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The generic existence of a core for q-rules*

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Summary. A q-rule is where a winning coalition has q or more of the n voters. It is important to understand when, generically, core points exist; that is, when does the core exist in other than highly contrived settings? As known, the answer depends upon the dimension of issue space. McKelvey and Schofield found bounds on these dimensions, but Banks found a subtle, critical error in their proofs. The sharp dimensional values along with results about the structure of the core are derived here. It is interesting how these dimensional values correspond to the number of issues that are needed to lure previously supporting voters into a new coalition.

JEL Classification Numbers: C7.

1 Introduction

A "q-rule" is where a winning coalition has at least q of the n voters. This makes the majority vote a $q = \left[\frac{n}{2}\right] + 1$ rule where $[x]$ is the "greatest integer function." The standard assumption that "if C is a winning coalition then the voters not in C cannot form another winning coalition" restricts q to range between the majority and unanimity ($q = n$) rules.

In order for q-rules to be useful for economics and spatial voting, we need to understand when they are stable. To illustrate with the selection of a pope for the Catholic Church, several times when a simple majority of the eligible Cardinals $\left(q = \left[\frac{n}{2}\right] + 1\right)$ sufficed, the precarious instability of the system caused the church to erupt into dissension and conflict with a pope and anti-pope vying for power. To achieve stability, in 1179 the Third Lateran

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Council adopted the current two-thirds $\left(q=\left[\frac{2n}{3}\right]+1\right)$ rule (Saari, [S1, p 15–16]).

A way to explain the resulting stability of the two-thirds rule is to use the *core*. Recall, $\mathbf{x}$ is a core point with a q-rule if, for all other proposals $\mathbf{y}$, it is impossible to find q voters who prefer $\mathbf{y}$ to $\mathbf{x}$; the core is the set of all core points. The flavor of this definition is captured from the pope selection problem by R. Kieckhefer's explanation for the two-thirds procedure ([S1, p. 16]). As he argues, for a candidate ($\mathbf{y}$) to replace a pope ($\mathbf{x}$), the pope would have to bungle affairs sufficiently badly on enough issues to alienate at least half of his original supporters. Eventually this happened.

Kieckhefer's explanation suggests that an increase in the number of concerns can threaten the stability of a q-rule. To express this mathematically, an issue defines a direction in "issue space," so the number of concerns is measured by k, the dimension of issue space. The sense is that stability is jeopardized with large enough k values. As developed here, this is accurate.

To see what can happen, let $\mathbf{x}_j$ be the ideal point for the jth agent with utility function $u_j(\mathbf{y}) = -\|\mathbf{y}-\mathbf{x}_j\|$. In words, with Euclidean distance preferences, the closer a point is to her ideal point, the more she likes it. For n odd, $q=\left[\frac{n}{2}\right]+1$, and $k=1$, the median voter's ideal point, $\mathbf{x}_{med}$, is the only core point. This standard fact is easy to prove. When the ideal points are placed on a line, $\left[\frac{n}{2}\right]$ of them are on each side of the median voter's bliss point $\mathbf{x}_{med}$ (the global maximum for her utility function). Any other alternative, $\mathbf{z}$, is on one side of $\mathbf{x}_{med}$ which means that the median voter and all voters on the other side of $\mathbf{x}_{med}$ prefer $\mathbf{x}_{med}$ to $\mathbf{z}$. Notice the robustness of this argument; it holds even if all voters' ideal points are altered.

Bliss points that are core points always exist for $k \geqq 2$, n odd, and $q=\left[\frac{n}{2}\right]+1$. A trivial example is unanimity where all ideal points agree. But this situation is so highly unlikely that it does nothing to justify the core. So, consider the Plott [P] construction of pairing voters' ideal points. Namely, place the first agent's ideal point, $\mathbf{x}_1$, somewhere in R^k and pass $\left[\frac{n}{2}\right]$ lines through $\mathbf{x}_1$. Each line is divided by $\mathbf{x}_1$ into two sides; place a voter's ideal point on each segment. (This is depicted in Fig. 1 for $k=2$). Any line through $\mathbf{x}_1$ separates the ideal points so that no more than $\left[\frac{n}{2}\right]$ of them are on each side. Thus, proving that $\mathbf{x}_1$ is a core point mimics the "median voter proof." Notice that if $\mathbf{x}$ is a core point for a q-rule, it is a core point for a q_1-rule where $q_1 > q$. (If q voters cannot be found to vote against $\mathbf{x}$, then it is impossible to find more that are willing to do so.) Consequently, Plott's construction establishes the existence of a core point for all q-rules for odd values of n and $k \leqq \frac{n-1}{2}$.

