Congruences for consecutive Gaussian polynomial coefficients with crank statistics.

Dennis Eichhorn
Department Of Mathematics
University of California, Irvine
Irvine, California 92697
deichhor@math.uci.edu

Lydia Engle
University of Minnesota
Minneapolis, Minnesota 55455
engle258@umn.edu

Brandt Kronholm
Department of Mathematics
University of Texas Rio Grande Valley
Edinburg, Texas 78539-2999
brandt.kronholm@utrgv.edu

June 7, 2021

Abstract

In this paper, we establish infinite families of congruences in consecutive arithmetic progressions modulo any odd prime ℓ for the function $p(n, m, N)$, which enumerates the partitions of n into at most m parts with no part larger than N. For $m \leqslant 4$, simple combinatorial statistics called "cranks" witness these congruences. We prove this analytically for $m = 4$, and then both analytically and combinatorially for $m = 3$. Our combinatorial proof relies upon explicit dissections of convex lattice polygons.

This paper is dedicated to the memory of Freeman Dyson, his contributions to the theory of partitions, and his contributions to science in general.

1 Introduction and Main Theorems

1.1 Partition Congruences

In 1919, Ramanujan [12] observed and proved the following congruences in arithmetic progressions for the ordinary partition function,

$$\begin{aligned} p\big(5n+4\big) &\equiv 0 \pmod{5}, \\ p\big(7n+5\big) &\equiv 0 \pmod{7}, \\ \text{and} \quad p\big(11n+6\big) &\equiv 0 \pmod{11}. \end{aligned} \tag{1}$$

Fifty years later, A. O. L. Atkin showed that $p(n)$ enjoys many more congruences in arithmetic progressions [3]. For example,

$$p(206839n + 2623) \equiv 0 \pmod{17}. \tag{2}$$

In 2000, Ono [10] showed that congruences like (2) are individual instances of infinitely many such congruences, by proving that for any prime $\ell \geq 5$, there exists integers A, B such that

$$p\big(An + B\big) \equiv 0 \pmod{\ell}. \tag{3}$$

Recently, the third author established several infinite families of congruences for the function $p(n,m)$ [7, 8, 9], which enumerates partitions of n into at most m parts. Below, we cite one of these families, where the congruences occur in intervals of consecutive arithmetic progressions of the form $p(An+B,m) \equiv 0 \pmod{\ell}$, where B can take on any value in a particular interval.

Theorem 1.1. *[8] Let ℓ be an odd prime. Then, for integers m, t and k such that $2 \leq m \leq \ell+1$, $1 \leq t \leq \binom{m+1}{2} - 1$, and $k \geq 0$, we have*

$$p\big(\ell \mathrm{lcm}(m)k - t, m\big) \equiv 0 \pmod{\ell}. \tag{4}$$

In this paper, we consider several refinements of Theorem 1.1.

Our first refinement treats the number of partitions of n into at most m parts with largest part at most N, denoted $p(n,m,N)$. This theorem shows that adding certain bounds on the largest part of our partitions preserves many of the congruences in Theorem 1.1.

Theorem 1.2. *Let ℓ be an odd prime. Then, for integers j, k, s and t, such that $2 \leq m \leq \ell+1$, $\frac{s(s-1)}{2} < t < ms - \frac{s(s-1)}{2}$, and $1 \leq s \leq m$, letting $A = \ell\mathrm{lcm}(m)$ and $C = \ell\mathrm{lcm}(m-1)$, we have*

$$p\big(Ak - t, m, Cj - s\big) \equiv 0 \pmod{\ell}. \tag{5}$$

We add a second level of refinement by treating $p\big(n, m, (a,b]\big)$, the number of partitions of n with at most m parts, largest part greater than a but at most b. Notice that

$$p\big(n, m, (a,b]\big) = p\big(n, m, b\big) - p\big(n, m, a\big). \tag{6}$$

To simplify our notation, we define I_j to be $(j-1, j]$ throughout the paper, $CI_j = (C(j-1), Cj]$ to be the dilation of I_j by C, and for any interval $I = (a,b]$, $I - s = (a-s, b-s]$ to be the translation of I by $-s$.

Corollary 1.3. *Let ℓ be an odd prime. Then, for integers j, k, s and t, such that $2 \leq m \leq \ell+1$, $\frac{s(s-1)}{2} < t < ms - \frac{s(s-1)}{2}$, and $1 \leq s \leq m$, letting $A = \ell\mathrm{lcm}(m)$ and $C = \ell\mathrm{lcm}(m-1)$, we have*

$$p\big(Ak - t, m, CI_j - s\big) \equiv 0 \pmod{\ell}. \tag{7}$$

Examples of Theorem 1.2 can be found in Section **??** and examples of Corollary 1.3 are in Section **??**.

In the following subsection, we discuss witnesses for these congruences which demonstrate a way in which these new refinements and the original Theorem 1.1 can be seen by directly examining the combinatorics of the associated sets of partitions.

1.2 Combinatorial Witnesses for Partition Congruences

We recall the definition of an integer partition.

Definition 1.4. A partition of a positive integer n is a finite nonincreasing sum of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i are called the *parts* of the partition. We write $\lambda \vdash n$ to denote "λ is a partition of n."

In 1944, Freeman Dyson [6] called for *direct* proofs of Ramanujan's congruences that show how the sets of partitions enumerated in (1) can be divided into five, seven, and eleven equinumerous subclasses, respectively. He remarked,

> "...it is unsatisfactory to receive no concrete idea of how the division is to be made. We require a proof which will not appeal to generating functions, but will demonstrate by cross-examination of the partitions themselves...

Dyson conjectured that a very simple statistic on partitions called the "rank" of a partition, the largest part of minus the smallest part, witnesses this division when considered modulo 5 and 7. Dyson denoted the number of partitions of n whose rank is congruent to r modulo ℓ by $N\big(r,\ell,n\big)$, and so he wrote his conjecture as

$$N\big(0,5,5n+4\big)=N\big(1,5,5n+4\big)=N\big(2,5,5n+4\big)=N\big(3,5,5n+4\big)=N\big(4,5,5n+4\big)$$

and

$$N\big(0,7,7n+5\big)=N\big(1,7,7n+5\big)=\cdots=N\big(6,7,7n+5\big).$$

Using analytic methods, Atkin and Swinnerton-Dyer [3] proved Dyson's conjecture. However, a combinatorial proof that the rank witnesses Ramanujan's first two congruences remains elusive. Dyson further hypothesized the existence of a different statistic, called the "crank", that would witness Ramanujan's congruence modulo 11 in the same way. In 1988, Andrews and Garvan [2] found a crank that not only witnessed Ramanujan's congruence modulo 11, but also witnessed Ramanujan's congruences modulo 5 and 7 with a new division into 5 and 7 classes, respectively. However, in both cases, the proofs were analytic, and they did not employ a cross-examination of the partitions themselves as Dyson had hoped.

In most of the literature, the Andrews-Garvan crank is referred to as "*the* crank." In addition, we may refer to any statistic on partitions (especially one that witnesses divisibilities) that is not Dyson's rank as "*a* crank." In Theorem 1.5 below, we consider the congruences modulo 3 in both Theorem 1.2 and Corollary 1.3, and we find that there is a simple crank that witnesses these congruences. Remarkably, this crank allows us to give a direct combinatorial proof of some of those congruences by cross-examination of the partitions themselves, in the way Dyson had imagined that his original conjecture would be treated.

When we have designated a crank other than Dyson's rank or the Andrews-Garvan crank, we define $M'\big(r,\ell,n,m\big)$ to be the number of partitions of n into at most m parts with crank value r modulo ℓ, $M'\big(r,\ell,n,m,N\big)$ to be the number of those partitions that have no part larger than N, and $M'\big(r,\ell,n,m,(a,b]\big)$ to be the number of those partitions with largest part confined to the interval $(a,b]$. We also define $\mathcal{M}'\big(r,\ell,n,m\big),\mathcal{M}'\big(r,\ell,n,m,N\big)$, and $\mathcal{M}'\big(r,\ell,n,m,(a,b]\big)$ to be the sets of partitions that those functions count, respectively. For each r between 0 and $\ell-1$, we refer to each of $\mathcal{M}'\big(r,\ell,n,m\big),\mathcal{M}'\big(r,\ell,n,m,N\big)$, and $\mathcal{M}'\big(r,\ell,n,m,(a,b]\big)$ as the rth *crank class modulo* ℓ of the partitions counted by $p\big(n,m\big),p\big(n,m,N\big)$, and $p\big(n,m,(a,b]\big)$, respectively.

Theorem 1.5. *For $\ell=3$, the second part of the partition is a crank witnessing the congruences of Theorem 1.2 and Corollary 1.3 when $m=2,3$. For $\ell=3$ and $m=4$, if $n\le 2N$, the second part of the partition is a crank witnessing the congruences of Theorem 1.2 and Corollary 1.3, whereas if $n>2N$, the* third part *of the partition is a crank witnessing those congruences.*

In Section 3, we prove Theorem 1.5 case by case according to m, the maximum number of parts. The proof for $m=2$ is direct and requires minimal background. For $m=3$, we offer two proofs; the first is a purely combinatorial realization where we treat partitions as integer lattice points, while the second proof uses generating functions to produce closed-form formulas for $M'\big(r,3,n,3,N\big)$. A highlight of the combinatorial proof is that we work

up from the smaller sets $\mathcal{M}'\big(r,3,n,3,(a,b]\big)$ to the larger sets $\mathcal{M}'\big(r,3,n,3,N\big)$ and then $\mathcal{M}'\big(r,3,n,3\big)$, which is the opposite order in which we treat these in the analytic proof. The case $m=4$ is treated with the same q-series procedure as $m=3$.

2 Examples, Background, and Proofs of Theorem 1.2 and Corollary 1.3.

2.1 Examples of Theorem 1.2 and Corollary 1.3.

The dynamics of the bounds of the largest part size, denoted by the variable N in Theorem 1.2 and Corollary 1.3, are on display in the following examples.

The examples below illustrate how the results of Theorem 1.2 and Corollary 1.3 change as we vary the maximum part sizes of our partitions.

Example 2.1 below illustrates how the intervals of divisibility change as we vary the maximum size of the parts in our partitions.

Example 2.1 illustrates the preservation of congruence properties of $p\big(n,m,N\big)$ in consecutive arithmetic progression within $\left[{N+m \atop m}\right]$ as N is changed by the parameter s.

