# ON THE CYCLIC HOMOLOGY OF CERTAIN UNIVERSAL DIFFERENTIAL GRADED ALGEBRAS

CHRISTOPHER DAVIS, JULIUS FRANK, AND IRAKLI PATCHKORIA

Abstract. Let $p$ be an odd prime and $R$ a $p$-torsion-free commutative $\mathbb{Z}_{(p)}$-algebra. We compute the periodic cyclic homology over $R$ of the universal differential graded algebra $R/\!\!/p$ which is obtained from $R$ by universally killing $p$. We furthermore compute the cyclic and negative cyclic homologies of $R/\!\!/p$ over $R$ in infinitely many degrees.

## 1. Introduction

For a fixed prime $p$, we can consider the ring $\mathbb{Z}[u]/(u(u-p))$. It is an interesting problem to compute the algebraic $K$-theory $K(\mathbb{Z}[u]/(u(u-p)))$ of this ring. It follows from a result of Land and Tamme [8, Theorem 1.1 and Example 4.31] that there is a homotopy pullback square of algebraic $K$-theory spectra

$$\begin{array}{ccc} K(\mathbb{Z}[u]/(u(u-p))) & \longrightarrow & K(\mathbb{Z}) \\ \downarrow & & \downarrow \\ K(\mathbb{Z}) & \longrightarrow & K(\mathbb{Z}/\!\!/p), \end{array}$$

where $\mathbb{Z}/\!\!/p$ is the universal differential graded algebra with one generator $x$ in degree 1 satisfying $dx = p$.

The differential graded algebra $\mathbb{Z}/\!\!/p$ has a different behaviour depending on whether the prime $p$ is even or odd. For $p = 2$, it follows from unpublished results of Krause and Nikolaus and [8, Example 4.32] that the algebra $\mathbb{Z}/\!\!/p$ is formal over the sphere spectrum. For $p$ odd, the results of the second author [6] imply that $\mathbb{Z}/\!\!/p$ is not formal as an $E_1$-ring spectrum and hence also not as a differential graded algebra.

More generally, one can consider any $p$-torsion-free commutative $\mathbb{Z}_{(p)}$-algebra $R$ and the differential graded algebra $R/\!\!/p$. The underlying graded algebra of $R/\!\!/p$ is $R[x]$ such that $x$ is in degree 1 and $dx = p$. It follows from the Dundas-McCarthy theorem [5, 10] that the commutative diagram

$$\begin{array}{ccc} K(R/\!\!/p) & \longrightarrow & \mathrm{TC}(R/\!\!/p) \\ \downarrow & & \downarrow \\ K(R/p) & \longrightarrow & \mathrm{TC}(R/p) \end{array}$$

is a homotopy pullback after $p$-adic completion, where TC denotes the topological cyclic homology and the horizontal maps are given by the cyclotomic trace map (see [3]). If one assumes the knowledge of $K(R/p)$ and $\mathrm{TC}(R/p)$, then using the latter square in order to compute $K(R/\!\!/p)$, it suffices to compute $\mathrm{TC}(R/\!\!/p)$. One can try to calculate $\mathrm{TC}(R/\!\!/p)$ using the approach of Nikolaus-Scholze [11]. Recall that the topological Hochschild homology spectrum $\mathrm{THH}(R/\!\!/p)$ has a circle action, and one denotes the homotopy fixed points with respect to this action by $\mathrm{TC}^-(R/\!\!/p)$ and the Tate construction by $\mathrm{TP}(R/\!\!/p)$. Using [11, Proposition II.1.9 and Lemma II.4.2.1], we know that after $p$-completion there exists a fiber sequence of spectra

$$\mathrm{TC}(R/\!\!/p) \longrightarrow \mathrm{TC}^-(R/\!\!/p) \xrightarrow{can-\varphi} \mathrm{TP}(R/\!\!/p),$$

where *can* is the canonical map and $\varphi$ the Frobenius. Thus in order to compute the topological cyclic homology $\mathrm{TC}(R/\!\!/p)$, one should compute $\mathrm{TC}^-(R/\!\!/p)$ and $\mathrm{TP}(R/\!\!/p)$ as well as the maps *can* and $\varphi$.

The ultimate goal of this project is to compute the $p$-completion of the spectrum $K(R/\!\!/p)$ for $R$ a $p$-torsion-free perfectoid ring in the sense of [1]. In this case it follows by [4] and the Dundas-McCarthy theorem [5, 10], that the $p$-completion of $K(R/\!\!/p)$ is the connective cover of the $p$-completion of $\mathrm{TC}(R/\!\!/p)$. Hence we need to compute $\mathrm{TC}^{-}(R/\!\!/p)$ and $\mathrm{TP}(R/\!\!/p)$ and eventually also $\mathrm{TC}(R/\!\!/p)$ for $R$ a $p$-torsion-free perfectoid ring. We do not recall here perfectoid rings since the definition of these will not be relevant in this paper. However, we do wish to recall that by [2], for any perfectoid ring $R$, there is a circle equivariant fiber sequence after $p$-completion:

$$\mathrm{THH}(R/\!\!/p)[2] \to \mathrm{THH}(R/\!\!/p) \to \mathrm{HH}^R(R/\!\!/p),$$

where $\mathrm{HH}^R(R/\!\!/p)$ denotes the Hochschild homology of $R/\!\!/p$ over $R$. By applying either the homotopy fixed points or Tate construction, one gets fiber sequences after $p$-completion:

$$\mathrm{TC}^{-}(R/\!\!/p)[2] \to \mathrm{TC}^{-}(R/\!\!/p) \to \mathrm{HC}^{R,-}(R/\!\!/p), \quad \mathrm{TP}(R/\!\!/p)[2] \to \mathrm{TP}(R/\!\!/p) \to \mathrm{HP}^R(R/\!\!/p),$$

where $\mathrm{HC}^{R,-}(R/\!\!/p)$ and $\mathrm{HP}^R(R/\!\!/p)$ denotes the negative and periodic cyclic homology over $R$, respectively [9, Chapter 5].

The goal of this paper is to compute $\mathrm{HC}^{R,-}(R/\!\!/p)$ and $\mathrm{HP}^R(R/\!\!/p)$ for any $p$-torsion-free commutative $\mathbb{Z}_{(p)}$-algebra $R$ which is not necessarily perfectoid. Having the above fiber sequences in mind, the hope is that we can solve extension problems and compute $\mathrm{TC}^{-}(R/\!\!/p)$ and $\mathrm{TP}(R/\!\!/p)$.

For simplicity we denote by $\mathrm{HC}_i(R/\!\!/p)$, $\mathrm{HC}_i^{-}(R/\!\!/p)$ and $\mathrm{HP}_i(R/\!\!/p)$, the cyclic, negative cyclic and periodic cyclic homology modules of $R/\!\!/p$ over $R$, respectively. We can now formulate the main results of this paper:

**Theorem 1.1.** *Let $p$ be an odd prime and $R$ a $p$-torsion-free commutative $\mathbb{Z}_{(p)}$-algebra. Then $\mathrm{HP}_i(R/\!\!/p) = 0$ for $i$ odd and for any even $i$, the $R$-module $\mathrm{HP}_i(R/\!\!/p)$ is isomorphic to*

$$R^{\wedge} \times R/1 \times R/3 \times R/5 \times \cdots,$$

*where $R^{\wedge}$ denotes the $p$-adic completion of $R$.*

Note that all the factors $R/n$ with $n$ coprime to $p$ vanish, and more generally, for any odd positive integer $n$, we have $R/n \cong R/p^{\nu_p(n)}$, where $\nu_p$ denotes the $p$-adic valuation.

To formulate the next result on the cyclic homology we need to define the following numbers: Let $A_1 := p$, and for each odd integer $j \geq 3$, we recursively define $A_j \in \mathbb{Q}$ as $A_j := \frac{p^2 A_{j-2}}{j}$. Corollary 3.6 below shows that these numbers belong to $\mathbb{Z}_{(p)}$ and hence the $p$-adic valuations $a_i = \nu_p(A_i)$ are non-negative integers.

**Theorem 1.2.** *Let $p$ be an odd prime and $R$ a $p$-torsion-free commutative $\mathbb{Z}_{(p)}$-algebra. Then $\mathrm{HC}_i(R/\!\!/p) = 0$ for $i$ odd and $\mathrm{HC}_0(R/\!\!/p) = R/p$. There furthermore exists an infinite set $Z$ of positive even integers, such that for any $i \in Z$, the $R$-module $\mathrm{HC}_i(R/\!\!/p)$ is isomorphic to*

$$R/p^{a_{i-1}+2} \oplus R/1 \oplus R/3 \oplus \cdots \oplus R/(i-1).$$

*The set $Z$ contains all even numbers of the form $\frac{p^a \pm 1}{2} + 1$ for $a > 0$.*

Finally, we have the following result calculating the negative cyclic homology:

**Theorem 1.3.** *Let $p$ be an odd prime and $R$ a $p$-torsion-free commutative $\mathbb{Z}_{(p)}$-algebra. Then $\mathrm{HC}_i^{-}(R/\!\!/p) = 0$ for $i$ odd and for any non-positive even $i$, the $R$-module $\mathrm{HC}_i^{-}(R/\!\!/p)$ is isomorphic to*

$$R^{\wedge} \times R/1 \times R/3 \times R/5 \times \cdots.$$

*There furthermore exists an infinite set $\overline{Z} \subset Z$, such that for any $i \in \overline{Z}$, the $R$-module $\mathrm{HC}_i^{-}(R/\!\!/p)$ is isomorphic to*

$$R^{\wedge} \times R/(i-1) \times R/(i+1) \times R/(i+3) \times \cdots.$$

*The set $\overline{Z}$ contains all even numbers of the form $\frac{p^a \pm 1}{2} + 1$ for $a > 0$.*

Additionally, we also compute the values of $\mathrm{HC}_{i-2}(R/\!\!/p)$ for $i \in \overline{Z}$ and $i > 2$. This is a consequence of the proof of the latter theorem from which it follows that for any $i \in \overline{Z}$ and $i > 2$, we have an isomorphism

$$\mathrm{HC}_{i-2}(R/\!\!/p) \cong R/p^{a_{i-1}} \oplus R/1 \oplus R/3 \oplus \cdots \oplus R/(i-3).$$

The values of $\mathrm{HC}_i(R/\!\!/p)$ and $\mathrm{HC}_i^{-}(R/\!\!/p)$ for a general $i$ remain still open. However, in the final section of this paper we estimate the sizes of the sets $Z$ and $\overline{Z}$ and conclude that for large primes they get asymptotically close to the set of all positive even integers.