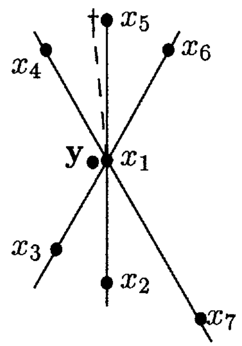

Figure 1. Plott's construction.

While Plott's pairing of ideal points ensures that $\mathbf{x}_1$ is a core point for the majority rule, it is not robust. To see why, in Fig. 1 I slightly varied voter five's preferences as denoted by the dagger on the dashed line. With this new configuration, $\mathbf{y}$ is preferred by a majority (voters two through five) to $\mathbf{x}_1$. To prove this, construct the convex hull of the ideal points of voters two through five, and draw a line perpendicular to an edge of the hull. When this coalition compares two points on the drawn line, each voter prefers the point closer to the hull. As $\mathbf{y}$ and $\mathbf{x}_1$ are on such a line, the conclusion follows.

Actually, the perturbed alignment has no core because the convex hulls for all four-voter coalitions have no common point. So, for any $\mathbf{x}$ a decisive coalition prefers a different $\mathbf{y}$. Consequently Plott's construction, which establishes the existence of a core point $\left(\text{for all } q\text{-rule for odd values of } n \text{ and } k \leq \frac{n-1}{2}\right)$, fails to indicate the general situation; his core points need not survive even the slightest change in preferences. As this fickle behavior requires larger k values, we need to find the dimensions of issue space which permit robust existence assertions. In mathematical terms, for a given (n, q)-rule, we seek those k values where, generically, the core is empty, and those k values where core points exist for an open set of preferences (i.e., they exist after any slight perturbation of preferences.) I resolve this problem for all n, q rules.

Because of its importance, this problem has received considerable research attention. Using a singularity theory argument (that carefully embeds various classes of singularities), Schofield [Sc] and then McKelvey and Schofield [MS1] asserted that the core is generically empty for certain k values. Because of their embedding approach, they computed the issue space dimension for an object different from the core. So, their conclusion describes only a subset of the relevant k values, but it was correctly greeted as a major advance.

Banks [B] found an error in these papers which, unfortunately, invalidated the conclusions and reopened this problem. (Banks also used the embedding approach.) I close this problem by finding the correct, sharp values. It is interesting that these values admit an interpretation compatible with Kieckhefer's comments about the pope problem. Moreover, the approach developed here can be used to derive related conclusions. I illustrate this by finding the k-values for Euclidean preferences. Also, as Salles correctly pointed out, it takes only minor modifications of this approach to answer these dimensional questions for games expressed in a more general format. In a third direction, Schofield suggested pointing out that while I consider only k-dimensional Euclidean spaces, all of my results and techniques extend immediately to k-dimensional smooth manifolds.

My improvements require characterizing the geometry of "core- singularities" (Prop. 1) in order to apply singularity theory to the "correct" geometry. But, while my approach differs significantly from that of [Sc], [MS1], and [B], Schofield was the first to use singularity theory to analyze cores. Earlier, however, Smale [Sm] and Saari and Simon [SS] used the same singularity approach to analyze closely related issues such as Pareto points.

As for creating examples, while there may be papers in addition to [P] and McKelvey and Schofield [MS2], I am unaware of any paper proposing general assertions about robust existence. Thus, the assertions and the approach developed in Sect. 5, are the first general results which are known to be best possible. Indeed, as Schofield called to my attention, the approach developed in Sect. 5 finally answers a question [Sc2] he raised about the construction of structurally stable cores.

2 Dimensions for the existence of the core

Assume that each agent has C^∞ smooth, strictly convex preferences. Namely, for each $\mathbf{x} \in R^k$, the set

$$M(\mathbf{x}) = \{\mathbf{y} \in R^k \mid u(\mathbf{y}) \geqq u(\mathbf{x})\} \tag{2.1}$$

is strictly convex. In the following theorem, "generically" means a residual set (i.e., a countable intersection of open-dense sets) of utility functions where the function space has the Whitney C^∞ topology (see Golubitsky and Guillemin [GG] and Saari and Simon [SS]). (When issues are restricted to a compact subset of R^k, the [SS] results allow "generic" to be replaced with "open-dense.") For a first reading, when "generic" refers to nonexistence, interpret it as meaning "everything except improbable, carefully concocted examples which are not indicative of what can happen because the conclusion can fail with even a slight change in the preferences." When "generic" describes existence, it means that examples exist where the conclusion holds even after an example is slightly modified; they hold for a "C^2 open set" in the Whitney topology. (The utility functions can be slightly perturbed along with its first and second derivatives and the conclusions remain.) A key phrase is that "examples exist;" this does not state that for any open set of utility functions, an example can be found. Indeed, creating examples is one of the difficulties addressed by this paper. Although the "smoothness" conditions can be relaxed (particularly when one considers preference profiles), I prefer to emphasize the structure of the core.

