WHICH FINITE SIMPLE GROUPS ARE UNIT GROUPS?

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Abstract. We prove that if G is a finite simple group which is the unit group of a ring, then G is isomorphic to either (a) a cyclic group of order 2; (b) a cyclic group of prime order $2^k - 1$ for some k; or (c) a projective special linear group $\mathrm{PSL}_n(\mathbb{F}_2)$ for some $n \geq 3$. Moreover, these groups do all occur as unit groups. We deduce this classification from a more general result, which holds for groups G with no non-trivial normal 2-subgroup.

Throughout this paper, rings will be assumed to be unital, but not necessarily commutative, and ring homomorphisms send 1 to 1. The finite groups G of odd order which occur as unit groups of rings were determined in [3]. We will prove similar results for a more general class of groups; the description of this class of groups uses the following.

Definition 1. *For a finite group G, the p-*core *of G is the largest normal p-subgroup of G. We denote this subgroup by $O_p(G)$. It is the intersection of all Sylow p-subgroups of G.*

We now state the main result. The authors[1] are most grateful to the anonymous referee for our earlier paper [2], who recognized that one of the results proved in that paper could be strengthened into the following.

Theorem 2. *Let G denote a finite group such that $O_2(G) = \{1\}$ and such that G is isomorphic to the unit group of a ring R. Then*

$$G \cong \mathrm{GL}_{n_1}(\mathbb{F}_{2^{k_1}}) \times \cdots \times \mathrm{GL}_{n_r}(\mathbb{F}_{2^{k_r}}).$$

Before proving Theorem 2, we record the following corollary.

Corollary 3. *The finite simple groups which occur as unit groups of rings are precisely the groups*

(a) $\mathbb{Z}/2\mathbb{Z}$,
(b) $\mathbb{Z}/p\mathbb{Z}$ *for a Mersenne prime* $p = 2^k - 1$,
(c) $\mathrm{PSL}_n(\mathbb{F}_2)$ *for* $n \geq 3$.

Proof. If G is a finite simple group, then either $O_2(G) = \{1\}$ or $O_2(G) = G$. If $O_2(G) = G$, then G is a 2-group, and because we are assuming G is simple, we must have $G \cong \mathbb{Z}/2\mathbb{Z}$, which for instance is isomorphic to the unit group of $\mathbb{Z}$.

Hence assume G is a finite simple group which is isomorphic to the unit group of a ring and further assume $O_2(G) = \{1\}$. By Theorem 2, we know

$$G \cong \mathrm{GL}_{n_1}(\mathbb{F}_{2^{k_1}}) \times \cdots \times \mathrm{GL}_{n_r}(\mathbb{F}_{2^{k_r}}).$$

These groups all occur as unit groups of the corresponding products of matrix rings, so we are reduced to determining which of them are simple; this forces

$$G \cong \mathrm{GL}_n(\mathbb{F}_{2^k}).$$

If $n > 1$ and $k > 1$, then the subgroup of invertible scalar matrices forms a nontrivial normal subgroup. Hence two possibilities remain. If $n = 1$, then $\mathrm{GL}_1(\mathbb{F}_{2^k})$ is cyclic of order $2^k - 1$; such a group is simple if and only if its order is prime. If $k = 1$, then $\mathrm{GL}_n(\mathbb{F}_2) = \mathrm{PSL}_n(\mathbb{F}_2)$. For the case $k = 1, n = 2$, we have $\mathrm{PSL}_2(\mathbb{F}_2) \cong S_3$ (see for example [4, Section 3.3.1]); this group is not simple. For the cases $k = 1, n \geq 3$, it is well-known that $\mathrm{PSL}_n(\mathbb{F}_2)$ is simple (see for example [4, Section 3.3.2]). This completes the proof. ∎

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Remark 4. *The simple groups A_8 and $\mathrm{PSL}_2(\mathbb{F}_7)$ also occur as unit groups. This follows immediately from the exceptional isomorphisms*

$$A_8 \cong \mathrm{PSL}_4(\mathbb{F}_2) \text{ and } \mathrm{PSL}_2(\mathbb{F}_7) \cong \mathrm{PSL}_3(\mathbb{F}_2).$$

See for instance [4, Section 3.12].

Having recorded the above consequences of the main result, we now gather the preliminary results used in its proof. We begin with the following observation.

Lemma 5. *Let G denote a finite group with $O_2(G) = \{1\}$, and let R denote a ring with $R^\times \cong G$. Then R has characteristic 2.*

Proof. The elements 1 and -1 are units in R and are in the center of R, hence are in the center of $R^\times$. By the assumption $O_2(G) = \{1\}$, the center of G cannot contain any elements of order 2. Hence $1 = -1$. □

Lemma 6. *Keep notation as in Lemma 5, and fix an isomorphism $R^\times \cong G$. Because R has characteristic 2, we have a natural map*

$$\varphi : \mathbb{F}_2[G] \to R$$

extending the fixed embedding of G into R. The image of φ is a ring with unit group isomorphic to G.

Proof. Write S for the image of φ. On one hand, we have that $S^\times \subseteq R^\times \cong G$. On the other hand, the induced map $\varphi : G \to S^\times \to R^\times$ is surjective. This shows that the unit group of S is isomorphic to G. □

Lemma 7. *Let R denote a finite ring of characteristic 2. If $J \subseteq R$ is a two-sided ideal such that $J^2 = 0$, then $1 + J$ is a normal elementary abelian 2-subgroup of $R^\times$.*

Proof. Note that for any $j, k \in J$ and $r \in R^\times$, we have

- $(1 + j)^2 = 1 + j^2 = 1$;
- $(1 + j)(1 + k) = 1 + j + k + jk = 1 + j + k = (1 + k)(1 + j)$;
- $r(1 + j)r^{-1} = 1 + rjr^{-1} \in 1 + J$.

The first of these calculations shows that $1 + J$ is a subset of $R^\times$, and the three calculations together show that it is a normal elementary abelian 2-group. □

We now use these preliminary results to prove our main theorem.

Proof of Theorem 2. By Lemma 6, we may assume R is a finite ring (and is in particular artinian) and has characteristic 2. Let J denote a two-sided ideal of R such that $J^2 = 0$. By Lemma 7, the set $1 + J$ is a normal 2-subgroup of $R^\times$, and so by the assumption $O_2(G) = \{1\}$, we have $J = \{0\}$. Thus the ring R has no non-zero two-sided ideals J with $J^2 = 0$, and hence R has no non-zero two-sided nilpotent ideals. By [1, Theorem 5.4.5], the artinian ring R is semisimple. By Wedderburn's Theorem [1, Theorem 5.3.4], we have

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$$

for some $n_1, \ldots, n_r \geq 1$ and some division algebras $D_1, \ldots, D_r$. Our ring R is finite and hence each D_i is finite. By another theorem of Wedderburn [1, Theorem 3.8.6], we have that each D_i is a finite field. Finally, because the ring R has characteristic 2, each field D_i has characteristic 2. This completes the proof. □

References

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