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## 2. Setup

Fix an odd prime $p$ and let $R$ be a commutative ring without $p$-torsion. We do not yet require $R$ to be a $\mathbb{Z}_{(p)}$-algebra.

**Definition 2.1.** We construct a differential graded algebra $R/\!\!/p$ over $R$ as follows: It has as its underlying graded algebra $R[x]$ with $x$ in degree 1, and the differential is given as $\delta x = p$.

*Remark* 2.2. We collect some immediate observations for this algebra:

- Note that while this is strictly commutative, it is not graded commutative.
- The differential in higher degrees is

$$\delta x^n = \begin{cases} px^{n-1} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

  Because $R$ has no $p$-torsion, the homology is then $H_*(R/\!\!/p) = R/p[u]$, with $u$ in degree 2.

To justify the notation, we give a more conceptual description of $R/\!\!/p$. Let $T_R : \mathrm{Ch}_R \to \mathrm{DGA}_R$ denote the tensor algebra functor that assigns to each chain complex $M$ the free differential graded $R$-algebra $T_R(M) = \bigoplus_{m \geq 0} M^{\otimes_R m}$.

**Proposition 2.3.** *There is a pushout square in* $\mathrm{DGA}_R$ *given by*

$$\begin{CD} T_R(R) @>{p}>> R \\ @VVV @VVV \\ T_R(CR) @>>> R/\!\!/p, \end{CD}$$

*where $CR$ is the cone of $R$ in $\mathrm{Ch}_R$, the left map is induced by the inclusion $R \hookrightarrow CR$, and the top map is adjoint to the multiplication $p : R \to R$.*

*Proof.* A morphism $\varphi : R/\!\!/p \to A$ of dg-$R$-algebras is precisely a ring homomorphism $\tilde{\varphi} : R \to A_0$ and the choice of an $a \in A_1$ such that $\tilde{\varphi}(p) = \delta a$. By identifying $T_R(CR) \cong R\langle x, \delta x\rangle$ with $|x| = 1$, and $T_R(R) \cong R\langle \delta x\rangle$, this data corresponds precisely to a commutative diagram

$$\begin{CD} R\langle \delta x\rangle @>{p}>> R \\ @VVV @VVV \\ R\langle x, \delta x\rangle @>>> A, \end{CD}$$

where $x$ is mapped to $a$.

□

*Remark* 2.4. This exhibits $R/\!\!/p$ as the $\mathbb{E}_1$-quotient of $R$ with respect to $p$, i.e. $R/\!\!/p$ is the initial $\mathbb{E}_1$-R-algebra whose homology has $p = 0$. More precisely, there is a model structure on $\mathrm{DGA}_R$ with quasi-isomorphisms as weak equivalences, under which the left map is a cofibration [7, 12]. Hence the square is a homotopy pushout of dg-$R$-algebras. The homotopy theory of dg-$R$-algebras is equivalent to the homotopy theory of $\mathbb{E}_1$-algebras in the derived $\infty$-category $\mathcal{D}(R)$ (see [13]). The homotopy pushout then corresponds to a pushout in the $\infty$-category of $\mathbb{E}_1$-algebras over $R$.

The aim of this paper is to understand cyclic, negative cyclic, and periodic homology of $R/\!\!/p$ over $R$.

As a first approach, note that there is a Connes long exact sequences connecting cyclic homology with Hochschild homology: This is obtained by considering the subcomplex of the cyclic bicomplex consisting only of the 0th and 1st column which is quasi-isomorphic to the Hochschild complex [9, 2.2.1]. The quotient complex then computes cyclic homology shifted by degree 2. The resulting long exact sequence for any algebra $A$ is therefore

$$\cdots \longrightarrow \mathrm{HH}_i(A) \longrightarrow \mathrm{HC}_i(A) \longrightarrow \mathrm{HC}_{i-2}(A) \longrightarrow \cdots$$

In our setting, $\mathrm{HC}_*$ vanishes in odd degrees, so we obtain for $i > 0$ even short exact sequences

$$0 \longrightarrow R/p^2 \longrightarrow \mathrm{HC}_i(R/\!\!/p) \longrightarrow \mathrm{HC}_{i-2}(R/\!\!/p) \longrightarrow 0,$$

and we recover $HC_0(R/\!\!/p) \cong R/p$.

Similarly, one gets a long exact sequence for negative cyclic homology:

$$\cdots \longrightarrow \mathrm{HC}^-_{i+2}(A) \longrightarrow \mathrm{HC}^-_i(A) \longrightarrow \mathrm{HH}_i(A) \longrightarrow \cdots .$$

Below in our setting, we obtain for even $i > 0$ short exact sequences

$$0 \longrightarrow \mathrm{HC}^-_{i+2}(R/\!\!/p) \longrightarrow \mathrm{HC}^-_i(R/\!\!/p) \longrightarrow R/p^2 \longrightarrow 0.$$

and for $i = 0$

$$0 \longrightarrow \mathrm{HC}^-_2(R/\!\!/p) \longrightarrow \mathrm{HC}^-_0(R/\!\!/p) \longrightarrow R/p \longrightarrow 0.$$

This looks innocent enough, but these extension problems will occupy the rest of this paper.

To get started, we will need a better understanding of the Hochschild bicomplex:

**Proposition 2.5.** *The Hochschild bicomplex of $R/\!\!/p$ is quasi-isomorphic to*

$$\begin{array}{ccccccccccc}
 & & R & \xleftarrow{p} & R & \xleftarrow{0} & R & \xleftarrow{p} & R & \xleftarrow{0} & \cdots \\
 & & {\scriptstyle 0}\downarrow & & {\scriptstyle 2}\downarrow & & {\scriptstyle 0}\downarrow & & {\scriptstyle 2}\downarrow & & \\
R & \xleftarrow{p} & R & \xleftarrow{0} & R & \xleftarrow{p} & R & \xleftarrow{0} & R & \xleftarrow{p} & \cdots
\end{array}$$

*Proof.* Note that the underlying graded algebra of $R/\!\!/p$ is a tensor algebra $T_R(V)$ over the free graded $R$-module $V = \langle x\rangle$ with $x$ in degree 1. For dg-$R$-algebras $A$ with underlying graded tensor algebra $T_R(V)$, the Hochschild bicomplex has a simplified description, [9, 5.3.8]: It is quasi-isomorphic to

$$\begin{array}{ccccccc}
(A\otimes V)_0 & \xleftarrow{\tilde\delta} & (A\otimes V)_1 & \xleftarrow{\tilde\delta} & (A\otimes V)_2 & \xleftarrow{\tilde\delta} & \cdots \\
{\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & \\
A_0 & \xleftarrow{\delta} & A_1 & \xleftarrow{\delta} & A_2 & \xleftarrow{\delta} & \cdots ,
\end{array}$$

where all tensor products are over $R$, and the maps are as follows: $\delta$ is the differential of $A$ and $b(a\otimes v) = [a, v]$. (Note that this is a graded commutator!) For $\tilde\delta$, consider first the map

$$\begin{aligned}
\varphi : A \otimes A &\longrightarrow A \otimes V \\
a \otimes (v_1 \otimes \cdots \otimes v_n) &\longmapsto \sum_{i=1}^{n} \pm (v_{i+1} \otimes \cdots \otimes v_n \otimes a \otimes v_1 \otimes \cdots \otimes v_{i-1}) \otimes v_i \\
a \otimes 1 &\longmapsto 0,
\end{aligned}$$

with the sign determined by the Koszul convention. Then $\tilde\delta$ is given by

$$\tilde\delta(a \otimes v) = \delta a \otimes v + (-1)^{|a|}\varphi(a \otimes \delta v).$$

In our case, $V = \langle x\rangle$ with $|x| = 1$. Hence $A_n = \langle x^n\rangle$, and for $n > 0$, $(A \otimes V)_n = \langle x^{n-1} \otimes x\rangle$. One can then easily identify the maps: We already know $\delta$. For $b$, we have

$$b(x^{n-1} \otimes x) = [x^{n-1}, x] = x^n - (-1)^{n-1}x^n = \begin{cases} 2x^n & n \text{ even} \\ 0 & n \text{ odd,} \end{cases}$$

and for $\tilde\delta$,

$$\tilde\delta(x^{n-1} \otimes x) = \delta x^{n-1} \otimes x + (-1)^{n-1}\varphi(x^{n-1} \otimes p) = \delta x^{n-1} \otimes x = \begin{cases} p x^{n-2} \otimes x & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

□

**Corollary 2.6.** *The total complex of the bicomplex of Proposition 2.5 that computes Hochschild homology is given by*

$$R \xleftarrow{\;p\;} R \xleftarrow{\;0\;} R^2 \xleftarrow{\left(\begin{smallmatrix} p & 2 \\ 0 & p \end{smallmatrix}\right)} R^2 \xleftarrow{\;0\;} R^2 \xleftarrow{\left(\begin{smallmatrix} p & 2 \\ 0 & p \end{smallmatrix}\right)} \cdots ,$$

*and if $R$ has no 2-torsion, the Hochschild homology is*

$$\mathrm{HH}_i(R/\!\!/p) = \begin{cases} R/p & i = 0 \\ R/p^2 & i > 0 \text{ even} \\ 0 & \text{else.} \end{cases}$$

*Proof.* The total complex is immediate. For the Hochschild homology, note that if there is no 2-torsion and $p$ is odd, the Smith normal form of $\left(\begin{smallmatrix} p & 2 \\ 0 & p \end{smallmatrix}\right)$ is calculated as

$$\begin{pmatrix} p & 2 \\ 0 & p \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 \\ -\lfloor \frac{p}{2} \rfloor p & p \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 2\lfloor \frac{p}{2} \rfloor p + p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix},$$

Where $\lfloor \frac{p}{2} \rfloor = \frac{p-1}{2}$ as $p$ is odd.