To simplify notation, call a core point that is some agent's bliss point a "bliss-core point." Other core points are called "nonbliss-core points." While the statements specify strictly convex preferences, they extend to all smooth preferences. (The interested reader can consult [SS] to see how to do this and how the conclusions change.) An interpretation of these results follows the statement of the theorem.

Theorem 1. *a. For a q-rule,* $\frac{n}{2} < q < n$, *bliss-core points*[1] *generically exist iff*

$$k \leqq 2q - n. \tag{2.2}$$

b. For any k and n, there exists a q-rule where core points exist generically. Indeed, if $n = q$ *(the unanimity rule), then, for any* $k \geqq 1$, *there exist open sets of preferences with bliss and nonbliss core points.*

c. Nonbliss core points exist, generically, for $k \leqq 2$ *when* $q = 3$, $n = 4$. *For* $n \geqq 5$, *let the "excess size of issue space dimension" be* $\beta = k - [2q - n]$. *Generically, nonbliss core points exist for a q-rule with* $\beta \geqq 0$ *iff* β *satisfies*

$$\frac{1}{2\beta + 4} + n\left(1 - \frac{1}{2\beta + 4}\right) \leqq q. \tag{2.3}$$

that is, iff $0 \leqq \beta \leqq \frac{4q - 3n - 1}{2(n - q)}$. *In this case, nonbliss-core points generically exist*[2] *iff*

$$k \leqq 2q - n + \frac{4q - 3n - 1}{2(n - q)}. \tag{2.4}$$

If Eq. 2.3 cannot be satisfied with a $\beta \geqq 0$, *then, generically, nonbliss-core points exist iff* $k \leqq 2q - n - 1$.

d. Consider the ratio $\alpha = \frac{q}{n}$ *where a candidate must receive at least* α *of the vote to be selected. For a given* α *rule,* $\frac{1}{2} < \alpha \leqq 1$, *and a specified dimension k, there exists a positive integer* n_0 *so that for all* $n \geqq n_0$, *generically, the core is nonempty.*

It is amusing to learn that the restrictions on the dimension k are closely connected with the representation of the q rule. To see this, start with the simple majority rule $q = \left[\frac{n}{2}\right] + 1$ where n is even. Just by expressing this $q = \frac{n}{2} + 1$ value as $2q - n = 2$, we obtain Eq. 2.2. Thus, robust, simple majority

[1] Remember, existence means that robust examples can be found. Incidentally, an assertion that this dimensional restriction is necessary is stated in [MS1]; they could not prove that it is a sufficient condition. We prove here that this estimate is both necessary and sufficient.

[2] This β correction terms is one aspect of what is needed to go beyond the [MS] results to obtain a full description of the issue space dimensions.

examples with a bliss point exist up to two-dimensional issue spaces. The first assertion of part c shows the simple majority rule also admits a nonbliss core point with two issues. However, once $n \geqq 6$, the right-hand side of Eq. 2.4 is $2 - \frac{n-3}{n-1} < 2$, so, for even $n > 4$ values, nonbliss simple majority core points exist (generically) only for a single issue.

Compare these conclusions with the behavior for odd n values. The $q = \frac{n+1}{2}$ value, expressed as $2q - n = 1$, again is Eq. 2.2, so with odd numbers of voters robust bliss-core point exist only for a single-dimensional issue space. For nonbliss core points, the dimensional bound is (Eq. 2.4) $1 - \frac{n-1}{n-1} = 0$; generically, simple majority nonbliss core points fail to exist for odd values of n.

To better understand this behavior, notice from Eqs. 2.2, 2.4 that $2q - n$ is a stability measure; the larger the value, the more we can expect from core points. (This is further developed in Thm. 2.) As the minimum value for this measure occurs when n is odd and $q = \left[\frac{n}{2}\right] + 1$, this recaptures the assertion that with odd numbers of voters, simple majority core points have a precarious existence.

