WHICH FINITE SIMPLE GROUPS ARE UNIT GROUPS?

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Abstract. We prove that if \( G \) is a finite simple group which is the unit group of a ring, then \( G \) is isomorphic to either (a) a cyclic group of order 2; (b) a cyclic group of prime order \( 2^k - 1 \) for some \( k \); or (c) a projective special linear group \( \text{PSL}_n(\mathbb{F}_2) \) for some \( n \geq 3 \). Moreover, these groups do all occur as unit groups. We deduce this classification from a more general result, which holds for groups \( G \) with no non-trivial normal \( 2 \)-subgroup.

Throughout this paper, rings will be assumed to be unital, but not necessarily commutative, and ring homomorphisms send 1 to 1. The finite groups \( G \) of odd order which occur as unit groups of rings were determined in [3]. We will prove similar results for a more general class of groups; the description of this class of groups uses the following.

Definition 1. For a finite group \( G \), the \( p \)-core of \( G \) is the largest normal \( p \)-subgroup of \( G \). We denote this subgroup by \( O_p(G) \). It is the intersection of all Sylow \( p \)-subgroups of \( G \).

We now state the main result. The authors are most grateful to the anonymous referee for our earlier paper [2], who recognized that one of the results proved in that paper could be strengthened into the following.

Theorem 2. Let \( G \) denote a finite group such that \( O_2(G) = \{1\} \) and such that \( G \) is isomorphic to the unit group of a ring \( R \). Then
\[
G \cong \text{GL}_{n_1}(\mathbb{F}_{2^{k_1}}) \times \cdots \times \text{GL}_{n_r}(\mathbb{F}_{2^{k_r}}).
\]

Before proving Theorem 2, we record the following corollary.

Corollary 3. The finite simple groups which occur as unit groups of rings are precisely the groups
(a) \( \mathbb{Z}/2\mathbb{Z} \),
(b) \( \mathbb{Z}/p\mathbb{Z} \) for a Mersenne prime \( p = 2^k - 1 \),
(c) \( \text{PSL}_n(\mathbb{F}_2) \) for \( n \geq 3 \).

Proof. If \( G \) is a finite simple group, then either \( O_2(G) = \{1\} \) or \( O_2(G) = G \). If \( O_2(G) = G \), then \( G \) is a 2-group, and because we are assuming \( G \) is simple, we must have \( G \cong \mathbb{Z}/2\mathbb{Z} \), which for instance is isomorphic to the unit group of \( \mathbb{Z} \).

Hence assume \( G \) is a finite simple group which is isomorphic to the unit group of a ring and further assume \( O_2(G) = \{1\} \). By Theorem 2 we know
\[
G \cong \text{GL}_{n_1}(\mathbb{F}_{2^{k_1}}) \times \cdots \times \text{GL}_{n_r}(\mathbb{F}_{2^{k_r}}).
\]

These groups all occur as unit groups of the corresponding products of matrix rings, so we are reduced to determining which of them are simple; this forces
\[
G \cong \text{GL}_n(\mathbb{F}_{2^k}).
\]

If \( n > 1 \) and \( k > 1 \), then the subgroup of invertible scalar matrices forms a nontrivial normal subgroup. Hence two possibilities remain. If \( n = 1 \), then \( \text{GL}_1(\mathbb{F}_{2^k}) \) is cyclic of order \( 2^k - 1 \); such a group is simple if and only if its order is prime. If \( k = 1 \), then \( \text{GL}_n(\mathbb{F}_2) = \text{PSL}_n(\mathbb{F}_2) \). For the case \( k = 1, n = 2 \), we have \( \text{PSL}_2(\mathbb{F}_2) \cong S_3 \) (see for example [4, Section 3.3.1]); this group is not simple. For the cases \( k = 1, n \geq 3 \), it is well-known that \( \text{PSL}_n(\mathbb{F}_2) \) is simple (see for example [4, Section 3.3.2]). This completes the proof. \( \square \)

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Remark 4. The simple groups $A_8$ and $PSL_2(\mathbb{F}_7)$ also occur as unit groups. This follows immediately from the exceptional isomorphisms

$$A_8 \cong PSL_4(\mathbb{F}_2) \text{ and } PSL_2(\mathbb{F}_7) \cong PSL_3(\mathbb{F}_2).$$


Having recorded the above consequences of the main result, we now gather the preliminary results used in its proof. We begin with the following observation.

Lemma 5. Let $G$ denote a finite group with $O_2(G) = \{1\}$, and let $R$ denote a ring with $R^\times \cong G$. Then $R$ has characteristic 2.

Proof. The elements 1 and $-1$ are units in $R$ and are in the center of $G$, hence are in the center of $R^\times$. By the assumption $O_2(G) = \{1\}$, the center of $G$ cannot contain any elements of order 2. Hence $1 = -1$. □

Lemma 6. Keep notation as in Lemma 5, and fix an isomorphism $R^\times \cong G$. Because $R$ has characteristic 2, we have a natural map

$$\varphi : F_2[G] \to R$$

extending the fixed embedding of $G$ into $R$. The image of $\varphi$ is a ring with unit group isomorphic to $G$.

Proof. Write $S$ for the image of $\varphi$. On one hand, we have that $S^\times \subseteq R^\times \cong G$. On the other hand, the induced map $\varphi : G \to S^\times \to R^\times$ is surjective. This shows that the unit group of $S$ is isomorphic to $G$. □

Lemma 7. Let $R$ denote a finite ring of characteristic 2. If $J \subseteq R$ is a two-sided ideal such that $J^2 = 0$, then $1 + J$ is a normal elementary abelian 2-subgroup of $R^\times$.

Proof. Note that for any $j, k \in J$ and $r \in R^\times$, we have

- $(1 + j)^2 = 1 + j^2 = 1$;
- $(1 + j)(1 + k) = 1 + j + k + jk = 1 + j + k = (1 + k)(1 + j);$
- $r(1 + j)r^{-1} = 1 + rjr^{-1} \in 1 + J$.

The first of these calculations shows that $1 + J$ is a subset of $R^\times$, and the three calculations together show that it is a normal elementary abelian 2-group. □

We now use these preliminary results to prove our main theorem.

Proof of Theorem 2. By Lemma 4, we may assume $R$ is a finite ring (and is in particular artinian) and has characteristic 2. Let $J$ denote a two-sided ideal of $R$ such that $J^2 = 0$. By Lemma 7, the set $1 + J$ is a normal 2-subgroup of $R^\times$, and so by the assumption $O_2(G) = \{1\}$, we have $J = \{0\}$. Thus the ring $R$ has no non-zero two-sided ideals $J$ with $J^2 = 0$, and hence $R$ has no non-zero two-sided nilpotent ideals. By [1] Theorem 5.4.5, the artinian ring $R$ is semisimple. By Wedderburn’s Theorem [1] Theorem 5.3.4], we have

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$$

for some $n_1, \ldots, n_r \geq 1$ and some division algebras $D_1, \ldots, D_r$. Our ring $R$ is finite and hence each $D_i$ is finite. By another theorem of Wedderburn [1] Theorem 3.8.6], we have that each $D_i$ is a finite field. Finally, because the ring $R$ has characteristic 2, each field $D_i$ has characteristic 2. This completes the proof. □

References


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