After dropping the summands that are matched by identities, the total complex becomes

$$R \xleftarrow{\;p\;} R \xleftarrow{\;0\;} R \xleftarrow{\;p^2\;} R \xleftarrow{\;0\;} R \xleftarrow{\;p^2\;} \cdots ,$$

so without $p$-torsion, the result follows. (If $p = 2$, then the differential instead becomes $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$, and $R/p^2$ is replaced by $R/p \oplus R/p$.) $\square$

For the cyclic complex, there is again a simplified description based on the simplified Hochschild complex [9, 5.3.9]: For a dg-$R$-algebra $A$ with underlying graded tensor algebra $T_R(V)$, the Connes operator can be identified as

$$\tilde{B} = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} : A_n \oplus (A \otimes V)_{n-1} \longrightarrow A_{n+1} \oplus (A \otimes V)_n$$

with $\gamma(a) = \varphi(1 \otimes a)$, where $\varphi : A \otimes A \to A \otimes V$ is the map from above.

Untangeling definitions for our setting, we obtain

$$\begin{aligned} \gamma : (R/\!\!/p)_n = \langle x^n \rangle &\longrightarrow \langle x^{n-1} \otimes x \rangle = (R/\!\!/p \otimes \langle x \rangle)_n \\ x^n &\longmapsto \sum_{i=1}^{n} (-1)^{(n-i)i} x^{n-1} \otimes x = \begin{cases} n x^{n-1} \otimes x & n \text{ odd} \\ 0 & n \text{ even.} \end{cases} \end{aligned}$$

This lets us describe the cyclic bicomplex for $R/\!\!/p$:

**Proposition 2.7.** *Let $p$ be odd, $R$ without $p$-torsion and $\frac{1}{2} \in R$. Then the cyclic bicomplex for the $R$-algebra $R/\!\!/p$ is quasi-isomorphic to*

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
{\scriptstyle p^2}\big\downarrow & & {\scriptstyle 0}\big\downarrow & & {\scriptstyle p^2}\big\downarrow & & {\scriptstyle 0}\big\downarrow & & {\scriptstyle p}\big\downarrow \\
R & \xleftarrow{\;3\;} & R & \xleftarrow{\;0\;} & R & \xleftarrow{\;1\;} & R & \xleftarrow{\;0\;} & R \\
{\scriptstyle 0}\big\downarrow & & {\scriptstyle p^2}\big\downarrow & & {\scriptstyle 0}\big\downarrow & & {\scriptstyle p}\big\downarrow & & \\
R & \xleftarrow{\;0\;} & R & \xleftarrow{\;1\;} & R & \xleftarrow{\;0\;} & R & & \\
{\scriptstyle p^2}\big\downarrow & & {\scriptstyle 0}\big\downarrow & & {\scriptstyle p}\big\downarrow & & & & \\
R & \xleftarrow{\;1\;} & R & \xleftarrow{\;0\;} & R & & & & \\
{\scriptstyle 0}\big\downarrow & & {\scriptstyle p}\big\downarrow & & & & & & \\
R & \xleftarrow{\;0\;} & R & & & & & & \\
{\scriptstyle p}\big\downarrow & & & & & & & & \\
R & & & & & & & &
\end{array}$$

*where the bottom $R$ is in bidegree $(0,0)$. Likewise, one obtains the periodic bicomplex, by continuing this to the left, and the negative bicomplex, by dropping the positive-degree columns from the periodic complex.*

*Proof.* By plugging the previous calculations into Loday's bicomplex, we obtain

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\pi\downarrow & & 0\downarrow & & \pi\downarrow & & 0\downarrow & & p\downarrow \\
R^2 & \xleftarrow{B_3} & R^2 & \xleftarrow{0} & R^2 & \xleftarrow{B_1} & R & \xleftarrow{0} & R \\
0\downarrow & & \pi\downarrow & & 0\downarrow & & p\downarrow & & \\
R^2 & \xleftarrow{0} & R^2 & \xleftarrow{B_1} & R & \xleftarrow{0} & R & & \\
\pi\downarrow & & 0\downarrow & & p\downarrow & & & & \\
R^2 & \xleftarrow{B_1} & R & \xleftarrow{0} & R & & & & \\
0\downarrow & & p\downarrow & & & & & & \\
R & \xleftarrow{0} & R & & & & & & \\
p\downarrow & & & & & & & & \\
R, & & & & & & & &
\end{array}
$$

where $\pi = \left(\begin{smallmatrix} p & 2 \\ 0 & p \end{smallmatrix}\right)$ and $B_k = \left(\begin{smallmatrix} 0 & 0 \\ k & 0 \end{smallmatrix}\right)$ for odd $k > 1$, and $B_1 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$.

From this, we can eliminate an acyclic subcomplex: Consider one of the $R^2$-entries in an odd total degree. This has generators $a_n = x^n$ and $b_n = x^{n-2} \otimes x$. Fix now $n$ odd, and define a new basis in degree $n$ as

$$\alpha_n = \frac{p}{2} b_n - a_n, \qquad \beta_n = \frac{1}{2} b_n.$$

In the even degree $n-1$, we define

$$\alpha_{n-1} = a_{n-1} + \frac{p}{2} b_{n-1}, \qquad \beta_{n-1} = \frac{1}{2} b_{n-1}.$$

Then the vertical differential becomes

$$\begin{aligned}
\pi(\alpha_n) &= \frac{p}{2}\pi(b_n) - \pi(a_n) = \frac{p}{2}(2a_{n-1} + pb_{n-1}) - pa_{n-1} = p^2 \beta_{n-1}, \\
\pi(\beta_n) &= \frac{1}{2}(2a_{n-1} + pb_{n-1}) = \alpha_{n-1}.
\end{aligned}$$

The horizontal differential is

$$\begin{aligned}
B_n(\alpha_n) &= \frac{p}{2} B_n(b_n) - B_n(a_n) = -nb_{n-1} = -2n\beta_{n-1}, \\
B_n(\beta_n) &= 0.
\end{aligned}$$

In even total degree, all differentials vanish. Therefore all odd-degree $\beta_n$ and even-degree $\alpha_{n-1}$ split off as an acyclic subcomplex, and the remaining generators are mapped as described in the proposition, up to a multiplication by the unit $-2$. □

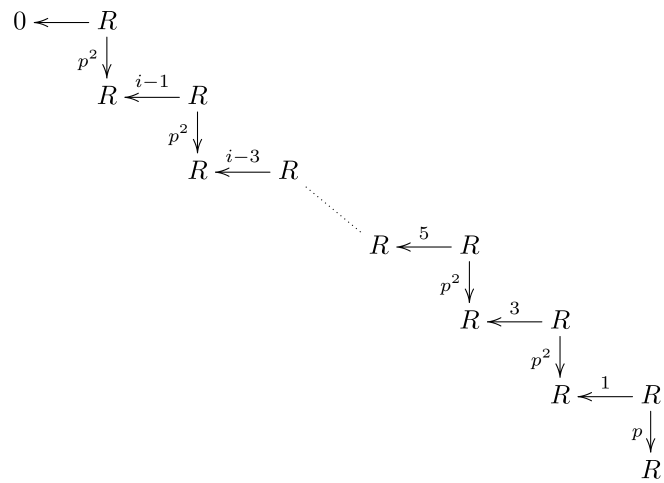
This allows for a more explicit description of the cyclic homology, periodic homology, and negative cyclic homology of $R/\!\!/p$ over $R$: Note first that for all three bicomplexes, the total differentials from even to odd total degrees are 0. If $R$ has no $n$-torsion for all positive $n$, the total differentials from odd to even degrees are injective. Hence we only need to compute the cokernels of the latter differentials. Explicitly, we have the following:

**Proposition 2.8.** *Let $p$ be odd, $R$ without $n$-torsion for all positive $n$, and $\frac{1}{2} \in R$. Then $\mathrm{HC}_0(R/\!\!/p) = R/p$, $\mathrm{HC}_i(R/\!\!/p) = 0$ for odd $i$, and for even positive $i$, $\mathrm{HC}_i(R/\!\!/p)$ is the cokernel of the map*

$$R^{\frac{i}{2}+1} \to R^{\frac{i}{2}+1}$$

*sending $(x_1, x_2, \ldots, x_{\frac{i}{2}+1})$ to $(px_1, x_1 + p^2 x_2, 3x_2 + p^2 x_3, \ldots, (i-1)x_{\frac{i}{2}} + p^2 x_{\frac{i}{2}+1})$.*

*This can be identified with the colimit of the following diagram of $R$-modules:*



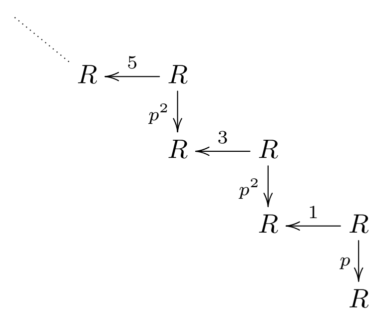

In the periodic situation, this diagram just continues to the left. However, there is a subtlety: Periodic and negative cyclic homology are computed by taking the product totalization of the respective bicomplexes [9, 5.1.2].

**Proposition 2.9.** *Let $p$ be odd, $R$ without $n$-torsion for all positive $n$, and $\frac{1}{2} \in R$. Then $\mathrm{HP}_i(R/\!\!/p) = 0$ for odd $i$, and for even $i$, $\mathrm{HP}_i(R/\!\!/p)$ is the cokernel of the map*

$$\prod_{\mathbb{N}} R \to \prod_{\mathbb{N}} R$$

*sending $(x_1, x_2, \dots)$ to $(px_1, x_1 + p^2x_2, 3x_2 + p^2x_3, 5x_3 + p^2x_4, \dots)$.*

*This can be identified with a completion of the colimit of the following diagram of $R$-modules (see* [9, 5.1.9]*):*




For negative cyclic homology, the periodic diagram is truncated at the other end:

**Proposition 2.10.** *Let $p$ be odd, $R$ without $n$-torsion for all positive $n$, and $\frac{1}{2} \in R$. Then $\mathrm{HC}^-_i(R/\!\!/p) = \mathrm{HP}_0(R/\!\!/p)$, for $i \leq 0$, and $\mathrm{HC}^-_i(R/\!\!/p) = 0$ for odd $i$, and for even positive $i$, $\mathrm{HC}^-_i(R/\!\!/p)$ is the cokernel of the map*

$$\prod_{\mathbb{N}} R \to \prod_{\mathbb{N}} R$$

*sending $(x_1, x_2, \dots)$ to $(p^2x_1, (i+1)x_1 + p^2x_2, (i+3)x_2 + p^2x_3, (i+5)x_3 + p^2x_4, \dots)$.*

*This can be identified with a completion of the colimit of the following diagram of $R$-modules (see* [9, 5.1.9]*):*

$$\begin{array}{cccccccc}
\ddots & & & & & & & \\
& R & \xleftarrow{i+5} & R & & & & \\
& & & \downarrow{\scriptstyle p^2} & & & & \\
& & & R & \xleftarrow{i+3} & R & & \\
& & & & & \downarrow{\scriptstyle p^2} & & \\
& & & & & R & \xleftarrow{i+1} & R \\
& & & & & & & \downarrow{\scriptstyle p^2} \\
& & & & & & & R
\end{array}$$

The rest of this paper is entirely devoted to identifying these cokernels.