Example 2.1. *Set $\ell=5$, $m=5$, and $j=4$ with $k\in\{0,1,\dots\}$. Looking to Theorem 1.2 and changing the parameter $s\in\{1,2,3\}$ changes the Gaussian polynomial and yet intervals of consecutive congruences in arithmetic progression remain, though of different length.*

- *Setting $s=1$ guarantees the following congruences in an interval of four ($0<t<5$) consecutive arithmetic progressions within the Gaussian polynomial $\left[{239+5 \atop 5}\right]=\sum_{n=0}^{1195}p\big(n,5,239\big)q^n$.*

$$\begin{aligned}p\big(296,5,239\big)&\equiv p\big(297,5,239\big)\equiv p\big(298,5,239\big)\equiv p\big(299,5,239\big),\\ \equiv p\big(596,5,239\big)&\equiv p\big(597,5,239\big)\equiv p\big(598,5,239\big)\equiv p\big(599,5,239\big),\\ \equiv p\big(896,5,239\big)&\equiv p\big(897,5,239\big)\equiv p\big(898,5,239\big)\equiv p\big(899,5,239\big)\equiv 0 \pmod 5.\end{aligned}$$

- *Setting $s=2$ guarantees the following congruences in an interval of seven ($1<t<9$) consecutive arithmetic progressions within the Gaussian polynomial $\left[{238+5 \atop 5}\right]=\sum_{n=0}^{1190}p\big(n,5,238\big)q^n$.*

$$\begin{aligned}p\big(292,5,238\big)&\equiv p\big(293,5,238\big)\equiv\cdots\equiv p\big(298,5,238\big),\\ \equiv p\big(592,5,238\big)&\equiv p\big(593,5,238\big)\equiv\cdots\equiv p\big(598,5,238\big),\\ \equiv p\big(892,5,238\big)&\equiv p\big(893,5,238\big)\equiv\cdots\equiv p\big(898,5,238\big)\equiv 0 \pmod 5.\end{aligned}$$

- *Setting $s=3$ guarantees the following congruences in an interval of eight ($3<t<12$) consecutive arithmetic progressions for the coefficients of the Gaussian polynomial $\left[{237+5 \atop 5}\right]=\sum_{n=0}^{1185}p\big(n,5,237\big)q^n$,*

$$\begin{aligned}p\big(289,5,237\big)&\equiv p\big(290,5,237\big)\equiv\cdots\equiv p\big(296,5,237\big),\\ \equiv p\big(589,5,237\big)&\equiv p\big(590,5,237\big)\equiv\cdots\equiv p\big(596,5,237\big),\quad \textit{and}\\ \equiv p\big(889,5,237\big)&\equiv p\big(890,5,237\big)\equiv\cdots\equiv p\big(896,5,237\big)\equiv 0 \pmod 5.\end{aligned}$$

Example 2.2 highlights an interaction between Theorem 1.1 and Corollary 1.3 of shared congruence properties. It is not surprising that a given set $\mathcal{P}\big(n,m\big)$ can be expressed as

a union of disjoint sets $\mathcal{P}\big(n,m,(a,b]\big)$ in many different ways. Example 2.2 considers a certain n such that, by Theorem 1.1, we have $p\big(n,m\big) \equiv 0 \pmod{\ell}$. Corollary 1.3 allows us a dissection of $p\big(n,m\big) \equiv 0 \pmod{\ell}$ *in three different ways* depending on s, as a sum of smaller partition numbers $p\big(n,m,(a,b]\big)$, such that each $p\big(n,m,(a,b]\big) \equiv 0 \pmod{\ell}$.

Example 2.2. *Set $\ell = 5,\ m = 5,\ k = 1$, and $t = 6$ so that $n = 5 \times 60 \times 1 - 6 = 294$. Theorem 1.1 tells us that $p\big(294,5\big) \equiv 0 \pmod 5$. Varying the parameter $s \in \{2,3,4\}$ in Corollary 1.3 allows us to give three different decompositions of $p\big(294,5\big)$ into sums of $p\big(294,5,(a,b]\big)$, where each number $p\big(294,5,(a,b]\big)$ is also congruent to 0 modulo 5.*

- $s = 2$

$$\begin{aligned}
p\big(294,5\big) &= \sum_{j\geq 1} p\big(294,5,60I_j - 2\big)\\
&= p\big(294,5,(-2,58]\,\big) + p\big(294,5,(58,118]\,\big) + p\big(294,5,(118,178]\,\big)\\
&+ p\big(294,5,(178,238]\,\big) + p\big(294,5,(238,298]\,\big)\\
&= 0 + 1{,}069{,}755 + 1{,}432{,}910 + 342{,}485 + 23{,}160\\
&= 2{,}868{,}310 \equiv 0 \pmod 5.
\end{aligned}$$

- $s = 3$

$$\begin{aligned}
p\big(294,5\big) &= \sum_{j\geq 1} p\big(294,5,60I_j - 3\big)\\
&= p\big(294,5,(-3,57]\,\big) + p\big(294,5,(57,117]\,\big) + p\big(294,5,(117,177]\,\big)\\
&+ p\big(294,5,(177,237]\,\big) + p\big(294,5,(237,297]\,\big)\\
&= 0 + 1{,}034{,}725 + 1{,}455{,}640 + 353{,}210 + 24{,}735\\
&= 2{,}868{,}310 \equiv 0 \pmod 5.
\end{aligned}$$

- $s = 4$

$$\begin{aligned}
p\big(294,5\big) &= \sum_{j\geq 1} p\big(294,5,60I_j - 4\big)\\
&= p\big(294,5,(-4,56]\,\big) + p\big(294,5,(56,116]\,\big) + p\big(294,5,(116,176]\,\big)\\
&+ p\big(294,5,(176,236]\,\big) + p\big(294,5,(236,296]\,\big)\\
&= 0 + 999{,}650 + 1{,}478{,}115 + 364{,}160 + 26{,}385\\
&= 2{,}868{,}310 \equiv 0 \pmod 5.
\end{aligned}$$

Read this sentence. → *Since for all j in Example 2.2, the values of $p\big(294,5,60I_j - s\big) \equiv 0 \pmod 5$, we have that $p\big(294,5,60j' - s\big) \equiv 0 \pmod 5$ for any j'. At the same time, since $p\big(294,5,60j' - s\big) \equiv 0 \pmod 5$ for any j', the values of $p\big(294,5,60I_j - s\big) \equiv 0 \pmod 5$ for any j.*

2.2 Background Material for Theorem 1.2 and Corollary 1.3.

We use the standard q-rising factorial notation throughout,

$$(q;q)_d = \prod_{i=1}^{d}(1-q^i).$$

It is well-known that the generating function for $p(n,m)$ is given by

$$\sum_{n=0}^{\infty} p(n,m)q^n = \frac{1}{(q;q)_m}.$$

Gaussian polynomials, denoted by $\left[{N+m \atop m}\right]$, are generating functions for $p(n,m,N)$.

$$\sum_{n=0}^{mN} p(n,m,N)q^n = \begin{bmatrix} N+m \\ m \end{bmatrix} = \frac{(q;q)_{N+m}}{(q;q)_m(q;q)_N} = \frac{(q^{N+1};q)_m}{(q;q)_m}. \tag{8}$$

Gaussian polynomials are reciprocal polynomials of degree mN.

Lemma 2.3. *[1]*

$$(z;q)_m = \sum_{h=0}^{m} (-1)^h \begin{bmatrix} m \\ h \end{bmatrix} q^{h(h-1)/2} z^h. \tag{9}$$

2.3 Poof of Theorem 1.2 and Corollary 1.3.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let ℓ be an odd prime, and for $2 \leq m \leq \ell+1$, consider

$$\begin{aligned}\sum_{n=0}^{mN} p(n,m,N)q^n &= \begin{bmatrix} N+m \\ m \end{bmatrix} = \frac{(q^{N+1};q)_m}{(q;q)_m} = \sum_{h=0}^{m} \frac{(-1)^h}{(q;q)_m} \begin{bmatrix} m \\ h \end{bmatrix} q^{hN+h(h+1)/2} \\ &= \sum_{h=0}^{m} \frac{(-1)^h q^{hN+h(h+1)/2}}{(q;q)_{m-h}(q;q)_h}, \end{aligned} \tag{10}$$

by the definition of the Gaussian polynomial and an application of Lemma 2.3 with $z = q^{N+1}$.

Let $N = \ell \mathrm{lcm}(m-1)j - s$. We now prove our theorem by showing that the desired congruences in arithmetic progressions hold for each individual term of the sum in (10). For $h=0$, the summand in (10) simplifies to $1/(q;q)_m$. By Theorem 1.1, we have that for every $n = \ell \mathrm{lcm}(m)k - t$, the summand in (10) is $0 \pmod{\ell}$ for $1 \leqslant t \leqslant \binom{m+1}{2} - 1$. Similarly, for $h=m$, the summand in (10) simplifies to $(-1)^m q^{m\ell \mathrm{lcm}(m-1)j - ms + m(m+1)/2}/(q;q)_m$. By Theorem 1.1, we have that for every $n = \ell \mathrm{lcm}(m)k - t$, the summand in (10) is then $0 \pmod{\ell}$ for $ms - \binom{m+1}{2} < t < ms$. Thus the sum of the $h=0$ and $h=m$ terms of the sum in (10) is $0 \pmod{\ell}$ for $0 < t < ms$ if $s \leqslant (m+1)/2$, and is $0 \pmod{\ell}$ for $ms - \binom{m+1}{2} < t < \binom{m+1}{2}$ if $s > (m+1)/2$.

For $h \neq 0, m$, we now also show congruences in an interval of consecutive arithmetic progressions for the summands in (10). In these cases, our arithmetic progressions have a much smaller common difference, and they fill a more narrow interval. We rewrite the summand in (10) as

$$\begin{aligned}\frac{E_h(q)q^{h(\ell \mathrm{lcm}(m-1)j - s) + h(h+1)/2}}{(1-q^{\mathrm{lcm}(m-1)})^\ell} &\equiv \frac{E_h(q)q^{h\ell \mathrm{lcm}(m-1)j + h(h+1)/2 - hs}}{(1-q^{\ell \mathrm{lcm}(m-1)})} \\ &\equiv E_h(q)q^{\frac{h(h+1)/2}{2} - hs} g(q^{\ell \mathrm{lcm}(m-1)}) \pmod{\ell}, \end{aligned} \tag{11}$$

for some function g, where $E_h(q) = \frac{(-1)^h(1-q^{\mathrm{lcm}(m-1)})^\ell}{(q;q)_{m-h}(q;q)_h}$. Notice that each of the m factors in the denominator of $E_h(q)$ divides $(1-q^{\mathrm{lcm}(m-1)})$, so in fact $E_h(q)$ is a polynomial, and

the degree of $E_h(q)$ is $\ell\mathrm{lcm}(m-1)+h(m-h)-m(m+1)/2$, which is always strictly less than $\ell\mathrm{lcm}(m-1)$. Thus the power series expansion of the right-hand side of (11) only has terms where the exponent of q is congruent to r modulo $\ell\mathrm{lcm}(m-1)$ for $r \in \{h(h+1-2s)/2, \ldots, \ell\mathrm{lcm}(m-1)-h(h-1-2(m-s))/2-m(m+1)/2\}$. Taking the union of these sets from $h=1$ to $m-1$, we have

$$\bigcup_{h=1}^{m-1}\left\{\frac{h(h+1-2s)}{2}, \ldots, \ell\mathrm{lcm}(m-1)-\frac{h(h-1-2(m-s))}{2}-\frac{m(m+1)}{2}\right\}$$
$$=\left\{-\frac{s(s-1)}{2}, \ldots, \ell\mathrm{lcm}(m-1)+\frac{s(s-1)}{2}-ms\right\}$$

This means that when $s(s-1)/2 < t < ms-s(s-1)/2$, exponents of the form $\ell\mathrm{lcm}(m-1)k'-t$ do not appear in the power series expansion of any summand in (11) for $h \neq 0, m$. In particular, for t in that same range, exponents of the form $\ell\mathrm{lcm}(m)k-t$ do not appear in the power series expansion of any summand in (11) for $h=0, m$.

Now, for any $1 \leq s \leq m$, we see that summing (10) over all h, the theorem follows.

Corollary 1.3 follows immediately from (6). □

3 Combinatorial Witnesses and the Proof of Theorem 1.5

The proof of Theorem 1.5 comes in three cases depending on $m=2,3$, and 4. For the case $m=2$, the proof is direct and requires almost no background information. For the case $m=3$, we supply two proofs, one analytic and another purely combinatorial. We offer an analytic proof for the case $m=4$ that follows the very same procedure as $m=3$.

We restate Theorem 1.5 with additional details.

Theorem 1.5. *For $\ell=3$, the second part of the partition is a crank witnessing the congruences of Theorem 1.2 and Corollary 1.3 when $m=2,3$. For $\ell=3$ and $m=4$, if $n \leq 2N$, the second part of the partition is a crank witnessing the congruences of Theorem 1.2 and Corollary 1.3, whereas if $n > 2N$, the* third part *of the partition is a crank witnessing those congruences.*

Equivalently, all of the following statements hold.

