L/K}^+ \subseteq \mathbf{E}_{L/K}^+$, and moreover $\mathbf{E}_{L/K}$ is a finite purely inseparable extension of $\mathrm{Frac}\left(\mathbf{A}_{L/K}^+/\pi\mathbf{A}_{L/K}^+\right)$.*

Proof. The inclusion $\mathbf{A}_{L/K}^+/\pi\mathbf{A}_{L/K}^+ \subseteq \mathbf{E}_{L/K}^+$ follows from Proposition 5.7. We know that $\mathbf{E}_{L/K}^+$ is abstractly isomorphic to $k[\![u]\!]$. We also know that the image of β contains the coefficient ring k by Proposition 5.9. By Proposition 7.14, we know that there exists $v \in \mathbf{A}_{L/K}^+$ such that $\beta(v) = u^{q^{i_1}}$. Because $\mathbf{A}_{L/K}^+ \subseteq \widetilde{\mathbf{A}}^+$ is weakly closed (Proposition 5.6), we deduce that the image of β contains $k[\![u^{q^{i_1}}]\!]$, which completes the proof. ∎

Proposition 7.15 immediately implies our second main theorem from the introduction, Theorem 1.4.

Remark 7.16. Continue to assume L/K is a φ-iterate extension. Here we provide a significantly simpler description of a subring of $W_\pi(\widetilde{\mathbf{E}}_{L/K}^+) \cong \varprojlim W_\pi(\mathscr{O}_{\widehat{L}})$ which lifts $\mathbf{E}_{L/K}^+$. Consider the continuous map

$$\iota_{\underline{\pi}} : W_\pi(k)[\![u]\!] \to \varprojlim W_\pi(\mathscr{O}_{\widehat{L}})$$

which sends u to $(\underline{x}_n)_n$, where $\underline{x}_n$ is the (unique) Witt vector in $W_\pi(\mathscr{O}_{\widehat{L}})$ with ghost components $(\pi_n, \pi_{n-1}, \ldots, \pi_0, \varphi(\pi_0), \ldots)$. (Such Witt vectors exist by the Dwork Lemma, Proposition 2.5.) We write $\mathbf{A}_{\underline{\pi}}$ for the image of $\iota_{\underline{\pi}}$. It is a π-Cohen ring for $\mathbf{E}_{L/K}^+$, and it is related to our ring $\mathbf{A}_{L/K}^+$ by $\varphi^{i_1}(\mathbf{A}_{\underline{\pi}}) \subseteq \mathbf{A}_{L/K}^+$, where i_1 is as in Proposition 7.14. The downside to $\mathbf{A}_{\underline{\pi}}$ is that it seems to depend heavily on the choice of the elements π_i, and hence it is not obvious from this point-of-view that $\mathbf{A}_{L/K}^+$ can be canonically associated to the φ-iterate extension L/K. In particular, if L/K is Galois, it is not clear that $\mathbf{A}_{\underline{\pi}}$ is stable under the action of G_K on $\varprojlim W_\pi(\mathscr{O}_{\widehat{L}})$. Also it is not clear how to define an analogue of $\mathbf{A}_{\underline{\pi}}$ for an arbitrary strictly APF extension L/K (i.e., for extensions other than φ-iterate extensions).

Remark 7.17. Our Proposition 7.15, in conjunction with Proposition 5.14, implies that if L/K is both a φ-iterate extension and Galois, then its Galois group is necessarily abelian. It would be interesting to prove this fact directly.

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