For readability, we will set up some notation for the staircase diagrams and homology groups:

Let $N_{-1} = \mathrm{HC}_0(R /\!\!/ p) = R/p$ and for each odd integer $i \geq 1$, and $N_i = \mathrm{HC}_{i+1}(R /\!\!/ p)$. By 2.8, $N_i$ is the colimit of the following diagram of $R$-modules:

$$\begin{array}{cccccccccccc}
0 & \longleftarrow & R & & & & & & & & & \\
& & \downarrow{\scriptstyle p^2} & & & & & & & & & \\
& & R & \xleftarrow{i} & R & & & & & & & \\
& & & & \downarrow{\scriptstyle p^2} & & & & & & & \\
& & & & R & \xleftarrow{i-2} & R & & & & & \\
& & & & & & & \ddots & & & & \\
& & & & & & & & R & \xleftarrow{5} & R & \\
& & & & & & & & & & \downarrow{\scriptstyle p^2} & \\
& & & & & & & & & & R & \xleftarrow{3} R \\
& & & & & & & & & & & \downarrow{\scriptstyle p^2} \\
& & & & & & & & & & & R \xleftarrow{1} R \\
& & & & & & & & & & & \qquad\ \downarrow{\scriptstyle p} \\
& & & & & & & & & & & \qquad\ R
\end{array}$$

So that we can refer to the individual copies of $R$ in this diagram more easily, we will use the following labels:

$$\begin{array}{cccccccccccc}
0 & \longleftarrow & R'_{i+2} & & & & & & & & & \\
& & \downarrow{\scriptstyle p^2} & & & & & & & & & \\
& & R_i & \xleftarrow{i} & R'_i & & & & & & & \\
& & & & \downarrow{\scriptstyle p^2} & & & & & & & \\
& & & & R_{i-2} & \xleftarrow{i-2} & R'_{i-2} & & & & & \\
& & & & & & & \ddots & & & & \\
& & & & & & & & R_5 & \xleftarrow{5} & R'_5 & \\
& & & & & & & & & & \downarrow{\scriptstyle p^2} & \\
& & & & & & & & & & R_3 & \xleftarrow{3} R'_3 \\
& & & & & & & & & & & \downarrow{\scriptstyle p^2} \\
& & & & & & & & & & & R_1 \xleftarrow{1} R'_1 \\
& & & & & & & & & & & \qquad\ \downarrow{\scriptstyle p} \\
& & & & & & & & & & & \qquad\ R_{-1}
\end{array} \tag{2.11}$$

All of the above terms $R'_1, R'_3, \ldots, R_{-1}, R_1, R_3, \ldots$ are equal to $R$.

We note the natural maps between the $R$-modules $N_i$ and record some of their properties.

**Lemma 2.12.** *Let $k \geq i \geq 1$ be odd integers, and let $N_k$ and $N_i$ be the colimits defined above. There is a natural surjective $R$-module homomorphism $\pi_{k,i} : N_k \to N_i$, and these are compatible in the sense that for $k' \geq k \geq i$, we have $\pi_{k,i} \circ \pi_{k',k} = \pi_{k',i}$. The kernel of $\pi_{k,i}$ is the $R$-submodule of $N_k$ generated by the images of the natural maps $R_j \to N_k$ for all $j > i$.*

*Proof.* There is a clear map $N_k \to N_i$ given by mapping the $R_j$ and $R'_{j+2}$ factors to zero for all $j > i$, and by mapping the remaining factors to themselves via the identity. Using the explicit description of an (unfiltered) colimit given, for example, in [14, Tag 00D5], we have that $N_i$ is a quotient of the direct sum $R_{-1} \oplus R_1 \oplus \cdots \oplus R_k \oplus \cdots \oplus R_i$, and similarly for $N_k$. The constructed map $N_k \to N_i$ is induced by the obvious projection between these direct sums. The rest of the claims follow easily. □

We let $N_\infty$ denote the inverse limit $\varprojlim N_i$. We let $K_i$ denote the kernel of the projection $N_\infty \to N_i$. It is immediate from 2.9 and 2.10 that $N_\infty = \mathrm{HP}_0(R/\!\!/p)$ and $K_i = \mathrm{HC}^-_{i+3}(R/\!\!/p)$ (see [9, 5.1.5 and 5.1.9]). Indeed, this follows since the cyclic homology of $R/\!\!/p$ is even (Proposition 2.8) and hence by [9, 5.1.5], we have a short exact sequence

$$0 \to \mathrm{HC}^-_l(R/\!\!/p) \to HP_l(R/\!\!/p) \to \mathrm{HC}_{l-2}(R/\!\!/p) \to 0$$

for any even integer $l$. Our goal is therefore to compute these terms $N_i, N_\infty, K_i$ as explicitly as possible.

Our explicit description of $N_i$ is quite complicated for general $i$. (We will see that for certain $i$, such as $i = \frac{p^a \pm 1}{2}$, there is a simple description.) Our strategy for calculating $N_i$ explicitly is to first construct a related $R$-module, that we call $M_i$, and that has a much more regular description in general than $N_i$.

## 3. Computation of a related colimit

From now on we will work with $p$-torsion-free commutative $\mathbb{Z}_{(p)}$-algebras. This will make our formulas and results easier to formulate. There is no restriction of generality since for any $p$-torsion-free $R$, the map $R \to R_{(p)}$ induces an isomorphism $\mathrm{HH}^R(R/\!\!/p) \xrightarrow{\cong} \mathrm{HH}^{R_{(p)}}(R_{(p)}/\!\!/p)$ by Corollary 2.6. Proposition 2.7 then implies that corresponding relative negative, periodic and cyclic homologies are isomorphic. For the rest of the paper $R$ will denote a $p$-torsion-free commutative $\mathbb{Z}_{(p)}$-algebra.

For every odd integer $i \geq 1$, we will prove that the $R$-module $M_i := R \oplus R/1 \oplus R/3 \oplus \cdots \oplus R/i$ is a colimit of the diagram

(3.1)
$$\begin{array}{ccccccccccc}
R_i & \xleftarrow{\;i\;} & R'_i & & & & & & & & \\
 & & \big\downarrow{\scriptstyle p^2} & & & & & & & & \\
 & & R_{i-2} & \xleftarrow{\;i-2\;} & R'_{i-2} & & & & & & \\
 & & & & & \ddots & & & & & \\
 & & & & & & R_5 & \xleftarrow{\;5\;} & R'_5 & & \\
 & & & & & & & & \big\downarrow{\scriptstyle p^2} & & \\
 & & & & & & & & R_3 & \xleftarrow{\;3\;} & R'_3 \\
 & & & & & & & & & & \big\downarrow{\scriptstyle p^2} \\
 & & & & & & & & & & R_1 \xleftarrow{\;1\;} R'_1 \\
 & & & & & & & & & & \qquad\quad \big\downarrow{\scriptstyle p} \\
 & & & & & & & & & & \qquad\quad R_{-1}
\end{array}$$

Here, as in diagram (2.11), all these terms $R_i$ and $R'_i$ are equal to $R$. We use the subscripts so we can refer to specific terms more easily. Note that this is the same as the diagram (2.11) defining $N_i$, except that the top row of the $N_i$ diagram has been removed. In particular, we have a surjective map $M_i \twoheadrightarrow N_i$ for every odd $i$.

For the proof that $M_i$ is a colimit of diagram (3.1), we first recursively define two sequences $a_j$ and $b_j$, and prove some inequalities related to them. Some of these results will be needed immediately, in the proof that $M_i$ is a colimit of diagram (3.1), while other of these results will not be needed until later. These sequences will also be used to describe the kernel of the surjection $M_i \to N_i$.

**Definition 3.2.** Define $A_1 := p$ and for each odd integer $j \geq 3$, recursively define $A_j \in \mathbb{Q}$ as $A_j := \frac{p^2 A_{j-2}}{j}$. Define $B_0 := 1$ and for each even integer $j \geq 2$, define $B_j \in \mathbb{Q}$ as $B_j := \frac{p^2 B_{j-2}}{j}$. Define $a_j := v_p(A_j)$ and $b_j := v_p(B_j)$.

We will see below in Corollary 3.6 that the elements $A_j$ and $B_j$ defined above in fact lie in $\mathbb{Z}_{(p)}$

Note that the sequence $b_j$ is not an increasing sequence. We can bound the value of $b_j$ as follows.

**Lemma 3.3.** *For every even integer $j \geq 0$, the number $b_j$ is equal to $j$ minus the $p$-adic valuation of $(j/2)!$. In particular, for every even integer $j \geq 2$, we have*

$$j - \frac{j}{2(p-1)} < b_j.$$

*Proof.* The statement about $(j/2)!$ is clear from the recursive definition of the $b$ sequence. From this and Legendre's formula, we have for all $j \geq 0$ that

$$b_j = j - \nu_p((j/2)!) = j - \sum_{k=1}^{\infty} \left\lfloor \frac{j}{2p^k} \right\rfloor.$$

So if $j > 0$, we have

$$b_j > j - \sum_{k=1}^{\infty} \frac{j}{2p^k} = j - \frac{j}{2(p-1)}.$$

This proves the inequality. □

The following coarse lower-bound on $b_j$ will be very useful.