When $m=2$ we have for $r=0,1,2$, integers j,k, and the ordered pairs $(s,t) \in \{(1,1),(2,2)\}$, one has

$$M'\big(r,3,6k-t,2,3I_j-s\big)=\frac{p\big(6k-t,2,3I_j-s\big)}{3}, \tag{12}$$

$$M'\big(r,3,6k-t,2,3j-s\big)=\frac{p\big(6k-t,2,3j-s\big)}{3}, \tag{13}$$

$$\textit{and} \qquad M'\big(r,3,6k-t,2\big)=\frac{p\big(6k-t,2\big)}{3}. \tag{14}$$

When $m=3$ we have for $r=0,1,2$, integers j,k, and the ordered pairs $(s,t) \in \{(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),(3,5)\}$, one has

$$M'\big(r,3,18k-t,3,6I_j-s\big)=\frac{p\big(18k-t,3,6I_j-s\big)}{3}, \tag{15}$$

$$M'\big(r,3,18k-t,3,6j-s\big)=\frac{p\big(18k-t,3,6j-s\big)}{3}, \tag{16}$$

$$\textit{and} \qquad M'\big(r,3,18k-t,3\big) = \frac{p\big(18k-t,3\big)}{3}. \tag{17}$$

For the case $m = 4$, when $36k - t \leq 2N$, M' denotes second part modulo 3 and if $36k - t > 2N$, M' denotes third part modulo 3.

When $m = 4$ and for $r = 0, 1, 2$, integers j, k, and the ordered pairs $(s,t) \in \{(1,1),(1,2),(1,3),(2,2),(2,3),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6),(3,7),(3,8),(4,7),(4,8),(4,9)\}$, one has

$$M'\big(r,3,36k-t,4,18I_j-s\big) = \frac{p\big(36k-t,4,18I_j-s\big)}{3}, \tag{18}$$

$$M'\big(r,3,36k-t,4,18j-s\big) = \frac{p\big(36k-t,4,18j-s\big)}{3}, \tag{19}$$

$$\textit{and} \qquad M'\big(r,3,36k-t,4\big) = \frac{p\big(36k-t,4\big)}{3}. \tag{20}$$

We prove the case $m = 2$ of Theorem 1.5 below.

Proof of Theorem 1.5 for $m = 2$. Since every partition of $6k-t$ into two parts is uniquely determined by its first part, λ_1, we see that for fixed r, k, t, and j, $M'\big(r,3,6k-t,2,3I_j-s\big) = 1$ so long as $\lceil (6k-t)/2 \rceil \leq 3(j-1)-s$ and $3j-s \leq 6k-t$, and is 0 otherwise. Examining the second parts of these single partitions as r ranges across $r = 0,1,2$ reveals that these partitions have cranks in each of the three residue classes modulo three. Further, for $(s,t) \in \{(1,1),(2,2)\}$ as j' ranges from $k+1$ to j in (12), we see that in the aggregate, this implies (13) with $j' = j$. Finally, by taking any j at least as large as $2k$ in (13), we see that in the aggregate, this implies (14). □

3.1 Integer Lattices and a Combinatorial/Bijective Proof of the case $m = 3$ of Theorem 1.5.

In this section, we give a direct proof of the case $m = 3$ from Theorem 1.5. To do this, we treat partitions into at most three parts as vectors in $\mathbb{Z}^3$. We then construct five triplets of vectors such that each triplet contains one partition from each of the three possible crank classes determined by λ_2 modulo 3. Then, we give an explicit covering of $\mathcal{P}\big(18k-t,3,6I_j-s\big)$ with translations of our five triplets, such that the translated triplets are disjoint.

We treat a partition $\lambda \vdash n = \lambda_1 + \lambda_2 + \lambda_3$ as an integer vector $\lambda = \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix} \in \mathbb{Z}^3$ so that the set of partitions of n into at most three parts becomes

$$\mathcal{P}\big(n,3\big) = \left\{ \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix} \in \mathbb{Z}^3 \mid \lambda_1+\lambda_2+\lambda_3 = n, \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0 \right\}. \tag{21}$$

For example, in Figure 1, we see the set $\mathcal{P}(51,3)$.

Remark 3.1. *Notice that when a set of partitions is displayed as in the right side of Figure 1, for each partition λ, the crank value $\lambda_2 \pmod 3$ corresponds to the apparent horizontal location of the lattice point/partition on the page.*

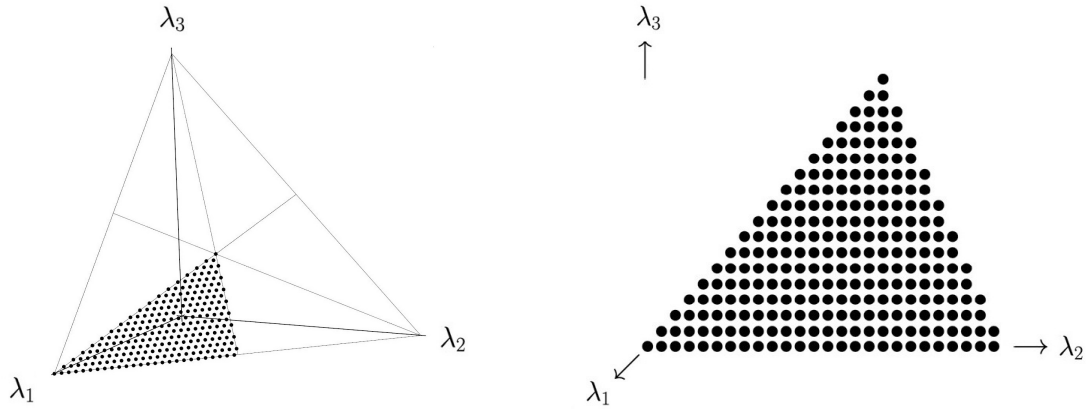
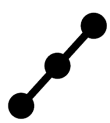


Figure 1: Two views of the set $\mathcal{P}(51,3) \subset \mathbb{Z}^3$. On the left we include the ambient large equilateral triangle $\begin{pmatrix}51\\0\\0\end{pmatrix}, \begin{pmatrix}0\\51\\0\end{pmatrix}, \begin{pmatrix}0\\0\\51\end{pmatrix}$ and its medians. On the right we display a planar version of $\mathcal{P}(51,3)$. The node on the bottom left is the partition $\begin{pmatrix}51\\0\\0\end{pmatrix}$, the bottom right is $\begin{pmatrix}26\\25\\0\end{pmatrix}$, and the node at the top is $\begin{pmatrix}17\\17\\17\end{pmatrix}$.

Combinatorial proof of Theorem 1.5 for $m = 3$. We begin by choosing triplets of integer lattice points that each span all three crank classes for $\lambda_2 \pmod 3$. We then cover $\mathcal{P}(18k-t, 3, 6I_j - s)$ with disjoint translations of these triplets, demonstrating that that those partitions are equally distributed among the three crank classes for $\lambda_2 \pmod 3$.

The five lattice point triplets are

$$A = \left\{ \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}-2\\1\\1\end{pmatrix}, \begin{pmatrix}-4\\2\\2\end{pmatrix} \right\}$$

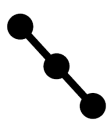


$$B = \left\{ \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}-2\\1\\1\end{pmatrix}, \begin{pmatrix}-2\\2\\0\end{pmatrix} \right\}$$

$$C = \left\{ \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\2\\0\end{pmatrix} \right\}$$

$$D = \left\{ \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\2\\-1\end{pmatrix} \right\}$$

$$E = \left\{ \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\2\\-2\end{pmatrix} \right\}$$



(22)

Note that the second coordinates in each of A, B, C, D and E creates a complete residue system modulo 3. In other words, once translated, each triplet of partitions spans the crank classes determined by $\lambda_2 \pmod 3$.

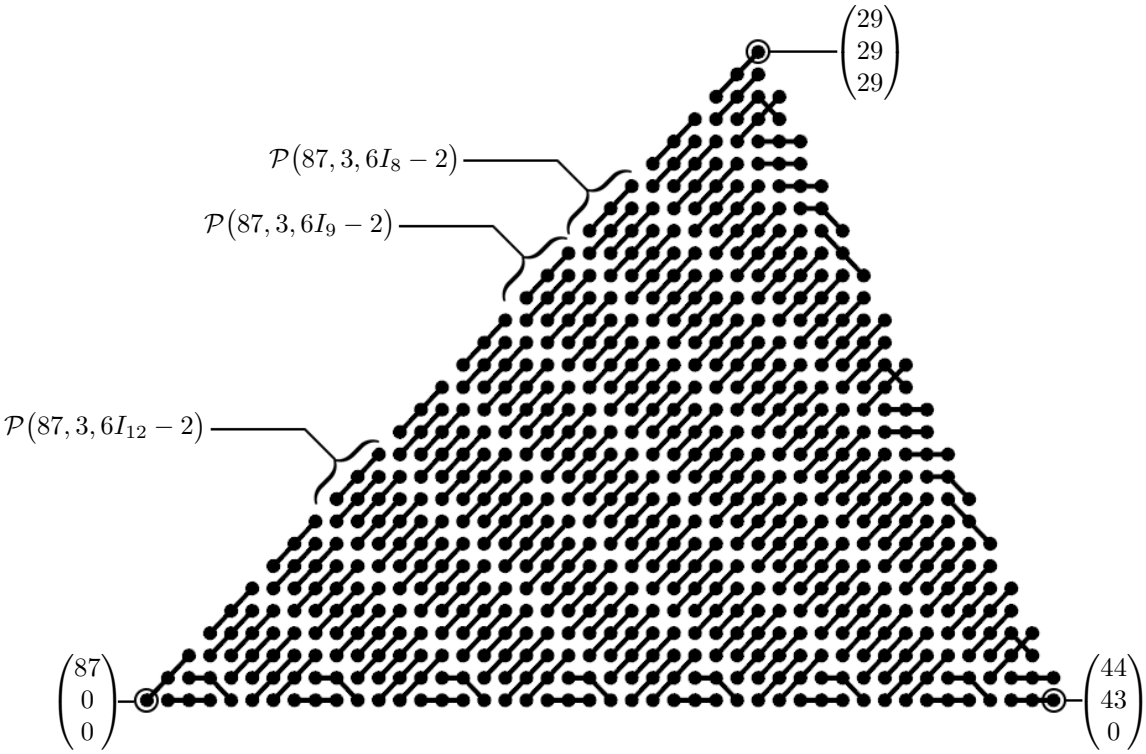


Figure 2: $\mathcal{P}(87,3) = \bigcup_{j=6}^{15} \mathcal{P}(87,3,6I_j-2)$, with each set of the form $\mathcal{P}(87,3,6I_j-2)$ covered by translations of triplets described in (22). Would this Figure be improved with a few labeled nodes? – or indicated nodes? ...the node at the far left is $\begin{pmatrix}87\\0\\0\end{pmatrix}$ and the top node is $\begin{pmatrix}29\\29\\29\end{pmatrix}$ and the rightmost pair of nodes are $\begin{pmatrix}44\\43\\0\end{pmatrix}$ and above it is $\begin{pmatrix}43\\43\\1\end{pmatrix}$...

Would this Figure be improved with a few labeled nodes? – or indicated nodes? ...the node at the far left is $\begin{pmatrix}87\\0\\0\end{pmatrix}$ and the top node is $\begin{pmatrix}29\\29\\29\end{pmatrix}$ and the rightmost pair of nodes are $\begin{pmatrix}44\\43\\0\end{pmatrix}$ and above it is $\begin{pmatrix}43\\43\\1\end{pmatrix}$...

We consider the sets $\mathcal{P}(18k-t,3,6I_j-s)$ in two separate regimes depending on j. The first regime is defined by $k+1 \le j \le \lceil \frac{3k}{2} \rceil$ and requires all five triplets A, B, C, D, E. It is detailed by Table 1 and accompanied by an example in Figure 3. The second regime is defined by $\lceil \frac{3k}{2} \rceil + 1 \le j \le 3k$ and requires the three triples A, C, and D. It is detailed by Table 2 and accompanied by Figure 4 as a supporting example. Let $j' = (j-k-1)$.

Table 1: $\mathcal{P}\big(18k-t,3,6I_j-s\big)$ for $k+1\le j\le\lceil\frac{3k}{2}\rceil$. Let $j'=(j-k-1)$.