As a way to understand the loss of stability, it seems reasonable to expect someone who voted for a proposal to defect only when presented an offer that cannot be refused. Suppose that each defecting voter needs a particular issue and that the voters have independent beliefs. (This independence plays an important role in the mathematics of Sect. 6.) This suggests that the maximal dimension of issue space allowing stability should roughly agree with the number of voters that need to defect to change the outcome. To relate this intuition to the $2q - n$ measure, observe that if k is the maximal admissible dimension of issue space, Eq. 2.2 becomes

$$2\left(q - \frac{n}{2}\right) = k. \tag{2.5}$$

Namely, a crude measure of the maximum dimension of issue space ensuring a robust core is twice the difference between q and the simple majority rule; it roughly agrees with the number of voters who have to change their opinions.

To interpret this statement with the pope-selection problem where $q \approx \frac{2n}{3}$, Eq. 2.5 allows us to expect stability for $k \leqq 2\left(\frac{2n}{3} - \frac{n}{2}\right) = \frac{n}{3}$. In words, without enough issues – one issue per defecting Cardinal – to alienate a third of the voting Cardinals, the original choice could remain stable. Instability, however, occurs with enough issues $\left(k > \frac{n}{3}\right)$ to entice half of the pope's original supporters to defect. The mathematics, then, closely parallels Kieckhefer's explanation.

Extending this argument to α rules (as defined in the theorem), we find that the *issue space dimension always roughly agrees with the number of voters that need to be alienated, or allured, away from* **x** *to support another proposal.* This is because an α rule admits stability should $k \leqq (2\alpha - 1)n$. Compare this dimensional limitation with the number of voters, x, a losing coalition of $(1-\alpha)n$ voters must lure away from a winning coalition of αn voters to form a new winning coalition; i.e., $(1-\alpha)n + x = \alpha n$, or $x = (2\alpha - 1)n$. As these numbers agree, the assertion follows. To illustrate with $\alpha = \frac{3}{4}$, the number of defecting voters must constitute half of all voters. Examples supporting this "issue per defecting voter" come from news reports about how "new concerns" (sometimes called "bribes") help entice Congressmen to abandon the status quo (**x**) and vote for proposal **y**.

The $\frac{3}{4}$ rule arises in a different context. To explain its special role in Theorem 1c, notice that a natural way to construct examples requires the hull of the voters' ideal points to have a full dimension. It turns out that to ensure the generic existence of nonbliss points, this construction imposes the bounds $k \leqq 2q - n$ for $q = n - 1$ rules and $k \leqq 2q - n - 1$ for all others. A surprising fact is that starting with the $\frac{3}{4}$ rule, stable situations exist when the hull of gradient vectors is in a lower dimensional subspace! This geometry allows the maximal k value which generically supports nonbliss core points to significantly increase.

Whenever two functions describe a particular setting where one is increasing faster than the other, there can be a turning point. In particular, assertions about the $q = 3$, $n = 4$ rule are governed by the geometric construction fully utilizing the $k = 2$ dimensional space (so $k \leqq 2q - n$). When $n \geqq 5$, $q = n - 1$ and $\beta = 0$ (so $n = 5, 6$) either construction provides the same dimensional bounds. For all other $\beta \geqq 0$ situations, placing gradients in lower dimensional subspaces is the optimal construction. Thus, the "excess dimension" assertion is operative only for q-rules starting with the $\frac{3}{4}$ rule. I define the value of the "excess dimension of issue space," β, in terms of $2q - n$ to emphasize (see Sect. 6) that $\beta > 0$ indicates when the optimal construction requires the gradients to be restricted to lower dimensional subspaces of issue space.

These comments about β have important consequences for Euclidean preferences. To explain, once a voter's ideal point $\mathbf{x}_j$ is specified, we know the gradient and *all* higher derivative terms of these preferences when they are evaluated at **x**. As this forces Euclidean preferences to belong to the excluded set of "non-generic behavior," Theorem 1 is not applicable. On the other hand, the wide use of Euclidean preferences makes it worth developing a parallel theory. To do so, recognize that the "ideal points" are the only relevant parameters, so "robustness" must involve movement of these points. "Generic" for Euclidean preferences, then, refers to conclusions which hold for an open set of choices for each voter's ideal point. For instance, the median voter conclusion for $k = 1$ is robust, but the example of Fig. 1 is not. In particular, the assertion of the last paragraph (that $\beta \geqq 0$ requires gradient vectors to be in a lower dimensional subspace) means that Euclidean robustness is violated.

Corollary 1. *For Euclidean preferences, bliss-core points exist generically iff* $k \leqq 2q - n$. *If* $q \leqq n - 2$, *then nonbliss core points exist generically iff* $k \leqq 2q - n - 1$. *For* $q = n - 1$, *nonbliss core points exist generically iff* $k \leqq 2q - n = n - 2, n \geqq 4$.

Additional consequences of Theorem 1 follow.