**Lemma 3.4.** *Assume $p^e \leq \frac{j}{2} + 1$, for an integer $e \geq 0$ and an even integer $j \geq 0$. Then $b_j \geq e$.*

*Proof.* The claim holds if $j = 0$ and $j = 2$, and for $j > 2$, we have $b_j > j - \frac{j}{2(p-1)} \geq \frac{j}{2} + 1 \geq p^e \geq e$. □

**Lemma 3.5.** *For every odd integer $j \geq 1$, we have $a_j = b_{2j} - b_{j-1} - 1$.*

*Proof.* The claim holds when $j = 1$. Now, assuming the result for some fixed value of $j - 2$, we compute (using that $p \neq 2$ several times)

$$\begin{aligned} a_j &= 2 + a_{j-2} - v_p(j) \\ &= 2 + \big(b_{2(j-2)} - b_{j-3} - 1\big) - v_p(j) \\ &= 2 + b_{2(j-2)} - v_p(2(j-1)) + v_p(j-1) - b_{j-3} - 1 - v_p(j) \\ &= b_{2j-2} - b_{j-1} + 1 - v_p(j) \\ &= b_{2j} - b_{j-1} - 1, \end{aligned}$$

which completes the induction. □

**Corollary 3.6.** *We have $b_j \geq 0$ for every even integer $j \geq 0$. In particular, the element $B_j \in \mathbb{Q}$ in fact lies in $\mathbb{Z}_{(p)}$, and so can be viewed as an element of $R$. Similarly, $A_j \in \mathbb{Z}_{(p)}$ for every odd integer $j \geq 1$.*

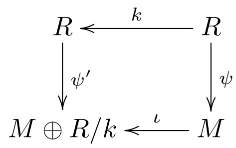
*Proof.* The statement about $b_j$ is clear from the above results. For the numbers $a_j$, the result holds for $j = 1$, and for all values of $j \geq 3$, we can use the bounds

$$a_j = b_{2j} - b_{j-1} - 1 \geq 2j - \frac{2j}{2(p-1)} - (j-1) - 1 = j - \frac{j}{p-1}.$$

This completes the proof. □

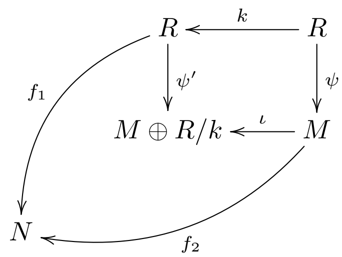
Our next preliminary result will provide a way to determine the colimit of the diagram (3.1) for a fixed value of $i$ in terms of the colimit for $i - 2$.

**Lemma 3.7.** *Fix an integer $k$. Assume $\psi : R \to M$ is an $R$-module homomorphism and that there exists $m_0 \in M$ such that $\psi(1) = km_0$. Let $\iota : M \to M \oplus R/k$ be given by $m \mapsto (m, 0)$ and let $\psi' : R \to M \oplus R/k$ be given by $r \mapsto (rm_0, r)$. Then*




*is a pushout diagram.*

*Proof.* It is clear that the given diagram commutes. Assume we have a commutative diagram

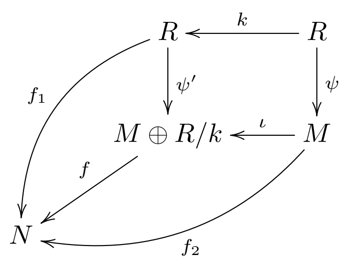



Define a map $f : M \oplus R/k \to N$ by $(m, r) \mapsto f_1(r) + f_2(m - rm_0)$. This map is well-defined because we have

$$f_1(r + sk) + f_2(m - rm_0 - skm_0) = f_1(r) + f_2(m - rm_0) + kf_1(s) - f_2(\psi(s)) = f_1(r) + f_2(m - rm_0).$$

This map is an $R$-module homomorphism.

It follows directly from the definitions that $f_1 = f \circ \psi'$ and $f_2 = f \circ \iota$. Hence the diagram



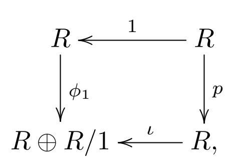

commutes.

Lastly we check that our map $f : M \oplus R/k \to N$ is the unique map making the diagram commute. Let $g$ be any map making the diagram commute. We then must have $g(m, 0) = f_2(m)$ for all $m \in M$, and we must have $g(rm_0, r) = f_1(r)$ for all $r \in R$. Thus we must have $g(m, r) = g(rm_0, r) + g(m - rm_0, 0) = f_1(r) + f_2(m - rm_0) = f(m, r)$ for all $m \in M, r \in R$, as required. □

We now use the above preliminary results to compute the colimit of the diagram (3.1).

**Proposition 3.8.** *For every odd positive integer $i$, set $M_i := R \oplus R/1 \oplus R/3 \oplus \cdots \oplus R/i$. Then $M_i$ is a colimit of the diagram (3.1).*

*Proof.* We prove this using induction on $i$. For the base case $i = 1$, consider the diagram of $R$-modules




with $\phi_1(1) = (p, 0)$ and $\iota(1) = (1, 0)$. In this case we have that $M_1$, with the indicated maps, is a colimit of the diagram (3.1). For later use in the induction, we note also that $\phi_1(1) \in R \oplus R/1$ can be represented by $(A_1, B_0)$, as defined in Definition 3.2.

Now assume the result has been proven for some fixed value of $i-2$, and let $\phi_{i-2}$ denote the corresponding map $R_{i-2} \to M_{i-2}$, where the notation $R_{i-2} = R$ refers to the upper-leftmost factor in diagram (3.1). Further, assume that $\phi_{i-2}(1)$ can be represented by the element $(A_{i-2}, B_{i-2-1}, B_{i-2-3}, \ldots, B_0)$.

Consider the diagram of $R$-modules

$$\begin{array}{ccc} R & \xleftarrow{\;i\;} & R \\ \big\downarrow{\scriptstyle \phi_i} & & \big\downarrow{\scriptstyle p^2\phi_{i-2}} \\ M_{i-2} \oplus R/i & \xleftarrow{\;\iota\;} & M_{i-2}, \end{array}$$

where $\iota(m) = (m, 0)$ and where $\phi_i(1) = (A_i, B_{i-1}, B_{i-3}, \ldots, B_2, B_0)$. By Lemma 3.7, to prove that $M_{i-2} \oplus R/i$ (with the above maps) is a colimit of the diagram

$$\begin{array}{ccc} R & \xleftarrow{\;i\;} & R \\ & & \big\downarrow{\scriptstyle p^2\phi_{i-2}} \\ & & M_{i-2}, \end{array}$$

it suffices to check that $iA_i = p^2 A_{i-2} \in R$ and that for every odd integer $n$ in the interval $1 \leq n \leq i-2$, we have $iB_{i-n} = p^2 B_{i-2-n} \in R/n$. The first equality is clear from the definition of $A_i$. For the second equality, it is clear again from the definition that $(i-n)B_{i-n} = p^2 B_{i-2-n}$, so the desired result follows, since $nB_{i-n} = 0 \in R/n$. □

We record an important aspect of the proof of Proposition 3.8 in the following corollary.

**Corollary 3.9.** *Fix odd integers $i \geq j \geq 1$. Because $R \oplus R/1 \oplus R/3 \oplus \cdots \oplus R/i$ is a colimit of the diagram (3.1), we have a corresponding map of $R$-modules $\phi_{j,i} : R_j \to R \oplus R/1 \oplus R/3 \oplus \cdots \oplus R/i$. This map has the property that $\phi_{j,i}(1)$ can be represented by an element $(c_{j,-1}, c_{j,1}, \ldots, c_{j,i})$, where each $c_{j,n} \in \mathbb{Z}_{(p)}$, and where these elements satisfy the following properties (with the $A$ and $B$ sequences defined as in Definition 3.2):*

*(1) We have $c_{j,-1} = A_j$.*
*(2) If $n$ is an odd integer in the range $j < n \leq i$, then $c_{j,n} = 0$.*
*(3) If $n$ is an odd integer in the range $1 \leq n \leq j$, then $c_{j,n} = B_{j-n} \in R/n$.*

*Proof.* This follows directly from the construction given in the proof of Proposition 3.8. □

**Corollary 3.10.** *Fix an odd positive integer $i$, and let $\phi_{i,i} : R_i \to M_i$ be as in Corollary 3.9. The $R$-module $N_i$ (as defined in Section 2) is isomorphic to the cokernel of $p^2\phi_{i,i} : R_i \to M_i$.*

*Proof.* This follows immediately from comparing the diagrams used to define $M_i$ and $N_i$. □

## 4. Computation of $N_i$ for certain $i$, and the computation of $N_\infty$

Recall again that $R$ denotes a $p$-torsion-free commutative $\mathbb{Z}_{(p)}$-algebra. Our goal in this section is to give an explicit description of $N_i$ for certain values of $i$. Namely, we define an infinite set $Z_1$ in Definition 4.4 below, and compute $N_i$ for all $i \in Z_1$. This will also enable us to compute the inverse limit $\varprojlim N_i =: N_\infty$.

By Corollary 3.10, to compute $N_i$, it is equivalent to compute the cokernel of the map $p^2\phi_{i,i} : R_i \to M_i$, where $\phi_{i,i}$ is defined as in Corollary 3.9. For certain values of $i$, this map $\phi_{i,i}$ turns out to have a particularly simple form. The description of those values of $i$ will involve the following parameter $g(n)$.

**Definition 4.1.** Let $n \geq 1$ denote an integer which is divisible by $p$. We define $g(n)$ to be the maximum even integer $j$ such that $b_j < v_p(n)$.

The motivation for the definition of $g(n)$ is the following. We will show that the map $\phi_{i,i} : R_i \to R \oplus R/1 \oplus R/3 \oplus \cdots \oplus R/i$ is non-zero in the $R/n$-factor only if $i$ is in the interval $[n, n + g(n)]$.

*Example* 4.2. We have $g(p) = g(p^2) = 0$, $g(p^3) = g(p^4) = 2$, $g(p^5) = 4$, and $g(3^6) = 6$. For example, if $\phi_{i,i}$ is non-zero in the $R/p^4$-factor, we will see below that the only possible values for $i$ are $i = p^4$ and $i = p^4 + 2$.

Let $c_{i,n}$ be defined as in Corollary 3.9, and let $b_j$ be defined as in Definition 3.2. The following result follows immediately from Corollary 3.9 and the definitions.

**Lemma 4.3.** *Let $n$ be an odd positive integer. Assume $c_{j,n} \neq 0 \in R/n$ for some $j > n$ (so in particular, $v_p(n) \geq 1$). Then $j - n \leq g(n)$.*

*Proof.* We know $v_p(c_{j,n}) = b_{j-n}$ by Corollary 3.9. Since we are assuming that $c_{j,n} \neq 0 \in R/n$, we must have $b_{j-n} < v_p(n)$. So $g(n) \geq j - n$ by the definition of $g(n)$, as claimed. □

**Definition 4.4.** Let $X$ denote the set of all odd positive integers, and let $X_1$ denote the set of all odd positive integers which are divisible by $p$. Define $Z_1$ to be the set

$$Z_1 := X \setminus \bigcup_{n \in X_1} [n, n + g(n)].$$

For all $i \in Z_1$, the map $\phi_{i,i} : R \to M_i$ has a particularly simple form.