Triplet	Translations	Conditions
A	$\overrightarrow{a_1}(x)=\begin{pmatrix}6k+6j'+6-s\\6k+6j'-s-x\\6k-12j'-6+2s-t+x\end{pmatrix}$	$0\le x\le 9j'+\lfloor\frac{t-3s}{2}\rfloor+3$
	$\overrightarrow{a_2}(y)=\begin{pmatrix}6k+6j'+5-s\\6k+6j'-1-s-y\\6k-12j'+2s-4-t+y\end{pmatrix}$	$0\le y\le 9j'+\lceil\frac{t-3s}{2}\rceil+1$
B	$\begin{pmatrix}6k+6j'+5-s\\6k+6j'-s\\6k-12j'-5+2s-t\end{pmatrix}$	
C	$\overrightarrow{c_1}=\begin{pmatrix}6k+6j'+5-s\\6k+6j'+1-s\\6k-12j'-6+2s-t\end{pmatrix}$	unless $j=\frac{3k+1}{2}$, in which case, for pairs (s,t), C is translated by $\begin{cases}\overrightarrow{c_1},\ \overrightarrow{c_2},\ \overrightarrow{c_3} & \text{for } (2,2),(3,4)\\ \overrightarrow{c_1},\ \overrightarrow{c_2} & \text{for } (1,1),(2,3),(3,5)\\ \overrightarrow{c_1} & \text{for } (1,2),(2,4)\end{cases}$
	$\overrightarrow{c_2}=\begin{pmatrix}6k+6j'+6-s\\6k+6j'+1-s\\6k-12j'-7+2s-t\end{pmatrix}$	
	$\overrightarrow{c_3}=\begin{pmatrix}6k+6j'+6-s\\6k+6j'+2-s\\6k-12j'-8+2s-t\end{pmatrix}$	
D	$\begin{pmatrix}6k+6j'+6-s\\6k+6j'+3-s\\6k-12j'-3+2s-t\end{pmatrix}$	unless $j=\frac{3k+1}{2}$, in which case D is not translated at all
E	$\begin{pmatrix}6k+6j'+6-s\\6k+6j'+4-s\\6k-12j'-4+2s-t\end{pmatrix}$	unless $j=\frac{3k+1}{2}$, in which case E is not translated at all

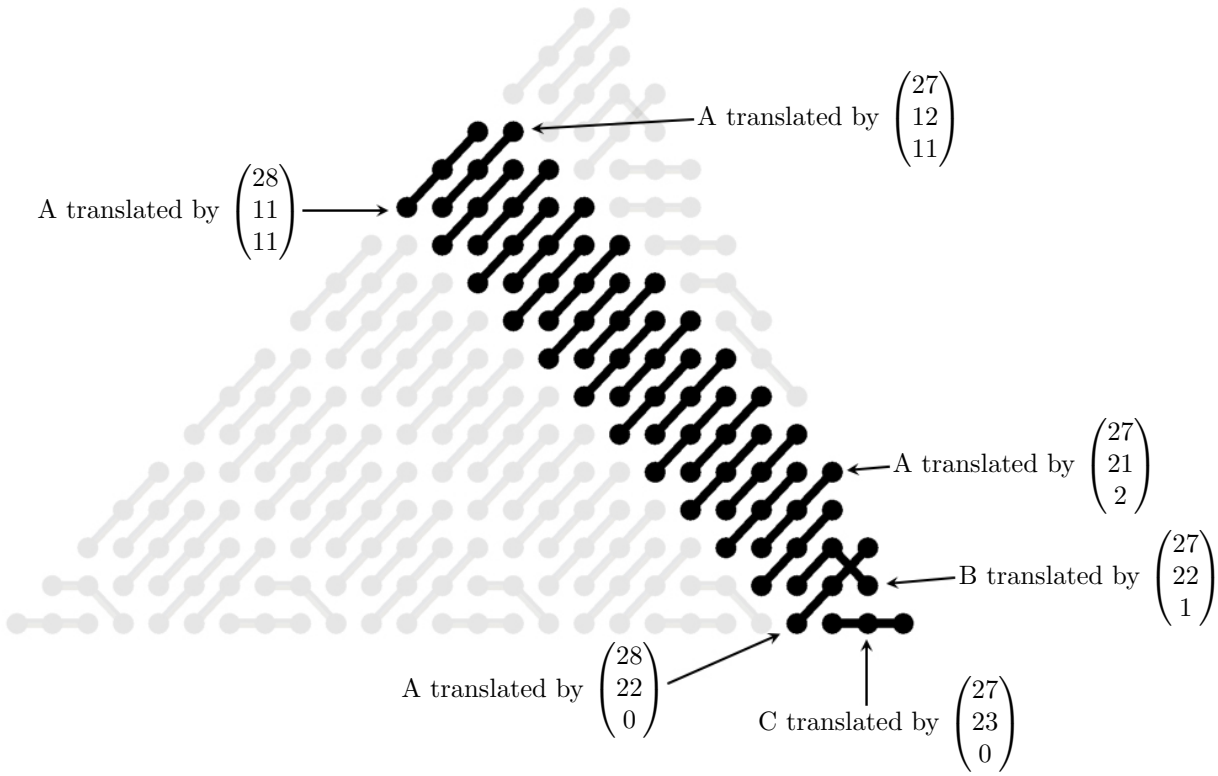


Figure 3: In this figure we have highlighted the set $\mathcal{P}\big(50,3,6I_5-2\big)=\mathcal{P}\big(50,3,(22,28]\big)$ within $\mathcal{P}\big(50,3\big)$. This region has parameter values $k=3, t=4$, $s=2$, and $j=5$, and since $3+1\leqslant 5\leqslant\lceil 3(3)/2\rceil$, we are in the first regime, described by Table 1. A is translated by $\overrightarrow{a_1}(0),\overrightarrow{a_1}(1),\ldots,\overrightarrow{a_1}(11)$ beginning with $\begin{pmatrix}28\\22\\0\end{pmatrix}$ at the bottom right and ending with $\begin{pmatrix}28\\11\\11\end{pmatrix}$, and again by $\overrightarrow{a_2}(0),\overrightarrow{a_2}(1),\ldots,\overrightarrow{a_2}(9)$ beginning with $\begin{pmatrix}27\\21\\2\end{pmatrix}$ and ending with $\begin{pmatrix}27\\12\\11\end{pmatrix}$ near the top left. B is translated once by $\begin{pmatrix}27\\22\\1\end{pmatrix}$, and C is translated once by $\begin{pmatrix}27\\23\\0\end{pmatrix}$.

Table 2: $\mathcal{P}\big(18k-t,3,6I_j-s\big)$ for $\lceil\frac{3k}{2}\rceil+1\le j\le 3k$.

Triplet	Translations	Conditions
A	$\vec{a_3}(x)=\begin{pmatrix}6j-1-s\\18k-6j+1+s-t-x\\x\end{pmatrix}$	$0\le x\le 9k-3j+\lceil\frac{s-t}{2}\rceil$
	$\vec{a_4}(y)=\begin{pmatrix}6j-s\\18k-6j+s-t-y\\y\end{pmatrix}$	$0\le y\le 9k-3j+\lfloor\frac{s-t}{2}\rfloor$
C	$\begin{pmatrix}6j-2-s\\18k-6j+2+s-t\\0\end{pmatrix}$	
D	$\begin{pmatrix}6j-4-s\\18k-6j+3+s-t\\1\end{pmatrix}$	

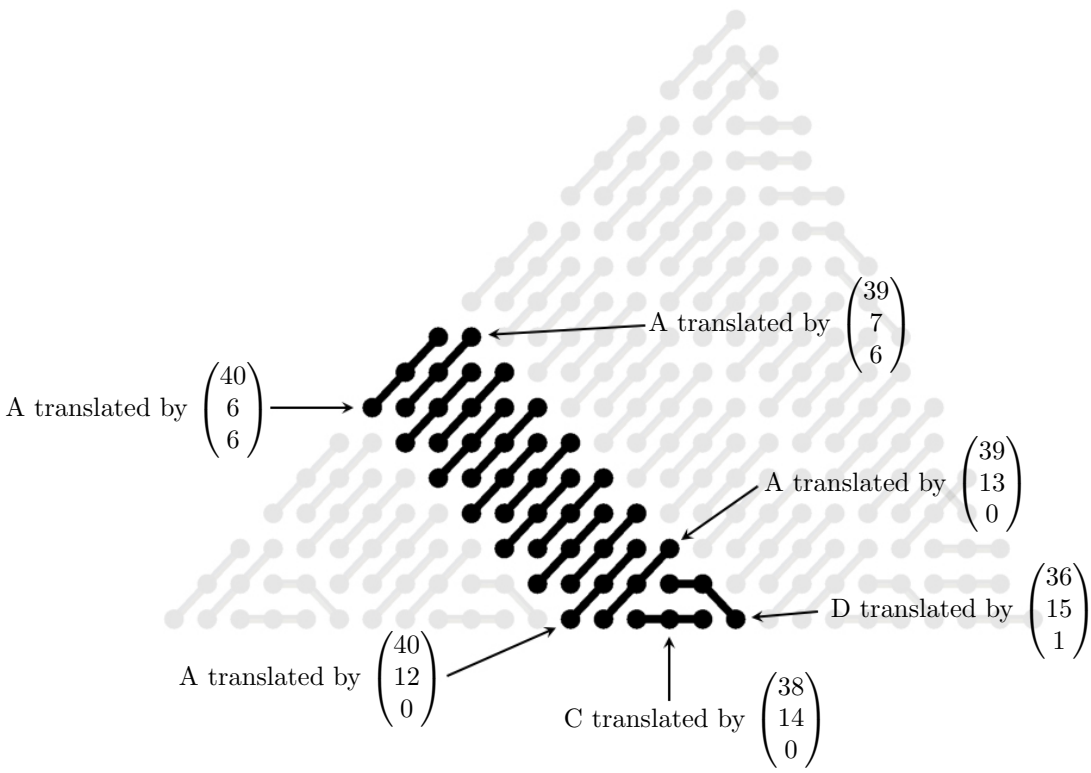


Figure 4: In this figure we have highlighted the set $\mathcal{P}\big(52,3,6I_7-2\big)=\mathcal{P}\big(52,3,(34,40]\big)$ within $\mathcal{P}\big(52,3\big)$. This region has parameter values $k=3,t=4$, $s=2$, and $j=7$, and since $\lceil 3(3)/2\rceil+1\le 7\le 3(3)$, we are in the second regime, described by Table 2. A is translated by $\vec{a}_3(0),\vec{a}_3(1),\ldots,\vec{a}_3(6)$, beginning with $\begin{pmatrix}39\\13\\0\end{pmatrix}$ and ending with $\begin{pmatrix}39\\7\\6\end{pmatrix}$, and again by $\vec{a}_4(0),\vec{a}_4(1),\ldots,\vec{a}_4(6)$, beginning with $\begin{pmatrix}40\\12\\0\end{pmatrix}$ and ending with $\begin{pmatrix}40\\6\\6\end{pmatrix}$. C is translated once by $\begin{pmatrix}38\\14\\0\end{pmatrix}$, and D is translated once by $\begin{pmatrix}36\\15\\1\end{pmatrix}$.

□

3.2 Counting Formulas for $p\big(18k-t,3,6j-s\big)$.

This is an ugly sentence. $\rightarrow$ The information in the combinatorial proof of the case $m=3$ from Theorem 1.5 allows us to directly count partitions and establish formulas for $p\big(18k-t,3,6I_j-s\big)$, $p\big(18k-t,3,6j-s\big)$, and $p\big(18k-t,3\big)$.

$$p\big(18k-t,3,6I_j-s\big)=\begin{cases}0 & \text{for } j<k+1\\ 3\big(18j'+12-3s+t\big) & \text{for } k+1\leq j<\lceil\frac{3k}{2}\rceil\\ 3\big(18j'+8-s\big)=27k-3(1+s) & \text{for } j=\frac{3k+1}{2}\\ 3\big(18k-6j+4+s-t\big) & \text{for } \lceil\frac{3k}{2}\rceil<j\leq 3k\\ 0 & \text{for } j>3k.\end{cases} \tag{23}$$

Summing the appropriate values from (23), we have proved the following proposition.