Examples. a. Part b follows immediately from c; we only need to determine whether $q = n$ (an admissible choice) satisfies Eq. 2.3. This involves checking the equivalent inequality

$$\frac{1}{2\beta + 4} \leqq \frac{n}{2\beta + 4}$$

which is trivially true for $n \geqq 1$.

b. By converting the $n\left(1 - \frac{1}{2\beta + 4}\right)$ term from Eq. 2.3 to the mixed number $a + \frac{x}{2\beta + 4}$, it follows that $q = a + 1$. Therefore, dropping the equality and the first term on the left side of Eq. 2.3, we obtain the equivalent

$$\frac{2\beta + 3}{2\beta + 4} < \frac{q}{n}. \tag{2.6}$$

To illustrate the use of Eq. 2.6, observe that all q-rules where issue space exceeds the $2q - n$ dimension by $\beta = 100$ are those satisfying

$$q > n\frac{203}{204}.$$

Using n values that are integer multiples of 204, say, $n = 4(204) = 816$, we have that the minimal q value is $1 + 4(203) = 813$, so the 813-rule generically admits core points when $k \leqq 2q - n + \beta = 1626 - 816 + 100 = 910$.

c. If q is the smallest value solving these inequalities for $n = \gamma(2\beta + 4)$, then $q = n - \gamma + 1$ and the dimensional restriction is

$$k \leqq n + (\beta - 2\gamma + 2) = (\beta + 1)(2\gamma + 1) + 1. \tag{2.7}$$

This inequality relates the k growth rates to the q-rule.

d. As described above, $\beta \geqq 0$ values indicate when it is advantageous for the gradients to be in a lower dimensional subspace. To find where these dimensional jumps occur, start with $n \geqq 5$ and $\beta = 0$ where Eq. 2.6 indicates that the $\frac{3}{4}$ rule is the bifurcation point. (More precisely, "one more than a three-fourths vote"). With $\beta = 1$, the bifurcation occurs at the $\frac{5}{6}$ rule. Continuing, we have that the bifurcations arise at the (one more than) $\frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots, \frac{\text{odd integer}}{1 + \text{odd integer}}, \ldots$ rules. Consequently, any $\beta \geqq 0$ admits supporting n and q values where $q < n$. But by specifying n and q, Eq. 2.4 bounds k. To illustrate with $q = 30$ and $n = 35$, it follows from Eq. 2.2 that bliss core points exist, generically, up to dimension

$$k \leqq 60 - 35 = 25$$

and nonbliss core points exist, generically, up to dimension

$$k \leqq (60 - 35) + \frac{120 - 106}{10} = 25 + \frac{14}{10},$$

or $k = 26$. This example allowing nonbliss points to exist longer than bliss points suggests finding when $(4q - 3n - 1)/2(n - q) < 0$ (so the bliss point persists longer than the nonbliss core points); it requires the decision rule to be bounded above by the $\frac{3}{4}$ rule.

To determine whether, generically, the nonbliss points could vanish dimensions earlier than the bliss points involves finding q and n values where

$$\frac{4q - 3n - 1}{2(n - q)} < -1.$$

As this $2q < n + 1$ phenomenon requires less than a majority vote, it is not relevant.

e. Part c of the theorem places special emphasis on the $q = 3$, $n = 4$ rule; it allows $k \leqq 2$. To show that the core is stable, place four points in the plane so that their hull has four sides. This figure has two diagonals; the intersection point is a nonbliss core point. Similar higher dimensional constructions hold for all other $q = n - 1$ rules with $k = 2q - n = n - 2$.

f. To prove part d of the theorem, express the optimal k values from Eq. 2.4 as

$$n(2\alpha - 1) - 1 < k \leqq n(2\alpha - 1) + \frac{4\alpha - 3 - \frac{1}{n}}{2(1 - \alpha)} < n(2\alpha - 1) + \frac{2\alpha}{1 - \alpha}, \quad \alpha = \frac{q}{n}. \tag{2.8}$$

The right- and left-hand sides are linear equations in n, so the conclusion follows. This assertion means that even α-rules close to the majority rule (i.e., $\alpha \approx \frac{1}{2}$) can be supported in, say, a 100 dimensional issue space with enough voters. To illustrate with $\alpha = 0.5001$, a core point can be supported in a hundred dimensional issue space with around a half million voters. So, for a city about the size of Minneapolis, as long as the number of issues doesn't exceed a hundred, stability could exist.