**Lemma 4.5.** *Assume $i \in Z_1$. Then the map $\phi_{i,i} : R_i \to M_i = R \oplus R/1 \oplus R/3 \oplus \cdots \oplus R/i$ has the property that $\phi_{i,i}(1) = (A_i, 0, 0, \ldots, 0)$, where $A_i$ is defined as in Definition 3.2. In particular, for each such $i$ we have an isomorphism of $R$-modules*

$$N_i \cong R/p^{a_i+2} \oplus R/1 \oplus R/3 \oplus \cdots \oplus R/(i-2) \oplus R/i.$$

*Proof.* This follows immediately from Lemma 4.3 and the definitions. □

**Corollary 4.6.** *Assume $i \in Z_1$ and let $k > i$ be an odd integer. Then the projection map $N_k \to N_i$ from Lemma 2.12 is induced by a map $M_k \to N_i$,*

$$R \oplus R/1 \oplus R/3 \oplus \cdots \oplus R/i \oplus \cdots \oplus R/k \to R/p^{a_i+2} \oplus R/1 \oplus R/3 \oplus \cdots \oplus R/i,$$

*of the form*

$$(r_{-1}, r_1, r_3, \ldots, r_i, \cdots, r_k) \mapsto (r_{-1} + C, r_1, r_3, \ldots, r_i),$$

*where $C$ is divisible by $p^a$, for $a = \min_{m \geq i+2} a_m$ and and it only depends on $(r_{i+2}, \ldots, r_k)$.*

*Proof.* Using the construction and notation from the proof of Lemma 2.12, the projection map $N_k \to N_i$ fits into a commutative diagram

$$\begin{array}{ccc}
R_{-1} \oplus R_1 \oplus \cdots \oplus R_i \oplus \cdots \oplus R_k & \longrightarrow & R_{-1} \oplus R_1 \oplus \cdots \oplus R_i \\
\downarrow{\scriptstyle \phi} & & \\
M_k = R \oplus R/1 \oplus R/3 \oplus \cdots \oplus R/i \oplus \cdots \oplus R/k & & \downarrow \\
\downarrow & & \\
N_k & \longrightarrow & N_i,
\end{array}$$

where all of the $R_j$ terms are equal to $R$. In this diagram, the top horizontal map is the obvious projection.

Consider first an element of the form

$$(r_{-1}, r_1, r_3, \ldots, r_i, 0, \cdots, 0) \in R \oplus R/1 \oplus R/3 \oplus \cdots \oplus R/i \oplus \cdots \oplus R/k \cong M_k.$$

Choose elements $s_{-1}, s_1, \ldots, s_i \in R$ so that the element

$$(s_{-1}, s_1, \ldots, s_i, 0, \ldots, 0) \in R_{-1} \oplus R_1 \oplus \cdots \oplus R_i \oplus \cdots \oplus R_k$$

maps under $\phi$ to $(r_{-1}, r_1, r_3, \ldots, r_i, 0, \cdots, 0) \in M_k$. This is possible from the construction of $\phi$ (Corollary 3.9), since we can choose

$$(s_{-1}, s_1, \ldots, s_i) \in R_{-1} \oplus R_1 \oplus \cdots \oplus R_i$$

mapping to $(r_{-1}, r_1, r_3, \ldots, r_i) \in M_i$, and hence also to an element with the same representative in $N_i$. Thus our claimed result is true for elements of the form $(r_{-1}, r_1, r_3, \ldots, r_i, 0, \cdots, 0) \in M_k$. (In fact, for these special elements, a stronger result is true, because we have showed that in this case, we have $C = 0$.)

Because the map $M_k \to N_i$ is an $R$-module map, it in particular is additive, so it suffices now to consider the "complementary" elements, i.e., those of the form

$$(0, \ldots, 0, r_{i+2}, \cdots, r_k) \in R \oplus R/1 \oplus R/3 \oplus \cdots \oplus R/i \oplus \cdots \oplus R/k \cong M_k.$$

We will show that this element is the image under $\phi$ of an element

$$(C, 0, \dots, 0, s_{i+2}, \dots, s_k) \in R_{-1} \oplus R_1 \oplus \cdots \oplus R_i \oplus \cdots R_k,$$

where $C$ is as in the statement. Under the top horizontal map, this element $(C, 0, \dots, 0, s_{i+2}, \dots, s_k)$ maps to $(C, 0, \dots, 0)$, and this will complete the proof.

We now carry out the construction just described. Let $(0, \dots, 0, r_{i+2}, \cdots, r_k) \in M_k$, and choose any pre-image under $\phi$, denoted $(y_{-1}, y_1, \dots, y_i, y_{i+2}, \dots, y_k)$. Because $i \in Z_1$, by Corollary 3.9 and Lemma 4.3, we know $c_{t,l} = 0$ for $1 \le l \le i$ and $i \le t$. Hence the image of $(0, \dots, 0, y_{i+2}, \dots, y_k)$ under $\phi$ must be $(-C, 0, \dots, 0, r_{i+2}, \cdots, r_k)$, for some $C \in R$ that is divisible by $p^a$, for $a = \min_{m \ge i+2} a_m$, and hence $(0, \dots, 0, r_{i+2}, \cdots, r_k) \in M_k$ is the image under $\phi$ of $(C, 0, \dots, 0, y_{i+2}, \dots, y_k)$. Under the top horizontal map in our diagram, this element maps to $(C, 0, \dots, 0)$, which in turn maps to an element in $N_i$ represented by $(C, 0, \dots, 0)$. This completes the proof. $\square$

Because of the $C$ terms in the above formula, the maps $N_j \to N_i$ for $i, j \in Z_1$ typically do not correspond to the obvious projections. (For example, the $R/(i+2)$-component will typically contribute something nonzero under the projection.) This is relevant, as our next goal is to compute the inverse limit $\varprojlim_k N_k$ by restricting to $\varprojlim_{i \in Z_1} N_i$, where these maps turn up.

To see that $Z_1 \subset \mathbb{N}$ is indeed cofinal, we first need the following:

**Lemma 4.7.** *For any $a > 0$, $g(p^a) < \frac{p^a - 1}{2}$.*

*Proof.* For $a = 1$ and 2, this is immediate since there $g(p^a) = 0$.

From Lemma 3.3, we have the linear lower bound

$$b_j > \left( \frac{2p-3}{2p-2} \right) j.$$

This yields an upper bound on $g(p^a)$:

$$g(p^a) = \max\{j : b_j < a\} \le \max \left\{ j : \left( \frac{2p-3}{2p-2} \right) j < a \right\} < \left( \frac{2p-2}{2p-3} \right) a.$$

After substituting this and rearranging the inequality, it therefore suffices to show

$$p^a - \frac{4p-4}{2p-3} a > 1.$$

Note that $\frac{4p-4}{2p-3}$ is strictly decreasing in $p$ for $p > 2$, and $p^a$ is strictly increasing in $p$ (assuming $a > 1$)–hence it suffices to show this for $p = 3$. That leaves us with

$$3^a - \frac{8}{3} a > 1,$$

which is true by elementary methods. (For example, it holds for $a = 2$, and the left hand side is strictly increasing for $a \ge 2$.) $\square$

**Corollary 4.8.** *For every odd $n$ that is divisible by $p$, $|\frac{p^a \pm 1}{2} - n| > g(n)$. In particular, all odd numbers of the form $\frac{p^a \pm 1}{2}$ are contained in $Z_1$, which therefore contains arbitrarily large numbers.*

*Proof.* By the previous lemma, we have

$$g(n) = g(p^{\nu_p(n)}) < \frac{p^{\nu_p(n)} - 1}{2},$$

therefore it suffices to show that

$$\left| \frac{p^a \pm 1}{2} - n \right| \ge \frac{p^{\nu_p(n)} - 1}{2}.$$

In the case that $\nu_p(n) \ge a$, we have

$$n - \frac{p^a \pm 1}{2} \ge p^{\nu_p(n)} - \frac{p^{\nu_p(n)} \pm 1}{2} \ge \frac{p^{\nu_p(n)} - 1}{2}.$$

Otherwise, we write $a = \tilde{a} + \nu_p(n)$, $n = mp^{\nu_p(n)}$, and observe

$$\left|\frac{p^a \pm 1}{2} - n\right| = \left|\frac{(p^{\tilde{a}} - 2m)p^{\nu_p(n)} \pm 1}{2}\right| \geq \left|\frac{p^{\nu_p(n)} \pm 1}{2}\right| \geq \frac{p^{\nu_p(n)} - 1}{2},$$

where the first inequality uses that $p$ is odd, hence $(p^{\tilde{a}} - 2m)$ is nonzero. □

**Proposition 4.9.** *The inverse limit $N_\infty := \varprojlim_k N_k$ is isomorphic to*

$$R^\wedge \times R/1 \times R/3 \times R/5 \times \cdots,$$

*where $R^\wedge$ denotes the $p$-adic completion of $R$.*

*Proof.* We consider the inverse limit of the final system $N_i$ for $i \in Z_1$. Define a map

$$\varprojlim_{i \in Z_1} N_i \to R^\wedge \times R/1 \times R/3 \times R/5 \times \cdots$$

as follows. Given a compatible sequence in $x = (x_i)_{i \in Z_1} \in \varprojlim_{i \in Z_1} N_i$, define its image to be $(y_{-1}, y_1, y_3, \ldots) \in R^\wedge \times R/1 \times R/3 \times \cdots$, where the coordinates $y_n$ are defined as follows. To define $y_n$ with $n \neq -1$, choose any $k \in Z_1$ such that $k \geq n$, and then write $x_k \in N_k$ as $(x_{k,-1}, x_{k,1}, \ldots, x_{k,k})$, and set $y_n := x_{k,n}$. It follows from Corollary 4.6 that this element $y_n$ is independent of the choice of $k$.

Defining $y_{-1}$ is more difficult, because of the elements denoted by $C$ in Corollary 4.6. Fix $i \in Z_1$ and let $a = \min_{m \geq i+2} a_m$. Then in the notation of the previous paragraph, we have that $x_{k,-1} \equiv x_{i,-1} \bmod p^a$ for all $k \in Z_1$ such that $k \geq i$. We also have that these values $\min_{m \geq i+2} a_m$ approach infinity as $i$ approaches infinity. In this way, we form a $p$-adic Cauchy sequence, and we define $y_{-1}$ to be the limit of this sequence in $R^\wedge$.