Proposition 3.2. *For integers j,k and the ordered pairs $(s,t)\in\{(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),(3,5)\}$, one has*

$$\begin{aligned}&p\big(18k-t,3,6j-s\big)\\ &=\begin{cases}0 & \textit{for } j<k+1\\ 3(k-j)(9k-9j-3+3s-t) & \textit{for } k+1\leq j<\lceil\frac{3k}{2}\rceil\\ 3(k-j)(9k-9j-3+3s-t)-3(1+s) & \textit{for } j=\frac{3k+1}{2}\\ 3\big(j(18k+s-t+1)-3j^2+k(-18k-3s+2t)\big) & \textit{for } \lceil\frac{3k}{2}\rceil<j<3k\\ 27k^2+3(t-3)(k)\ =\ p\big(18k-t,3\big) & \textit{for } j\geq 3k.\end{cases}\end{aligned} \tag{24}$$

□

3.3 An interlude prior to the analytic proofs for the cases $m=3$ and 4 of Theorem 1.5 and a definition.

A collection of 36 closed-term polynomial formulas describing $p\big(n,3,N\big)$ for all n and N can be found in the Appendix of [4]. The 36 formulas were established using generating function methods that are not dissimilar from the methods we will use to prove the cases $m=3,4$ of Theorem 1.5. The entire collection of the 36 formulas is called a quasipolynomial.

Definition 3.3. A function $f(n)$ is a *quasipolynomial* if there exist d polynomials $f_0(n),\ldots,f_{d-1}(n)$ such that

$$f(n)=\begin{cases}f_0(n) & \text{if } n\equiv 0 \pmod d\\ f_1(n) & \text{if } n\equiv 1 \pmod d\\ \vdots & \vdots\\ f_{d-1}(n) & \text{if } n\equiv d-1 \pmod d\end{cases}$$

for all $n\in\mathbb{Z}$. The polynomials f_i are called the *constituents* of the quasipolynomial f and the number of them, d, is the *period* of f.

The method used to generate such quasipolynomials obligates us to adhere to a strict interpretation of binomial coefficients for the constituents of $p(n,m,N)$. For a and b natural numbers, when $a<b$ then $\binom{a}{b}=0$, and when $a\geqslant b$, then $\binom{a}{b}=\frac{a!}{b!(a-b)!}$.

With the constituents from the Appendix of [4], it is possible to express $p\big(18k-t,3,6j-s\big)$ for all $k,j\in\mathbb{Z}_{\geq 0}$ for the seven ordered pairs $(s,t)\in\{(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),(3,5)\}$

in one formula (compare (25) to (24)).

$$\begin{aligned} p\big(18k-t,3,6j-s\big) &= (6-t)\tbinom{3k+1}{2} + t\tbinom{3k}{2} - 3(s-t+4)\tbinom{3k-j+1}{2} \\ &\quad - 3(6-(s-t+4))\tbinom{3k-j}{2} + 3(2s-t+2)\tbinom{3k-2j+1}{2} \\ &\quad + 3(6-(2s-t+2))\tbinom{3k-2j}{2} - (t-3s)\tbinom{3k-3j+1}{2} \\ &\quad - (6-(t-3s))\tbinom{3k-3j}{2}. \end{aligned} \tag{25}$$

From the analytic proof of the case $m = 3$, for Theorem 1.5 we obtain a similarly condensed result for the relevant functions $M'\big(r,3,18k-t,3,6j-s\big)$ for s and t listed in the seven ordered pairs above. We do not attempt to condense the formulas for the case $m = 4$.

3.4 Analytic Proof of the cases $m = 3$ from Theorem 1.5.

We treat the case $m = 3$ in detail. The proof for $m = 4$ follows the same procedure.

Here we follow a procedure detailed in [4] for producing formulae for $p\big(n,m,N\big)$ to similarly establish formulae for $M'\big(r,3,n,3,N\big)$. The procedure begins with a generating function and the end result is a quasipolynomial for $M'\big(r,3,n,3,N\big)$ for all r and n.

For each $0 \le r \le 2$, we produce a quasipolynomial such that for $x = 0,1,2$ and the ordered pairs $(s,t) \in \{(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),(3,5)\}$, we find the following constituents to be equivalent by inspection:

$$M'(0,3,18k-t,3,18j-s+6x) = M'(1,3,18k-t,3,18j-s+6x) = M'(2,3,18k-t,3,18j-s+6x) \tag{26}$$

Because the sequence $\{6j-s\}_{j\ge 0}$ is equivalent to the three subsequences $\{18j-s, 18j-s+6, 18j-s+12\}_{j\ge 0}$, the constituents give us (16) in Theorem 1.5.

Analytic proof of Theorem 1.5 for $m = 3$. Our first goal is to establish a generating function for the partitions of n with crank value $r \pmod 3$ into at most 3 parts, no part larger than N, where the crank value is determined by the second part of the partition. We begin with the generating function for $p\big(n,3,N\big)$,

$$f(q) = \sum_{n\ge 0} p\big(n,3,N\big)q^n = \frac{(q^{18j-1};q)_3}{(q;q)_3}. \tag{27}$$

We construct a generating function for partitions having certain crank vales from (27) determined by $\lambda_2 \pmod 3$. This is done by inserting the variable z into (27) to keep track of the part sizes of the partitions.

$$f(z,q) = \sum_{n\ge 0}\sum_{r=0}^{2} M'(r,3,n,3,N)z^r q^n \tag{28}$$

Combinatorial arguments produce the generating function:

$$\begin{aligned} f(z,q) &= \sum_{n,r=0}^{\infty} M'(r,3,n,3,N)z^r q^n \\ &= \sum_{j=0}^{N} \left(q^j + q^{j+1} + \cdots + q^N\right) z^j q^j \left(1 + q + \cdots + q^j\right) \end{aligned} \tag{29}$$

We rewrite (29) as the following rational function.

$$f(z,q) = \sum_{n,r=0}^{\infty} M'(r,3,n,3,N) z^r q^n$$
$$= \frac{1 - q^{N+1} - zq + zq^{N+4} + z^{N+2}q^{2N+3} - z^{N+2}q^{3N+6} - z^{N+3}q^{2N+6} + z^{N+3}q^{3N+7}}{(1-q)(1-zq)(1-zq^2)(1-zq^3)} \quad (30)$$

We replace z with $\zeta = e^{2\pi i/3}$, a primitive third root of unity, into (30) so that we can split the generating function into three seperate generating functions depending on ζ^0, ζ^1, and ζ^2.

$$f(\zeta,q) = \sum_{n\geq 0}\sum_{r=0}^{2} M'(r,3,n,3,N)\zeta^r q^n$$
$$= \frac{1 - q^{N+1} - \zeta q + \zeta q^{N+4} + \zeta^{N+2}q^{2N+3} - \zeta^{N+2}q^{3N+6} - \zeta^{N+3}q^{2N+6} + \zeta^{N+3}q^{3N+7}}{(1-q)(1-\zeta q)(1-\zeta q^2)(1-\zeta q^3)}$$
$$= \sum_{n\geq 0} M'(0,3,n,3,N)q^n + \zeta \sum_{n\geq 0} M'(1,3,n,3,N)q^n + \zeta^2 \sum_{n\geq 0} M'(2,3,n,3,N)q^n \quad (31)$$

Our goal in this procedure is to recast (31) as a product of a polynomial and a generating function for binomial coefficients. We multiply the far right side of (31) by $E(\zeta,q)/E(\zeta,q)$ where

$$E(\zeta,q) = \sum_{i=0}^{17} q^i \times \sum_{i=0}^{17} (\zeta q)^i \times \sum_{i=0}^{8} \left(\zeta q^2\right)^i \times \sum_{i=0}^{5} \left(\zeta q^3\right)^i.$$

We note that the polynomial $E(\zeta,q)$ is constructed specifically so that $E(\zeta,q)\times(1-q)(1-\zeta q)(1-\zeta q^2)(1-\zeta q^3) = (1-q^{18})^4$. Please see [4] for more details including a generalization of $E(q)$.

$$\frac{E(\zeta,q)}{E(\zeta,q)} \times \frac{1 - q^{N+1} - \zeta q + \zeta q^{N+4} + \zeta^{N+2}q^{2N+3} - \zeta^{N+2}q^{3N+6} - \zeta^{N+3}q^{2N+6} + \zeta^{N+3}q^{3N+7}}{(1-q)(1-\zeta q)(1-\zeta q^2)(1-\zeta q^3)} \quad (32)$$
$$= \frac{A(\zeta,q)}{(1-q^{18})^4} = A(\zeta,q) \times \sum_{k\geq 0} \binom{k+3}{3} q^{18k} \quad (33)$$

Where we write $A(\zeta,q)$ for the numerator on the left side of (32). The last equality comes from the generating function for binomial coefficients.

$$\frac{1}{(1-q)^b} = \sum_{a\geq 0} \binom{a+b-1}{b-1} q^a.$$

We now write $A(\zeta,q)$ as a sum of three polynomials organized by third roots of unity: $A(\zeta,q) = A_0(q) + \zeta A_1(q) + \zeta^2 A_2(q)$. This allows us to express the far right side of (33) as

$$\left(A_0(q) + \zeta A_1(q) + \zeta^2 A_2(q)\right) \times \sum_{k\geq 0} \binom{k+3}{3} q^{18k}. \quad (34)$$

Hence,

$$f(\zeta,q)=\sum_{n\geq 0}\sum_{r=0}^{2}M'(r,3,n,3,N)\zeta^r q^n=\sum_{n\geq 0}M'(0,3,n,3,N)q^n$$
$$+\sum_{n\geq 0}M'(1,3,n,3,N)\zeta q^n+\sum_{n\geq 0}M'(2,3,n,3,N)\zeta^2 q^n$$

$$=\ A_0(q)\times\sum_{k\geq 0}\binom{k+3}{3}q^{18k}+\zeta A_1(q)\times\sum_{k\geq 0}\binom{k+3}{3}q^{18k}+\zeta^2 A_2(q)\times\sum_{k\geq 0}\binom{k+3}{3}q^{18k}. \tag{35}$$

Multiplying and collecting like terms from each of the series in the right side of (35), we are able to build three period 18 quasipolynomials.