This assertion significantly extends a result by Schofield and Tovey [ST] providing conditions so that, in the sense of probability, the core exists as $n \to \infty$. Their limit result, then, is applicable only for an unspecified but very large numbers of voters. Because the assertion offered here holds for any n and q and it is based on sharp dimensional values, it can provide sharp estimates with practical examples. □

3 Comments on the structure of the core

While Theorem 1 tells us when the core exits, it tells us nothing about its structure. For instance, can the core be the union of disjoint sets? (Not with strict convexity, but it can when convexity is dropped.) To fill this gap, certain structural results explaining how the core changes with different k, n, q values are described next. They support the intuition that the stability of a core improves with larger q values.

For intuition about what to expect, start with a R^1 example of eleven voters, where the jth voter's ideal point is located at the integer $j, j = 1, \ldots, 11$. For the simple majority rule ($q = 6$), $x_6 = 6$ is the only core point. However, with $q \geqq 7$, the core is the interval $[12 - q, q]$ where the endpoints are bliss core points. This suggests that the q_1 core, $q_1 > q$, strictly contains the q core. In general, this is true.

Another stability measure is the topological dimension of the core. For instance, in the eleven voter example with $q = 8$, the fact that the core includes the interval $(4, 8)$ provides a strong sense of stability – near any core point there is another one. Similarly, because the core for the seven voter example of Fig. 2 is the shaded region, near any core point in the interior is another core point. Thus, a larger dimensional core suggests added resilience. To expand on this notion, if $\mathbf{x}$ is in the interior of a core, then $\mathbf{x}$ remains a core point even after the preferences are slightly varied. But when the core does not have an interior (such as for bliss-core points or the nonbliss core point for $k = 2, q = 3, n = 4$), we should expect the position of the core to move with changes in preferences, but the structure to remain as asserted. This underscores the importance of Theorem 2b, c.

In Fig. 2, the core is any point in the closure of the shaded region. (There are no bliss-core points.) Observe that a boundary line connects the bliss points of two agents, and a vertex is where two bliss-connecting lines intersect, so they represent specialized core points where the preferences must line up appropriately. Singularity theory ensures that this picture characterizes the general situation. Namely, singularities form a "stratified structure" where a restricted setting is in the closure of the previous setting. Thus, while the boundary of a core need not have straight lines, generically it consists of core points of restricted types where preferences line up as indicated.

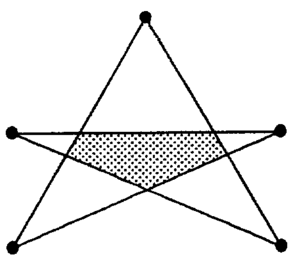

Figure 2. Core for $n = 5$, $q = 4$, $k = 2$

Implicit in the assertions of Theorem 1 is that the size of the core should decrease with an increase in the k value. To show this is true, for a k dimensional issue space with a k_1-dimension subspace, $k > k_1 > 0$, let P_{k,k_1} be the natural projection mapping. (For instance, if $k = 3$ and $k_1 = 2$ represents the x-y plane, then $P_{3,2}((x, y, z)) = (x, y)$ where the z component is dropped.) The k_1-dimensional preferences are assumed to be the k-dimensional preferences restricted to the lower dimensional plane. If this lower dimensional plane is obtained by dropping $k - k_1$ issues, then it is a coordinate plane of the k-dimensional space. Otherwise, the plane represents where certain issues are combined into a single issue. (With minor technical changes, this holds for a k_1 dimensional smooth manifold.) The following tells us that a core point persists when issues are restricted.

Theorem 2. *a. Suppose n and k are such that non-bliss core points exist generically for a q-rule; $q < n$. The core for the $q + 1$ rule always contains the q-rule core. It is generically unlikely that the two cores are the same.*

b. Generically, bliss-core points are isolated points. Generically for those q, n, and k values that do not satisfy Eq. 2.3 for $\beta \geqq 0$, the set of nonbliss core points has a nonempty interior.

c. Suppose $\beta > 0$ is admitted by Eq. 2.3 for specified n, q values. While the dimension of issue space is $k = (2q - n) + \beta$, generically the core is a union of submanifolds with dimension no larger than $k - (\beta + 1)(2n - 2q + 1)$.

d. If $\mathbf{x}$ is a core point for a q-rule in a k dimensional space and if there is a k_1 dimensional plane passing through $\mathbf{x}$, then $P_{k,k_1}(\mathbf{x})$ is a core point for the k_1 dimensional issue space.

Observe the interesting conflict; because larger q values provide added stability with a wider selection of core points, larger q-values appear to be "better." So, why don't we always use unanimity as the deciding rule? The reason, of course, is obvious; the "stability" of unanimity can require "rigidity" where very little can be accomplished. This instinct is supported by the theorem. With unanimity and Euclidean preferences, the core is the convex hull defined by the voters' ideal points. In general, the unanimity core strictly contains the q-rule core (Thm. 2a), so it is easier for the status quo to be in the unanimity core than in the core of any other q-rule. But once the status quo is in a core, change is impossible. So, rather than being a desirable feature, maybe we should avoid procedures allowing a large core with their danger of fixing the outcome at the status quo.