It is clear that the given map $\varprojlim_{i \in Z_1} N_i \to R^\wedge \times R/1 \times R/3 \times R/5 \times \cdots$ is an $R$-module homomorphism. To see that it is injective, consider a sequence $(x_i)$ mapping to 0. By the construction, we have that $x_{i,k} = 0$ for all $i$ and all $k \neq -1$. But now in this case, with all other terms being 0, the $C$ term from Corollary 4.6 is also equal to 0, so the terms $x_{i,-1}$ must also equal 0, for all $i \in Z_1$.

For surjectivity, consider any element $(y_{-1}, y_1, y_3, \ldots) \in R^\wedge \times R/1 \times R/3 \times \cdots$. Note first that some element of the form $(*, y_1, y_3, \ldots) \in R^\wedge \times R/1 \times R/3 \times \cdots$ is in the image. (All components except for the $x_{i,-1}$-components are determined, and the $x_{i,-1}$ components can be calculated inductively, one at a time, for $i \in Z_1$.) Also, for any element $r' \in R^\wedge$, the element $(r', 0, 0, \ldots)$ is in the image. Surjectivity now follows from the fact that the map is additive. □

*Example* 4.10. When $p = 3$, the set $Z_1$ from Definition 4.4 contains the following elements: 5, 7, 11, 13, 17, 19, 23, 25, 31, 35, 37, 41, 43, 47, 49, 53, 55, 59, 61, 65, 67, 71, 73, 77, 79, 85, 89, 91, 95, 97, 101, 103, 107, 109, 113, 115, 119, 121, 125, 127, 131, 133, 139, 143, 145, 149, 151, 155, 157, 161, 163, 167, 169, 173, 175, 179, 181, 185, 187, 193, 197, 199.

In other words, in addition to multiples of 3, it only excludes the elements $29, 83, 137, 191$ from the odd numbers up to 200.

For any odd integers $i \geq j \geq 1$, the map $\phi_{j,i} : R_j \to M_i$ considered in Corollary 3.9 induces a map $\psi_{j,i} : R_j \to N_i$. When $i \in Z_1$, then by Lemma 4.5, we have $\psi_{j,i}(1) = (c_{j,-1}, c_{j,1}, \ldots, c_{j,i})$, where the values $c_{j,k}$ are as in Corollary 3.9, with the only difference being that the initial component $c_{j,-1}$ here is considered as an element of $R/p^{a_i+2}$, rather than as an element of $R$. These maps $\psi_{j,i}$ can also be used to define a map to the inverse limit, $R_j \to N_\infty$, as in the following corollary.

**Corollary 4.11.** *For any odd integer $j \geq 1$, the maps $\psi_{j,i}$, for varying $i \geq j$ with $i \in Z_1$, determine a map $\psi_j : R_j \to N_\infty$, such that $\psi_j(1) = (c_{j,-1}, c_{j,1}, c_{j,3}, \ldots)$. These values $c_{j,k}$ are the same as described in Corollary 3.9.*

*Proof.* This follows from the preceding comments, together with Corollary 4.6 and the construction in the proof of Proposition 4.9. □

## 5. Computation of $K_i$ for certain $i$

Recall from Section 2 that $K_i$ is defined to be the kernel of the projection map

$$N_\infty = \varprojlim N_j \to N_i.$$

In this section, we compute this kernel for certain values of $i$ in a certain set $Z_2$; this set $Z_2$ is a proper subset of the set $Z_1$ considered above.

**Definition 5.1.** Let $X$ denote the set of all odd positive integers, and let $X_1$ denote the set of all odd positive integers which are divisible by $p$. Let $g(n)$ be as in Definition 4.1. Define $Z_2$ to be the set

$$Z_2 := X \setminus \bigcup_{n \in X_1} [n - g(n), n + g(n)].$$

The maps $\psi_j : R_j \to N_\infty$ from Corollary 4.11 can be used to describe the $R$-modules $K_m$, as in the following lemma. It turns out the value of $m+2$ is more important in the eventual computation than $m$ itself, so that is why we phrase this lemma in terms of $K_{i-2}$.

**Lemma 5.2.** *For any odd integer $i > 1$, the kernel $K_{i-2}$ of the natural projection $N_\infty \to N_{i-2}$ is the closure (in the inverse limit topology) of the $R$-submodule of $N_\infty$ generated by $\psi_j(1)$ for all $j \geq i$.*

*Proof.* This follows immediately from the definitions and the computations in Lemma 2.12. □

We are interested in identifying values of $i$ for which we can provide an explicit description of the submodule of $N_\infty$ described in Lemma 5.2. The elements

$$\psi_i(1), \psi_{i+2}(1), \psi_{i+4}(1), \ldots \in N_\infty \cong R^\wedge \times R/1 \times R/3 \times \ldots$$

have first components in $R^\wedge$ with $p$-adic valuations $a_i, a_{i+2}, a_{i+4}, \ldots$, respectively, where $a_m$ is defined in Definition 3.2. It is convenient to know when the initial value $a_i$ is the minimum of the set $\{a_i, a_{i+2}, a_{i+4}, \ldots\}$. The following lemma (in which the parameters $g(n)$ from Definition 4.1 again appear) provides cases where this value $a_i$ is indeed the minimum.

**Lemma 5.3.** *Let $i$ be an odd integer which is not divisible by $p$. Let $a_j$ and $b_j$ be defined as in Definition 3.2. Assume there exists an odd integer $n$ such that $i < n$ and $a_i > a_n$. Then there exists such an $n$ with $v_p(n) \geq 3$ and $b_{n-i} < v_p(n)$. In particular, $n - i \leq g(n)$.*

*Proof.* Choose a minimal $n$ such that $i < n$ and $a_i > a_n$. In particular, we have $a_{n-2} > a_n$, so $v_p(n) \geq 3$.

There cannot exist odd $j$ such that $i < j < n$ and $v_p(j) \geq v_p(n)$. Indeed, assume towards contradiction that there exists such a $j$. Let $e = \max\{v_p(k) \colon i < k < n, \text{with } k \text{ odd}\}$ and let $j = \max\{k \colon i < k < n, \text{ with } k \text{ odd and } v_p(k) = e\}$. We will show that $a_j \leq a_n$, which contradicts the minimality of $n$.

For $j$ and $n$ as defined above, by the inductive definition of the $a_k$ sequence, we have

$$a_n - a_j = (2 - v_p(j+2)) + (2 - v_p(j+4)) + (2 - v_p(j+6)) + \cdots + (2 - v_p(j + n - j)).$$

Observe that $j + 2 < n$. Indeed, if $j + 2 = n$, then $\nu_p(2) = \nu_p(n-j) \geq \nu_p(n) \geq 3$ which is a contradiction. Next, by the definition of $j$, we know that $v_p(j) > v_p(k)$ for all odd $j + 2 \leq k < n$. This implies that $v_p(j+2) = v_p(2)$, $v_p(j+4) = v_p(4), \ldots, v_p(n-2) = v_p(n-j-2)$. We also have $v_p(n) \leq v_p(n-j)$. In total, we have

$$a_n - a_j \geq (2 - v_p(2)) + (2 - v_p(4)) + (2 - v_p(6)) + \cdots + (2 - v_p(n-j)).$$

This latter expression is equal to $b_{n-j}$, and we have already seen that all terms in the $b_k$ sequence are non-negative. Thus $a_n \geq a_j$, which is a contradiction. We conclude that for all odd $j$ in the range $i < j < n$, we have $v_p(j) < v_p(n)$.

Write $\sum'$ to deduce the sum over the *even* values in the specified interval. Because $a_i > a_n$, we deduce that

$$n - i - \sum_{k=2}^{n-i}{}' v_p(i+k) < 0.$$

Rewriting,

$$n - i - \sum_{k=2}^{n-i-2}{}' v_p(n-k) < v_p(n).$$

By our comments above, we have $v_p(n-k) = v_p(k)$ for all $k$ in the above range, so we have the inequality

$$n - i - \sum_{k=2}^{n-i-2}{}' \, v_p(k) < v_p(n).$$

Because $i$ is not divisible by $p$, we also have that $n-i$ is not divisible by $p$, so the above inequality is the same as

$$b_{n-i} < v_p(n).$$

The result follows. ☐

**Proposition 5.4.** *For any odd $l \geq 1$, let $e_l \in N_\infty \cong R^\wedge \times R/1 \times R/3 \times \cdots$ be the element which is 1 in the $R/l$ component and which is zero in all other coordinates. Also write $e_{-1}$ for the element $(1, 0, 0, \ldots)$. Let $Z_2$ be as in Definition 5.1 and assume $i \in Z_2$. Let $A_i \in R$ be defined as in Definition 3.2. Then the $R$-submodule of $N_\infty$ generated by*

$$\psi_i(1), \psi_{i+2}(1), \psi_{i+4}(1), \ldots$$

*is equal to the $R$-submodule generated by*

$$A_i e_{-1}, e_i, e_{i+2}, e_{i+4}, \ldots.$$

*Proof.* We show using induction that for any fixed even $j \geq 0$, the $R$-submodule of $N_\infty$ generated by

$$\psi_i(1), \psi_{i+2}(1), \psi_{i+4}(1), \ldots, \psi_{i+j}(1)$$

is equal to the $R$-submodule generated by

$$A_i e_{-1}, e_i, e_{i+2}, e_{i+4}, \ldots, e_{i+j}.$$

The claim of the proposition then follows immediately.

For the base case $j = 0$, note that $\psi_i(1) = A_i e_{-1}$ (by Lemma 4.5, because $Z_2 \subseteq Z_1$) and that $e_i = 0 \in R/i$ (because $i \in Z_2$, and hence $i$ is not divisible by $p$).