Example 3.4. *Setting $(s,t)=(2,4)$, with $x=2$, so that $n=18k=4$ and $N=18j+10$, we compute the constituent $M'\big(1,3,18k-4,3,18j+10\big)$.*

$$\sum_{k\geq 1}M'\big(1,3,18k-4,3,18j+10\big)\zeta q^{18k-4}=\sum_{k\geq 1}M'\big(1,3,18(k-1)+14,3,18j+10\big)\zeta q^{18k-4}$$
$$=\zeta\left(8q^{14}+2q^{32}-10q^{50}-2q^{18j+14}-\cdots+10q^{54j+86}\right)\times\sum_{k\geq 1}\binom{k+2}{3}q^{18k}. \tag{36}$$

Hence, we arrive at

$$\begin{aligned}M'(1,3,18k-4,3,18j+10)&=8\tbinom{k+2}{3}+2\tbinom{k+1}{3}-10\tbinom{k}{3}-2\tbinom{k+2-j}{3}-36\tbinom{k+2-(j+1)}{3}\\&+24\tbinom{k+2-(j+2)}{3}+14\tbinom{k+2-(j+3)}{3}+10\tbinom{k+2-(2j+1)}{3}\\&+30\tbinom{k+2-(2j+2)}{3}-36\tbinom{k+2-(2j+3)}{3}-4\tbinom{k+2-(2j+4)}{3}\\&-8\tbinom{k+2-(3j+2)}{3}-2\tbinom{k+2-(3j+3)}{3}+10\tbinom{k+2-(3j+4)}{3}.\end{aligned} \tag{37}$$

By examining the three quasipolynomials for $M'(r,3,n,3,18j+6x-s)$ for $r=0,1,2$ we are able to show that for $x=0,1,2$ and the ordered pairs
$(s,t)\in\{(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),(3,5)\}$ the following constituents are equal

$$\boldsymbol{n=18k-t,\ N=18j-s}$$

$$\begin{aligned}&M'(0,3,18k-t,3,18j-s)=M'(1,3,18k-t,3,18j-s)=M'(2,3,18k-t,3,18j-s)\\&=-(t-12)\binom{k+2}{3}+2(t-3)\binom{k+1}{3}-(t+6)\binom{k}{3}\\&\quad+3(t-s-10)\binom{k-j+2}{3}-6(t-s-1)\binom{k-j+1}{3}+3(t-s+8)\binom{k-j}{3}\\&\quad-3(t-2s-8)\binom{k-2j+2}{3}+6(t-2s+1)\binom{k-2j+1}{3}-3(t-2s+10)\binom{k-2j}{3}\\&\quad+(t-3s-6)\binom{k-3j+2}{3}-2(t-3s+3)\binom{k-3j+1}{3}+(t-3s+12)\binom{k-3j}{3}\\&\qquad=\frac{p\big(18k-4,3,18j-s\big)}{3}\end{aligned} \tag{38}$$

$$\boldsymbol{n = 18k - t,\ N = 18j - s + 6}$$

$$\begin{aligned}
&M'(0,3,18k-t,3,18j-s+6) = M'(1,3,18k-t,3,18j-s+6) = M'(2,3,18k-t,3,18j-s+6)\\
&= -(t-12)\binom{k+2}{3} + 2(t-3)\binom{k+1}{3} - (t+6)\binom{k}{3}\\
&\quad + 2(t-s-7)\binom{k-j+2}{3} - 3(t-s+8)\binom{k-j+1}{3} + 36\binom{k-j}{3} + (t-s+2)\binom{k-j-1}{3}\\
&\quad - (t-2s-2)\binom{k-2j+2}{3} + 36\binom{k-2j+1}{3} + 3(t-2s-8)\binom{k-2j}{3} - 2(t-2s+7)\binom{k-2j-1}{3}\\
&\quad + (t-3s-6)\binom{k-3j+1}{3} - 2(t-3s+3)\binom{k-3j}{3} + (t-3s+12)\binom{k-3j-1}{3}\\
&\qquad = \frac{p\big(18k-3,3,18j-s+6\big)}{3} \qquad (39)
\end{aligned}$$

$$\boldsymbol{n = 18k - t,\ N = 18j - s + 12}$$

$$\begin{aligned}
&M'(0,3,18k-t,3,18j-s+12) = M'(1,3,18k-t,3,18j-s+12) = M'(2,3,18k-t,3,18j-s+12)\\
&= -(t-12)\binom{k+2}{3} + 2(t-3)\binom{k+1}{3} - (t+6)\binom{k}{3}\\
&\quad + (t-s-4)\binom{k-j+2}{3} - 36\binom{k-j+1}{3} - 3(t-s-10)\binom{k-j}{3} + 2(t-s+5)\binom{k-j-1}{3}\\
&\quad - 2(t-2s-5)\binom{k-2j+1}{3} + 3(t-2s+10)\binom{k-2j}{3} - 36\binom{k+2j-1}{3} - (t-2s+4)\binom{k-2j-2}{3}\\
&\quad + (t-3s-6)\binom{k-3j}{3} - 2(t-3s+3)\binom{k-3j-1}{3} + (t-3s+12)\binom{k-3j-2}{3}\\
&\qquad = \frac{p\big(18k-4,3,18j-s+12\big)}{3} \qquad (40)
\end{aligned}$$

Thus, (38), (39), and (40) together, amount to an analytic proof of the case $m = 3$ from Theorem 1.5. □

3.5 An Analytic Proof of case of Theorem 1.5 for $m = 4$.

In the case $m = 4$, since the crank is defined differently depending on whether or not $n \leqslant 2N$, we require the following proposition to deduce the truth of Theorem 1.5 in the case $n \geqslant 2N$ from the case when $n \leqslant 2N$.

Proposition 3.5. *If the coefficient of $z^r q^n$ in $f(z,q)$ is the number of partitions of n with largest part at most N, number of parts at most m, and $\lambda_a = r$, then the coefficient of $z^r q^{mN-n}$ in $f(z,q)$ is the number of partitions of n with largest part at most N, number of parts at most m, and $\lambda_{m+1-a} = N - r$.*

Proof. □

Analytic proof of Theorem 1.5 for $m = 4$. We follow the same procedure here that was done for the case $m = 3$ in Section 3.4, however, we claim 16 ordered pairs (s,t):

$$\begin{aligned}
(s,t) \in \{&(1,1),(1,2),(1,3),(2,2),(2,3),(2,4),(2,5),(2,6),\\
&(3,4),(3,5),(3,6),(3,7),(3,8),(4,7),(4,8),(4,9)\}.
\end{aligned}$$

For $n \leqslant 2N$ where the crank is λ_2, combinatorial arguments produce the generating function

$$f(z,q) = \sum_{n,r=0}^{\infty} M'(r,3,n,4,N) z^r q^n = \sum_{j=0}^{N} \left(q^j + q^{j+1} + \cdots + q^N\right) z^j q^j \begin{bmatrix} j+2 \\ 2 \end{bmatrix}. \tag{41}$$

We replace z with $\zeta = e^{2\pi i/3}$, a primitive third root of unity, into (41) so that we can split the generating function into three separate generating functions depending on ζ^0, ζ^1, and ζ^2. As in the analytic proof for the case $m = 3$, we produce three quasipolynomials. Taking Proposition 3.5 into consideration, the relevant constituents corresponding to each of the 16 ordered pairs (s,t) are compiled in Appendix A.

We note that our quasipolynomials are in terms of $N = 36j - s$ and $N = 36j + 18 - s$ giving us 32 expressions in Appendix A. Taken in pairs they prove the result. For example, the pair of constituents $M'\big(r,3,36k-7,4,36j-3\big)$ and $M'\big(r,3,36k-7,4,36j+15\big)$ in lines (64) and (65), show $M'\big(r,3,36k-7,4,18j-3\big) = \frac{p\left(36k-7,4,18j-3\right)}{3}$. This establishes (19). For a given k, taking j large enough establishes (20) and finally, taking differences of the constituents for different values of j we obtain (18). Thus, Theorem 1.5 is proved for $n \leqslant 2N$.

For $n \geq 2N$ where the crank is λ_3, we apply Proposition 3.5, and the other half of Theorem 1.5 follows.

□

Example 3.6. *Consider the Gaussian polynomial* $\begin{bmatrix} 232+4 \\ 4 \end{bmatrix}$. *It can be shown that*

$$\begin{aligned} p\big(36k-5,4,36j+16\big) \;=\; & 11\binom{3k+5}{3} + 50\binom{3k+4}{3} + 11\binom{3k+3}{3} \\ & -4\binom{3k+4-3j}{3} - 131\binom{3k+3-3j}{3} - 146\binom{3k+2-3j}{3} - 7\binom{3k+1-3j}{3} \\ & +55\binom{3k+2-6j}{3} + 286\binom{3k+1-6j}{3} + 91\binom{3k-6j}{3} \\ & -2\binom{3k+1-9j}{3} - 115\binom{3k-9j}{3} - 160\binom{3k-1-9j}{3} - 11\binom{3k-2-9j}{3} \\ & +6\binom{3k-1-12j}{3} + 48\binom{3k-2-12j}{3} + 18\binom{3k-3-12j}{3}. \end{aligned} \tag{42}$$

We note that $N = 232 = 36(6) + 16$. *With* (42), *we can compute the values* $p\big(36k-5,4,232\big)$ *for* $0 \leq k \leq 24$. *For example, we set* $k = 7$ *and* $j = 6$ *and compute*

$$p\big(283,4,232\big) = 161616.$$

Setting $k = 20$ *and* $j = 6$, *one may further compute*

$$p\big(751,4,232\big) = 41085.$$

Now, from Appendix A we examine (55) *and compute the values of* $M'\big(r,3,36k-$

$5, 4, 36j+16\big)$ *for* $0 \le k \le 24$.

$$M'\big(r,3,36k-5,4,36j+16\big) = \begin{cases} 107\binom{k+2}{3} + 434\binom{k+1}{3} + 107\binom{k}{3} - 49\binom{k+2-j}{3} & \text{for } 36k-5 \le 72j+32 \\ -1220\binom{k+1-j}{3} - 1265\binom{k-j}{3} - 58\binom{k-1-j}{3} & \text{crank } \lambda_2 \pmod 3 \\ \\ 126\binom{4j-k+4}{3} + 432\binom{4j-k+3}{3} + 90\binom{4j-k+2}{3} & \text{for } 36k-5 \ge 72j+32 \\ -68\binom{3j-k+4}{3} - 1309\binom{3j-k+3}{3} - 1174\binom{3j-k+2}{3} & \text{crank } \lambda_3 \pmod 3 \\ -41\binom{3j-k+1}{3} & \end{cases} \tag{55}$$

Setting $k=7$ *and* $j=6$ *in* (55), *we have*

$$M'\big(r,3,283,4,232\big) = 53872 = \frac{161616}{3} = \frac{p\big(283,4,232\big)}{3}.$$

In this case, since $n = 283 \le 464 = 2N$, *the crank is determined by* $\lambda_2 \pmod 3$.

Setting $k=20$ *and* $j=6$, *we compute*

$$M'\big(r,3,751,4,232\big) = 13695 = \frac{41085}{3} = \frac{p\big(751,4,232\big)}{3}.$$

In this case, however, since $n = 751 \ge 464 = 2N$, *the crank is determined by* $\lambda_3 \pmod 3$.

4 Future Work

Numerical evidence suggests that $\lambda_2 \pmod \ell$ is a crank witnessing the congruences of Theorem 1.2 and Corollary 1.3 for larger values of m and ℓ than what Theorem 1.5 implies. For example, in Theorem 1.2 let $\ell = 7, m = 4, k = j = 5$ with $s = 1$ and $t = 6$, or let $\ell = 5, m = 4, k = j = 7$ with $s = 1$ and $t = 6$. In either case, we are considering partitions of 414 into at most $m = 4$ parts, each part no bigger than 209. Hence, Theorem 1.2 is doubly satisfied, both modulo 5 and modulo 7:

$$p\big(414,4,209\big) = 262{,}675 = 7 \times 37{,}525 = 5 \times 52{,}535 \equiv 0 \pmod{35}. \tag{43}$$

Furthermore, we find that

$$M'\big(r,7,414,4,209\big) = 37{,}525 = \frac{p\big(414,4,209\big)}{7} \tag{44}$$

$$M'\big(r,5,414,4,209\big) = 52{,}535 = \frac{p\big(414,4,209\big)}{5} \tag{45}$$

for all r. Thus our crank $\lambda_2 \pmod \ell$ witnesses congruences from Theorem 1.2 in some cases where $\ell > 3$. This and other numerical evidence leads us to a conjecture.

Conjecture 4.1. *Let* ℓ *be any odd prime. For* $m = 2, 3$, *the crank* $\lambda_2 \pmod \ell$ *witnesses the congruences of Theorem 1.2 and Corollary 1.3 for all* n. *For* $m > 3$, *the crank* $\lambda_2 \pmod \ell$ *witnesses the congruences of Theorem 1.2 and Corollary 1.3 for* $n \le mN/2$. *For* $n \ge mN/2$, *the crank* $\lambda_{m-1} \pmod \ell$ *witnesses the congruences of Theorem 1.2 and Corollary 1.3.*

In contrast to Conjecture 4.1 about a single crank witnessing congruences for doubly restricted partition functions $p(n, m, N)$, in a forthcoming paper [5], it will be shown that for all odd primes ℓ, there are *two fundamentally different* cranks; the now familiar $\lambda_2 \pmod{\ell}$, and also $\lambda_1 - \lambda_{\ell+1} \pmod{\ell}$, both of which are witnesses for the congruences for partitions into at most m parts of Theorem 1.1. The crank $\lambda_1 - \lambda_{\ell+1} \pmod{\ell}$ does not appear to witness the congruences of Theorem 1.2 and Corollary 1.3 presented here.