Restating this concern, why care about the core? We all should; with an empty core and *any* $\mathbf{x}_1$ some winning coalition has a preferred $\mathbf{x}_2$. Similarly, another decisive coalition has $\mathbf{x}_3$ preferred to $\mathbf{x}_2$. The argument continues—forever. Without core stability, then, we suffer the "chaos" behavior of spatial voting described by Kramer [K], McKelvey [M] and others. (Richards [R] provides a new explanation. Tataru [T] significantly advances this literature with her estimates on the number of steps required to move from any initial option to any other one.) Namely, without core stability, an outcome cannot be trusted as reflecting the views of the voters.

We need a compromise between enjoying the stability of the core while preventing gridlock which arises should the core usually contain the status quo. According to Thm. 2, this requires choosing a q value where the robust core is not "too large." If we know the number of issues typically involved in a decision process, this k value determines the minimal q-rule with core stability but, maybe, without gridlock. The containment assertions of Thm. 2 suggest that the refined core of this q-rule may more accurately capture the voters' intent. Alternatively, maybe the real problem is caused by restricting attention to binary comparisons and q-rules. This question is explored in detail elsewhere.

4 A characterization of core points

To motivate the basic technical tool of this paper, start with Euclidean preferences. As indicated in the Sect. 1, the core for the unanimity rule is the convex hull defined by the voters' ideal points. Similarly, with a q-rule and a specified q-voter coalition, "their core" is the convex hull of their q ideal points. So, the core for a q-rule is the intersection of the convex hulls defined by all possible q-voter coalitions. If this intersection is empty, the core is empty.

This geometry explains why we should expect the core to vanish with larger k values. For instance, with $n = 7$ and $q = 4$, because $k = 1$ requires all points to lie on a straight line, the convex hulls must have a non-empty intersection; the core must exist. However, once $k \geqq 2$, we cannot expect more than two lines to cross or more than two points to be on any line. The extra freedom to position ideal points, then, makes it difficult for the convex hulls to have a common intersection point. One remedy is to use higher dimensional hulls where intersections are more likely. As this requires adding vertices to the hulls, it is equivalent to increasing the q value.

This intuition extends to strictly convex smooth preferences. ("Smoothness" ensures that the curved utility surfaces can be approximated by the planes defined by the derivative conditions; strict convexity requires all preferred points to be on one side of this plane.) Thus, basic to our arguments is Prop. 1 which characterizes core points for q-rules in terms of the derivative properties the utility functions must have for the intersection to be nonempty. In this description, $Co_{\mathbf{x}}(\{\mathbf{v}_j\}_{j\in C})$ denotes the convex hull of the (vertices of the) vectors $\{\mathbf{x} + \mathbf{v}_j\}_{j\in C}$ and $\overline{Co_{\mathbf{x}}}(\{\mathbf{v}_j\}_{j\in C})$ is $Co_{\mathbf{x}}(\{\mathbf{v}_j\}_{j\in C})$ minus the vertices. For this proposition, "smooth" can be relaxed to C^2 smoothness.

Proposition 1. *Assume the voters have smooth, strictly convex preferences. A necessary condition for* $\mathbf{x}$ *to be a core point is if for any set of q agents, C,*

$$\mathbf{x} \in Co_{\mathbf{x}}(\{\nabla u_j(\mathbf{x})\}_{j\in C}) \tag{4.1a}$$

A sufficient condition is if either

$$\mathbf{x} \in \overline{Co_{\mathbf{x}}}(\{\nabla u_j(\mathbf{x})\}_{j\in C}) \tag{4.1b}$$

or if $\mathbf{x}$ *is both a vertex of* $Co_{\mathbf{x}}(\{\mathbf{v}_j\}_{j\in C})$ *and a bliss point for some voter in C.*

This assertion makes sense; if q or more voters prefer alternatives that are in the same general direction from $\mathbf{x}$, then they can block the selection of $\mathbf{x}$. The directional derivative is determined by a scalar product, $\frac{\partial u(\mathbf{x})}{\partial \mathbf{v}} = (\nabla u(\mathbf{x}), \mathbf{v})$, so the sense of "the same general direction" is captured by passing a plane through $\mathbf{x}$ where, for all agents in this coalition, the $\nabla u_j(\mathbf{x})$ vectors are strictly on the same side of the plane. Indeed, if $\mathbf{v}$ is the normal for this plane where $\mathbf{v}$ is on the same side as the gradients, the positive values of $(\nabla u_j(\mathbf{x}), \mathbf{v})$ require that all utilities are improved by moving in this direction. So, to ensure that $\mathbf{x}$ is a core point we need to find conditions to prevent this scenario from occurring. (The proof is a formal expression of this intuition.) Notice how Prop. 1 extends the Sect. 1 argument used for the median voter and Plott's constructions.