Now assume the result has been shown for $j-2$. We write

$$\psi_{i+j}(1) = A_{i+j}e_{-1} + c_{i+j,1}e_1 + c_{i+j,3}e_3 + \cdots + c_{i+j,i+j-2}e_{i+j-2} + e_{i+j}.$$

We must have $c_{i+j,n} = 0$ for all $-1 < n < i$. (If not, so $c_{i+j,n} \neq 0$ for some $-1 < n < i$, then by Lemma 4.3, we have $i + j - n \leq g(n)$, but then $n < i \leq n + g(n)$, which contradicts our assumption that $i \in Z_2$.) So we have

$$\psi_{i+j}(1) = A_{i+j}e_{-1} + c_{i+j,i}e_i + c_{i+j,i+2}e_{i+2} + \cdots + c_{i+j,i+j-2}e_{i+j-2} + e_{i+j}.$$

We have that $A_{i+j}$ is a multiple of $A_i$ by Lemma 5.3, so $\psi_{i+j}(1)$ is in the $R$-submodule of $N_\infty$ generated by

$$A_i e_{-1}, e_i, e_{i+2}, e_{i+4}, \ldots, e_{i+j}.$$

Conversely, by the induction hypothesis, we have that $A_i e_{-1}, e_i, \ldots, e_{i+j-2}$ are in the $R$-submodule generated by

$$\psi_i(1), \psi_{i+2}(1), \psi_{i+4}(1), \ldots, \psi_{i+j-2}(1),$$

and again note that by Lemma 5.3, the number $A_{i+j}$ is a multiple of $A_i$. Therefore

$$e_{i+j} = \psi_{i+j}(1) - A_{i+j}e_{-1} - c_{i+j,i}e_i - c_{i+j,i+2}e_{i+2} - \cdots - c_{i+j,i+j-2}e_{i+j-2}$$

is in the $R$-submodule generated by

$$\psi_i(1), \psi_{i+2}(1), \psi_{i+4}(1), \ldots, \psi_{i+j}(1).$$

This completes the induction. ☐

**Proposition 5.5.** *Let $i > 1$ be an odd integer, and assume $i \in Z_2$, where $Z_2$ is the set defined in Definition 5.1. Then $K_{i-2}$ is isomorphic as an $R$-module to*

$$R^\wedge \times R/(i) \times R/(i+2) \times R/(i+4) \times \cdots.$$

*Proof.* It's clear that the closure in $N_\infty$ of the $R$-submodule generated by $A_i e_{-1}, e_i, e_{i+2}, \ldots$ is isomorphic to the given module, so the claim follows from Proposition 5.4. ☐

**Corollary 5.6.** *Let $i > 1$ be an odd integer, and assume $i \in Z_2$, where $Z_2$ is the set defined in Definition 5.1. Then $N_{i-2}$ is isomorphic as an $R$-module to*

$$R/p^{a_i} \oplus R/1 \oplus R/3 \oplus \cdots \oplus R/(i-2).$$

As a reality check we point out that this matches with Lemma 4.5 and the first short exact sequence on page 4.

*Example* 5.7. When $p = 3$, the set $Z_2$ from Definition 5.1 contains the following elements: 5, 7, 11, 13, 17, 19, 23, 31, 35, 37, 41, 43, 47, 49, 53, 55, 59, 61, 65, 67, 71, 73, 77, 85, 89, 91, 95, 97, 101, 103, 107, 109, 113, 115, 119, 121, 125, 127, 131, 139, 143, 145, 149, 151, 155, 157, 161, 163, 167, 169, 173, 175, 179, 181, 185, 193, 197, 199.

In other words, in addition to multiples of 3, it only excludes the elements 25, 29, 79, 83, 133, 137, 187, 191.

*Remark* 5.8. It follows from Corollary 4.8 that all odd numbers of the form $\frac{p^a \pm 1}{2}$ for $a > 0$ are contained in $Z_2$.

## 6. The size of the sets $Z_1$ and $Z_2$

We give lower-bounds on the proportion of positive odd integers in the sets $Z_1$ and $Z_2$ from Definition 4.4 and Definition 5.1.

The following lemma should be intuitively clear. It roughly says that $2/p^e$ of odd integers are of the form $cp^e \pm b$, where $b$ is some fixed constant. The point of the lemma is to give a precise version of this claim.

**Lemma 6.1.** *Let $X$ denote the set of all odd positive integers, and for any positive number $N$, let $X_N := X \cap [0, N]$. Fix a positive integer $e$, and an even positive integer $b$, where $b < \frac{p^e}{2}$. Let*

$$Y_{e,N} := \{x \in X_N : x = cp^e \pm b, \text{ some } c \in \mathbb{Z}\}.$$

*We have*

$$\frac{\#Y_{e,N}}{\#X_N} < \frac{2}{p^e} + \frac{2}{\#X_N}.$$

*Proof.* Let $Y_{e,N,-} := \{x \in X_N : x = cp^e - b, \text{ some } c \in \mathbb{Z}\}$. This set contains approximately half of the elements of $Y_{e,N}$, and it clearly suffices to prove that

$$\#Y_{e,N,-} < \frac{\#X_N}{p^e} + 1.$$

To ease notation, write $X$ instead of $X_N$ and write $Y$ for the set obtained from $Y_{e,N,-}$ obtained by removing its minimal element $p^e - b$ (we are ignoring the trivial case that $Y_{e,N,-}$ is the empty set). In terms of this new notation, we must prove that

$$\#Y < \frac{\#X}{p^e}.$$

Consider the following translates of $Y$:

$$Y, Y-2, Y-4, \ldots, Y-2p^e+2.$$

We have listed $p^e$ of these sets, and they are pairwise disjoint. Their union is strictly contained in $X$ (because the minimal element in $Y$ is $3p^e - b$). Because these sets all have cardinality $\#Y$, we deduce that $p^e \cdot \#Y < \#X$, as required. □

**Proposition 6.2.** *Let $X$ denote the set of all odd positive integers. Let $Z_1$ be as in Definition 4.4 and let $Z_2$ be as in Definition 5.1. Fix any positive integer $N$, and write $\lambda := \frac{2p-3}{2p-2}$. We have*

$$\frac{\#(Z_1 \cap [0,N])}{\#(X \cap [0,N])} \geq 1 - \frac{1}{p} - \frac{1}{p^3} - \frac{1}{p^5} - \left( \sum_{k \geq 6 \text{ even}} \frac{1}{p^{\lambda k}} \right) - \frac{\log_p(N) + 1}{\lambda \#(X \cap [0,N])}.$$

*and*

$$\frac{\#(Z_2 \cap [0,N])}{\#(X \cap [0,N])} \geq 1 - \frac{1}{p} - \frac{2}{p^3} - \frac{2}{p^5} - \left( \sum_{k \geq 6 \text{ even}} \frac{2}{p^{\lambda k}} \right) - \frac{2\log_p(N) + 2}{\lambda \#(X \cap [0,N])}.$$

*Proof.* We prove the result for $Z_2$. The argument for $Z_1$ is the same.

To obtain $Z_2$, we can proceed as follows:

(1) Begin with the set $X_N$ of all odd positive integers up to $N$.
(2) Remove all multiples of $p$.
(3) For every even positive integer $k$, let $e_k$ be the minimal positive integer such that $g(p^{e_k}) \geq k$. Remove all elements of the form $cp^{e_k} \pm k$, i.e., the set $Y_{e_k,N}$.

The second step removes at most $\frac{\#X_N}{p}$ many elements. For the third step, we explicitly know that $e_2 = 3$ and $e_4 = 5$. The values $e_j$ for $j \geq 6$ depend on the odd prime $p$, and instead of finding these values exactly, we use the bound from Lemma 4.7. From that lemma's proof, we have

$$e_k > \frac{2p-3}{2p-2}\,k.$$

We can then apply Lemma 6.1 to estimate

$$\frac{Y_{e_k,N}}{\#X_N} < \frac{2}{p^{e_k}} + \frac{2}{\#X_N}.$$

While the first summand assembles into a geometric series, the second one is constant in $X_N$. This requires us to bound the occurrence of $k$ such that $Y_{e_k,N}$ is nonempty. The smallest possible element in $Y_{e_k,N}$ is given by $p^{e_k} - k$, which is bounded below by

$$p^{\frac{2p-3}{2p-2}\,k} - k.$$

We denote $\lambda = \frac{2p-3}{2p-2}$, and claim that for $\lambda k > \log_p(N) + 1$, the inequality $p^{\lambda k} - k > N$ holds as desired: For this, note first that $N = p^{\log_p N + 1 - 1}$, so it suffices to show that

$$p^{\lambda k} - k > p^{\lambda k - 1} \Leftrightarrow (p-1)p^{\lambda k - 1} - k > 0.$$

As $k$ ranges over even positive integers, we first check this manually for $k = 2$:

$$(p-1)p^{\lambda 2 - 1} - 2 = (p-1)p^{\frac{(2p-3)-(p-1)}{p-1}} - 2 = \overbrace{(p-1)}^{\geq 2}\,\overbrace{p^{\frac{p-2}{p-1}}}^{>1} - 2 > 0.$$

Next, note that for $f(k) = (p-1)p^{\lambda k - 1} - k$, we have the derivative $f'(k) = (p-1)\log(p)p^{\lambda k - 1} - 1$. By the same reason as above, this is non-negative for $k \geq 0$, so $f(k) > 0$ for $k \geq 2$.

The upshot is then that we only need to remove the sets $Y_{e_k,N}$ for $k \leq \frac{\log_p(N)+1}{\lambda}$. Putting together our observations, this leaves us with the result:

$$\begin{aligned}
\frac{\#(Z_2 \cap [0,N])}{\#(X \cap [0,N])} &\geq 1 - \frac{1}{p} - \sum_{k \geq 2 \text{ even}} \frac{\#Y_{e_k,N}}{\#X_N} \\
&> 1 - \frac{1}{p} - \frac{2}{p^3} - \frac{2}{p^5} - \left( \sum_{k \geq 6 \text{ even}} \frac{2}{p^{\lambda k}} \right) - \frac{2\log_p(N) + 2}{\lambda \#X_N}.
\end{aligned}$$

□

*Remark* 6.3. The proportion of elements in $Z_1$ and $Z_2$ is larger when the (odd) prime $p$ is larger. For example, when $p = 3$, we have

$$\liminf \frac{\#(Z_1 \cap [0,N])}{\#(X \cap [0,N])} \geq 0.61 \text{ and } \liminf \frac{\#(Z_2 \cap [0,N])}{\#(X \cap [0,N])} \geq 0.58,$$

and when $p = 101$, we have

$$\liminf \frac{\#(Z_1 \cap [0,N])}{\#(X \cap [0,N])} \geq 0.99 \text{ and } \liminf \frac{\#(Z_2 \cap [0,N])}{\#(X \cap [0,N])} \geq 0.99.$$

# References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