5 Acknowledgments

A recently published result due to Dylan Pentland [11] establishes similar congruence properties for $p\big(n, m, N\big)$. However, Pentland's results do not coincide with the results presented here, nor do his methods reproduce them. It may be worthwhile to explore Pentland's methods with the goal of expanding the results of both papers.

Lastly, the authors would like to thank George Andrews for help on an earlier draft of this paper.

References

[1] G. E. Andrews. *The Theory of Partitions.* Cambridge University Press, 1998.
[2] G. E. Andrews and F. G. Garvan. Dyson's crank of a partition. *Bull. Amer. Math. Soc. (N.S.)*, 18(2):167–171, 1988.
[3] A. O. L. Atkin and P. Swinnerton-Dyer. Some properties of partitions. *Proc. London Math. Soc. (3)*, 4:84–106, 1954.
[4] A. Castillo, A. Hernandez, S. Flores, B. Kronholm, A. Larsen, and A. Martinez. Quasipolynomials and maximal coefficients of gaussian polynomials. *Ann. Comb.*, 2019.
[5] B. K. Dennis Eichhorn and A. Larsen. Multiple combinatorial witnesses and cranks for restricted partition functions.
[6] F. J. Dyson. Some guesses in the theory of partitions. *Eureka*, (8):10–15, 1944.
[7] B. Kronholm. On congruence properties of $p(n, m)$. *Proc. Amer. Math. Soc.*, 133(10):2891–2895, April 2005.
[8] B. Kronholm. On congruence properties of consecutive values of p(n,m). *INTEGERS*, #A16, 2007.
[9] B. Kronholm. A result on ramanujan-like congruence properties of the restricted partition function p(n, m) across both variables. *INTEGERS*, (#A63):1–6, November 2012.
[10] K. Ono. Distribution of the partition function modulo m. *Annals of Mathematics*, 151(1):293–307, 2000.
[11] D. Pentland. Coefficients of gaussian polynomials modulo n. *The Electronic Journal of Combinatorics*, 27(2), 2020.
[12] S. Ramanujan. *Collected papers of Srinivasa Ramanujan.* AMS Chelsea Publishing, Providence, RI, 2000. Edited by G. H. Hardy, P. V. Seshu Aiyar and B. M. Wilson, Third printing of the 1927 original, With a new preface and commentary by Bruce C. Berndt.

A Constituents for $M'\big(r,3,36k-t,4,18j-s\big)$

There are 32 relevant constituents required the proof of Theorem 1.5 for the case $m=4$. For $n \le 2N$, the crank is $\lambda_2 \pmod 3$ and for $n \ge 2N$, the crank is $\lambda_3 \pmod 3$.

A.1 $s=1$, $1 \le t \le 3$

$$M'\big(r,3,36k-3,4,36j-1\big) = \begin{cases} 126\binom{k+2}{3} + 432\binom{k+1}{3} + 90\binom{k}{3} & \text{for } 36k-3 \le 72j-2 \\ -449\binom{k+2-j}{3} - 1730\binom{k+1-j}{3} - 413\binom{k-j}{3} & \\ & \\ 147\binom{4j-k+2}{3} + 426\binom{4j-k+1}{3} + 75\binom{4j-k}{3} & \text{for } 36k-3 \ge 72j-2 \\ -527\binom{3j-k+2}{3} - 1718\binom{3j-k+1}{3} - 347\binom{3j-k}{3} & \end{cases} \tag{46}$$

$$M'\big(r,3,36k-3,4,36j+17\big) = \begin{cases} 126\binom{k+2}{3} + 432\binom{k+1}{3} + 90\binom{k}{3} - 58\binom{k+2-j}{3} & \text{for } 36k-3 \le 72j+34 \\ -1265\binom{k+1-j}{3} - 1220\binom{k-j}{3} - 49\binom{k-1-j}{3} & \\ & \\ 147\binom{4j-k+4}{3} + 426\binom{4j-k+3}{3} + 75\binom{4j-k+2}{3} & \text{for } 36k-3 \ge 72j+34 \\ -79\binom{3j-k+4}{3} - 1352\binom{3j-k+3}{3} - 1127\binom{3j-k+2}{3} & \\ -34\binom{3j-k+1}{3} & \end{cases} \tag{47}$$

$$M'\big(r,3,36k-2,4,36j-1\big) = \begin{cases} 137\binom{k+2}{3} + 432\binom{k+1}{3} + 83\binom{k}{3} & \text{for all } k \\ -487\binom{k+2-j}{3} - 1726\binom{k+1-j}{3} - 379\binom{k-j}{3} & \end{cases} \tag{48}$$

$$M'\big(r,3,36k-2,4,36j+17\big) = \begin{cases} 137\binom{k+2}{3} + 432\binom{k+1}{3} + 83\binom{k}{3} - 68\binom{k+2-j}{3} & \text{for all } k \\ -1309\binom{k+1-j}{3} - 1174\binom{k-j}{3} - 41\binom{k-1-j}{3} & \end{cases} \tag{49}$$

$$M'\big(r,3,36k-1,4,36j-1\big) = \begin{cases} 147\binom{k+2}{3} + 426\binom{k+1}{3} + 75\binom{k}{3} & \text{for } 36k-1 \le 72j-2 \\ -527\binom{k+2-j}{3} - 1718\binom{k+1-j}{3} - 347\binom{k-j}{3} & \\ & \\ 126\binom{4j-k+2}{3} + 432\binom{4j-k+1}{3} + 90\binom{4j-k}{3} & \text{for } 36k-1 \ge 72j-2 \\ -449\binom{3j-k+2}{3} - 1730\binom{3j-k+1}{3} - 413\binom{3j-k}{3} & \end{cases} \tag{50}$$

$$M'\big(r,3,36k-1,4,36j+17\big)$$
$$=\begin{cases}147\binom{k+2}{3}+426\binom{k+1}{3}+75\binom{k}{3}-79\binom{k+2-j}{3} & \text{for } 36k-1\le 72j+34\\ -1352\binom{k+1-j}{3}-1127\binom{k-j}{3}-34\binom{k-1-j}{3} & \\ & \\ 126\binom{4j-k+4}{3}+432\binom{4j-k+3}{3}+90\binom{4j-k+2}{3} & \text{for } 36k-1\ge 72j+34\\ -58\binom{3j-k+4}{3}-1265\binom{3j-k+3}{3}-1220\binom{3j-k+2}{3} & \\ -49\binom{3j-k+1}{3} & \end{cases} \tag{51}$$

A.2 $s=2$**,** $2\le t\le 6$

$$M'\big(r,3,36k-6,4,36j-2\big)$$
$$=\begin{cases}99\binom{k+2}{3}+432\binom{k+1}{3}+117\binom{k}{3} & \text{for } 36k-6\le 72j-4\\ -379\binom{k+2-j}{3}-1726\binom{k+1-j}{3}-487\binom{k-j}{3} & \\ & \\ 137\binom{4j-k+2}{3}+428\binom{4j-k+1}{3}+83\binom{4j-k}{3} & \text{for } 36k-6\ge 72j-4\\ -527\binom{3j-k+2}{3}-1718\binom{3j-k+1}{3}-347\binom{3j-k}{3} & \end{cases} \tag{52}$$

$$M'\big(r,3,36k-6,4,36j+16\big)$$
$$=\begin{cases}99\binom{k+2}{3}+432\binom{k+1}{3}+117\binom{k}{3}-41\binom{k+2-j}{3} & \text{for } 36k-6\le 72j+32\\ -1174\binom{k+1-j}{3}-1309\binom{k-j}{3}-68\binom{k-1-j}{3} & \\ & \\ 137\binom{4j-k+4}{3}+428\binom{4j-k+3}{3}+83\binom{4j-k+2}{3} & \text{for } 36k-6\ge 72j+32\\ -79\binom{3j-k+4}{3}-1352\binom{3j-k+3}{3}-1127\binom{3j-k+2}{3} & \\ -34\binom{3j-k+1}{3} & \end{cases} \tag{53}$$

$$M'\big(r,3,36k-5,4,36j-2\big)$$
$$=\begin{cases}107\binom{k+2}{3}+434\binom{k+1}{3}+107\binom{k}{3} & \text{for } 36k-5\le 72j-4\\ -413\binom{k+2-j}{3}-1730\binom{k+1-j}{3}-449\binom{k-j}{3} & \\ & \\ 126\binom{4j-k+2}{3}+432\binom{4j-k+1}{3}+90\binom{4j-k}{3} & \text{for } 36k-5\ge 72j-4\\ -487\binom{3j-k+2}{3}-1726\binom{3j-k+1}{3}-379\binom{3j-k}{3} & \end{cases} \tag{54}$$

$$M'\big(r,3,36k-5,4,36j+16\big)$$
$$=\begin{cases}107\binom{k+2}{3}+434\binom{k+1}{3}+107\binom{k}{3}-49\binom{k+2-j}{3} & \text{for } 36k-5\le 72j+32\\ -1220\binom{k+1-j}{3}-1265\binom{k-j}{3}-58\binom{k-1-j}{3} & \\ & \\ 126\binom{4j-k+4}{3}+432\binom{4j-k+3}{3}+90\binom{4j-k+2}{3} & \text{for } 36k-5\ge 72j+32\\ -68\binom{3j-k+4}{3}-1309\binom{3j-k+3}{3}-1174\binom{3j-k+2}{3} & \\ -41\binom{3j-k+1}{3} & \end{cases} \tag{55}$$

$$M'\big(r,3,36k-4,4,36j-2\big)=\begin{cases}117\binom{k+2}{3}+432\binom{k+1}{3}+99\binom{k}{3} & \text{for all } k\\ -449\binom{k+2-j}{3}-1730\binom{k+1-j}{3}-413\binom{k-j}{3}\end{cases} \tag{56}$$

$$M'\big(r,3,36k-4,4,36j+16\big)$$
$$=\begin{cases}117\binom{k+2}{3}+432\binom{k+1}{3}+99\binom{k}{3}-58\binom{k+2-j}{3} & \text{for all } k\\ -1256\binom{k+1-j}{3}-1220\binom{k-j}{3}-49\binom{k-1-j}{3}\end{cases} \tag{57}$$

$$M'\big(r,3,36k-3,4,36j-2\big)$$
$$=\begin{cases}126\binom{k+2}{3}+432\binom{k+1}{3}+90\binom{k}{3} & \text{for } 36k-3\le 72j-4\\ -487\binom{k+2-j}{3}-1726\binom{k+1-j}{3}-379\binom{k-j}{3} & \\ & \\ 107\binom{4j-k+2}{3}+434\binom{4j-k+1}{3}+107\binom{4j-k}{3} & \text{for } 36k-3\ge 72j-4\\ -413\binom{3j-k+2}{3}-1730\binom{3j-k+1}{3}-449\binom{3j-k}{3} & \end{cases} \tag{58}$$

$$M'\big(r,3,36k-3,4,36j+16\big)$$
$$=\begin{cases}126\binom{k+2}{3}+432\binom{k+1}{3}+90\binom{k}{3}-68\binom{k+2-j}{3} & \text{for } 36k-3\le 72j+32\\ -1309\binom{k+1-j}{3}-1174\binom{k-j}{3}-41\binom{k-1-j}{3} & \\ & \\ 107\binom{4j-k+4}{3}+434\binom{4j-k+3}{3}+107\binom{4j-k+2}{3} & \text{for } 36k-3\ge 72j+32\\ -49\binom{3j-k+4}{3}-1220\binom{3j-k+3}{3}-1265\binom{3j-k+2}{3} & \\ -58\binom{3j-k+1}{3} & \end{cases} \tag{59}$$

$$M'\big(r,3,36k-2,4,36j-2\big)$$
$$=\begin{cases}137\binom{k+2}{3}+428\binom{k+1}{3}+83\binom{k}{3} & \text{for } 36k-2\le 72j-4\\ -527\binom{k+2-j}{3}-1718\binom{k+1-j}{3}-347\binom{k-j}{3} & \\ & \\ 99\binom{4j-k+2}{3}+432\binom{4j-k+1}{3}+117\binom{4j-k}{3} & \text{for } 36k-2\ge 72j-4\\ -379\binom{3j-k+2}{3}-1726\binom{3j-k+1}{3}-487\binom{3j-k}{3} & \end{cases} \tag{60}$$

$$M'\big(r,3,36k-2,4,36j+16\big)$$
$$=\begin{cases}137\binom{k+2}{3}+428\binom{k+1}{3}+83\binom{k}{3}-79\binom{k+2-j}{3} & \text{for } 36k-2\le 72j+32\\ -1352\binom{k+1-j}{3}-1127\binom{k-j}{3}-34\binom{k-1-j}{3} & \\ & \\ 99\binom{4j-k+4}{3}+432\binom{4j-k+3}{3}+117\binom{4j-k+2}{3} & \text{for } 36k-2\ge 72j+32\\ -41\binom{3j-k+4}{3}-1174\binom{3j-k+3}{3}-1309\binom{3j-k+2}{3} & \\ -68\binom{3j-k+1}{3} & \end{cases} \tag{61}$$

A.3 $s = 3$, $4 \le t \le 8$

$$M'\big(r,3,36k-8,4,36j-3\big) = \begin{cases} 83\binom{k+2}{3} + 428\binom{k+1}{3} + 137\binom{k}{3} & \text{for } 36k-8 \le 72j-6 \\ -347\binom{k+2-j}{3} - 1718\binom{k+1-j}{3} - 527\binom{k-j}{3} & \\ & \\ 117\binom{4j-k+2}{3} + 432\binom{4j-k+1}{3} + 99\binom{4j-k}{3} & \text{for } 36k-8 \ge 72j-6 \\ -487\binom{3j-k+2}{3} - 1726\binom{3j-k+1}{3} - 379\binom{3j-k}{3} & \end{cases} \tag{62}$$

$$M'\big(r,3,36k-8,4,36j+15\big) = \begin{cases} 83\binom{k+2}{3} + 428\binom{k+1}{3} + 137\binom{k}{3} - 34\binom{k+2-j}{3} & \text{for } 36k-8 \le 72j+30 \\ -1127\binom{k+1-j}{3} - 1352\binom{k-j}{3} - 79\binom{k-1-j}{3} & \\ & \\ 117\binom{4j-k+4}{3} + 432\binom{4j-k+3}{3} + 99\binom{4j-k+2}{3} & \text{for } 36k-8 \ge 72j+30 \\ -68\binom{3j-k+4}{3} - 1309\binom{3j-k+3}{3} - 1174\binom{3j-k+2}{3} & \\ -41\binom{3j-k+1}{3} & \end{cases} \tag{63}$$

$$M'\big(r,3,36k-7,4,36j-3\big) = \begin{cases} 90\binom{k+2}{3} + 432\binom{k+1}{3} + 126\binom{k}{3} & \text{for } 36k-7 \le 72j-6 \\ -379\binom{k+2-j}{3} - 1726\binom{k+1-j}{3} - 487\binom{k-j}{3} & \\ & \\ 107\binom{4j-k+2}{3} + 434\binom{4j-k+1}{3} + 107\binom{4j-k}{3} & \text{for } 36k-7 \ge 72j-6 \\ -449\binom{3j-k+2}{3} - 1730\binom{3j-k+1}{3} - 413\binom{3j-k}{3} & \end{cases} \tag{64}$$

$$M'\big(r,3,36k-7,4,36j+15\big) = \begin{cases} 90\binom{k+2}{3} + 432\binom{k+1}{3} + 126\binom{k}{3} - 41\binom{k+2-j}{3} & \text{for } 36k-7 \le 72j+30 \\ -1174\binom{k+1-j}{3} - 1309\binom{k-j}{3} - 68\binom{k-1-j}{3} & \\ & \\ 107\binom{4j-k+4}{3} + 434\binom{4j-k+3}{3} + 107\binom{4j-k+2}{3} & \text{for } 36k-7 \ge 72j+30 \\ -58\binom{3j-k+4}{3} - 1265\binom{3j-k+3}{3} - 1220\binom{3j-k+2}{3} & \\ -49\binom{3j-k+1}{3} & \end{cases} \tag{65}$$

$$M'\big(r,3,36k-6,4,36j-3\big) = \begin{cases} 99\binom{k+2}{3} + 432\binom{k+1}{3} + 117\binom{k}{3} & \text{for all } k \\ -413\binom{k+2-j}{3} - 1730\binom{k+1-j}{3} - 449\binom{k-j}{3} & \end{cases} \tag{66}$$

$$M'\big(r,3,36k-6,4,36j+15\big) = \begin{cases} 99\binom{k+2}{3} + 432\binom{k+1}{3} + 117\binom{k}{3} - 49\binom{k+2-j}{3} & \text{for all } k \\ -1220\binom{k+1-j}{3} - 1265\binom{k-j}{3} - 58\binom{k-1-j}{3} & \end{cases} \tag{67}$$

$$M'\big(r,3,36k-5,4,36j-3\big)$$
$$=\begin{cases}107\binom{k+2}{3}+434\binom{k+1}{3}+107\binom{k}{3} & \text{for } 36k-5\le 72j-6\\ -449\binom{k+2-j}{3}-1730\binom{k+1-j}{3}-413\binom{k-j}{3} & \\ & \\ 90\binom{4j-k+2}{3}+432\binom{4j-k+1}{3}+126\binom{4j-k}{3} & \text{for } 36k-5\ge 72j-6\\ -379\binom{3j-k+2}{3}-1726\binom{3j-k+1}{3}-487\binom{3j-k}{3} & \end{cases} \tag{68}$$

$$M'\big(r,3,36k-5,4,36j+15\big)$$
$$=\begin{cases}107\binom{k+2}{3}+434\binom{k+1}{3}+107\binom{k}{3}-58\binom{k+2-j}{3} & \text{for } 36k-5\le 72j+30\\ -1265\binom{k+1-j}{3}-1220\binom{k-j}{3}-49\binom{k-1-j}{3} & \\ & \\ 90\binom{4j-k+4}{3}+432\binom{4j-k+3}{3}+126\binom{4j-k+2}{3} & \text{for } 36k-5\ge 72j+30\\ -41\binom{3j-k+4}{3}-1174\binom{3j-k+3}{3}-1309\binom{3j-k+2}{3} & \\ -68\binom{3j-k+1}{3} & \end{cases} \tag{69}$$

$$M'\big(r,3,36k-4,4,36j-3\big)$$
$$=\begin{cases}117\binom{k+2}{3}+432\binom{k+1}{3}+99\binom{k}{3} & \text{for } 36k-4\le 72j-6\\ -487\binom{k+2-j}{3}-1726\binom{k+1-j}{3}-379\binom{k-j}{3} & \\ & \\ 83\binom{4j-k+2}{3}+428\binom{4j-k+1}{3}+137\binom{4j-k}{3} & \text{for } 36k-4\ge 72j-6\\ -347\binom{3j-k+2}{3}-1718\binom{3j-k+1}{3}-527\binom{3j-k}{3} & \end{cases} \tag{70}$$

$$M'\big(r,3,36k-4,4,36j+15\big)$$
$$=\begin{cases}117\binom{k+2}{3}+432\binom{k+1}{3}+99\binom{k}{3}-68\binom{k+2-j}{3} & \text{for } 36k-4\le 72j+30\\ -1309\binom{k+1-j}{3}-1174\binom{k-j}{3}-41\binom{k-1-j}{3} & \\ & \\ 83\binom{4j-k+4}{3}+428\binom{4j-k+3}{3}+137\binom{4j-k+2}{3} & \text{for } 36k-4\ge 72j+30\\ -34\binom{3j-k+4}{3}-1127\binom{3j-k+3}{3}-1352\binom{3j-k+2}{3} & \\ -79\binom{3j-k+1}{3} & \end{cases} \tag{71}$$

A.4 $s=4$, $7\le t\le 9$

$$M'\big(r,3,36k-9,4,36j-4\big)$$
$$=\begin{cases}75\binom{k+2}{3}+426\binom{k+1}{3}+147\binom{k}{3} & \text{for } 36k-9\le 72j-8\\ -347\binom{k+2-j}{3}-1718\binom{k+1-j}{3}-527\binom{k-j}{3} & \\ & \\ 90\binom{4j-k+2}{3}+432\binom{4j-k+1}{3}+126\binom{4j-k}{3} & \text{for } 36k-9\ge 72j-8\\ -413\binom{3j-k+2}{3}-1730\binom{3j-k+1}{3}-449\binom{3j-k}{3} & \end{cases} \tag{72}$$

$$M'\big(r,3,36k-9,4,36j+14\big)$$
$$=\begin{cases}75\binom{k+2}{3}+426\binom{k+1}{3}+147\binom{k}{3}-34\binom{k+2-j}{3} & \text{for } 36k-9\le 72j+28\\ -1127\binom{k+1-j}{3}-1352\binom{k-j}{3}-79\binom{k-1-j}{3} & \\ & \\ 90\binom{4j-k+4}{3}+432\binom{4j-k+3}{3}+126\binom{4j-k+2}{3} & \text{for } 36k-9\ge 72j+28\\ -49\binom{3j-k+4}{3}-1220\binom{3j-k+3}{3}-1265\binom{3j-k+2}{3} & \\ -58\binom{3j-k+1}{3} & \end{cases} \tag{73}$$

$$M'\big(r,3,36k-8,4,36j-4\big)=\begin{cases}83\binom{k+2}{3}+428\binom{k+1}{3}+137\binom{k}{3} & \text{for all } k\\ -379\binom{k+2-j}{3}-1726\binom{k+1-j}{3}-487\binom{k-j}{3} & \end{cases} \tag{74}$$

$$M'\big(r,3,36k-8,4,36j+14\big)$$
$$=\begin{cases}83\binom{k+2}{3}+428\binom{k+1}{3}+137\binom{k}{3}-41\binom{k+2-j}{3} & \text{for all } k\\ -1174\binom{k+1-j}{3}-1309\binom{k-j}{3}-68\binom{k-1-j}{3} & \end{cases} \tag{75}$$

$$M'\big(r,3,36k-7,4,36j-4\big)$$
$$=\begin{cases}90\binom{k+2}{3}+432\binom{k+1}{3}+126\binom{k}{3} & \text{for } 36k-7\le 72j-8\\ -413\binom{k+2-j}{3}-1730\binom{k+1-j}{3}-449\binom{k-j}{3} & \\ & \\ 75\binom{4j-k+2}{3}+426\binom{4j-k+1}{3}+147\binom{4j-k}{3} & \text{for } 36k-7\ge 72j-8\\ -347\binom{3j-k+2}{3}-1718\binom{3j-k+1}{3}-527\binom{3j-k}{3} & \end{cases} \tag{76}$$

$$M'\big(r,3,36k-7,4,36j+14\big)$$
$$=\begin{cases}90\binom{k+2}{3}+432\binom{k+1}{3}+126\binom{k}{3}-49\binom{k+2-j}{3} & \text{for } 36k-7\le 72j+28\\ -1220\binom{k+1-j}{3}-1265\binom{k-j}{3}-58\binom{k-1-j}{3} & \\ & \\ 75\binom{4j-k+4}{3}+426\binom{4j-k+3}{3}+147\binom{4j-k+2}{3} & \text{for } 36k-9\ge 72j+28\\ -34\binom{3j-k+4}{3}-1127\binom{3j-k+3}{3}-1352\binom{3j-k+2}{3} & \\ -79\binom{3j-k+1}{3} & \end{cases} \tag{77}$$