The proof uses an important relationship between local and global comparisons of alternatives. Call $\mathbf{x}$ an "infinitesimal core point" if it is a core point when the admissible choices are restricted to a sufficiently small open neighborhood of $\mathbf{x}$. While a core point always is an infinitesimal core point, it is easy to construct examples where an infinitesimal core point is not a core point. (The idea is similar to constructing examples where a local maximum is not a global maximum; just use utility functions where the level sets have "wiggles.") On the other hand, just as appropriate convexity assumptions force a local maximum of a function to be a global maximum, the following lemma asserts that our strict convexity assumption on utility functions forces infinitesimal core points to be core points.

Lemma 1. *If all agents have smooth, strictly convex preferences, then an infinitesimal core point is a core point.*

Proof of Lemma 1. Assume $\mathbf{x}$ is an infinitesimal core point, but not a core point because a decisive coalition C prefers $\mathbf{y}$; i.e., $\mathbf{y} \in \bigcap_{i \in C} M_i(\mathbf{x})$. (See Eq. 2.1) As it is the intersection of convex sets, $\bigcap_{i \in C} M_i(\mathbf{x})$ is convex. Consequently, any point $\mathbf{y}_t = t\mathbf{x} + (1-t)\mathbf{y}$ on the straight line connecting $\mathbf{y}$ and $\mathbf{x}$ is in this set. For t sufficiently close to unity (so $\mathbf{y}_t$ is sufficiently close to $\mathbf{x}$), this same coalition prefers $\mathbf{y}_t$ to $\mathbf{x}$. Thus $\mathbf{x}$ is not an infinitesimal core point. The contradiction completes the proof. □

Proof of the proposition. To prove that the stated condition is necessary, assume that $\mathbf{x} \notin Co_{\mathbf{x}}(\{\nabla u_j(\mathbf{x})\}_{j \in C})$ for coalition C. Thus, a plane passes through $\mathbf{x}$ with the convex hull strictly on one side. If $\mathbf{v}$ is a normal vector for the plane pointing toward the side with the convex hull, then $\frac{\partial u_j(\mathbf{x})}{\partial \mathbf{v}}$ is positive for all voters in the decisive coalition C. Consequently, $\mathbf{x}$ is not a core point.

The proof of sufficiency involves three cases; the first has $\mathbf{x} \in Co_{\mathbf{x}}(\{\nabla u_j(\mathbf{x})\}_{j \in C})$ as an interior point for all decisive coalitions; the other two allow $\mathbf{x}$ to be a boundary point for some decisive coalition. For the interior point situation, any plane passing though $\mathbf{x}$ must have vertices of $Co_{\mathbf{x}}(\{\nabla u_j(\mathbf{x})\}_{j \in C})$ strictly on each side of the plane – these vertices are defined by gradient vectors of the agents from this coalition. If $\mathbf{x}$ is not a core point, there exists an alternative $\mathbf{y}$ that this coalition prefers to $\mathbf{x}$. Let $\mathbf{v} = \mathbf{y} - \mathbf{x}$ be a normal vector for a plane passing though $\mathbf{x}$. For those gradient vectors on the side of the plane opposite of $\mathbf{v}$, the directional derivative $\frac{\partial u}{\partial \mathbf{v}} = (\nabla u, \mathbf{v})$ is negative. As these agents have a lower utility for any such change, they will not vote for such a move. Thus, $\mathbf{x}$ is an infinitesimal core point.

The analysis is essentially the same when $\mathbf{x}$ is a boundary point. If $\mathbf{x}$ is a vertex of $Co_{\mathbf{x}}(\{\nabla u_j(\mathbf{x})\}_{j \in C})$ for a q-voter coalition C, then, by assumption in the Proposition, $\mathbf{x}$ is a "bliss point" for an agent in this coalition. As this voter prefers $\mathbf{x}$ to anything else, this coalition cannot change the outcome.

For the remaining case, if $\mathbf{x}$ is on the boundary of $\overline{Co_{\mathbf{x}}}(\{\nabla u_j(\mathbf{x})\}_{j \in C})$ but not a vertex (but, it could be a bliss point), then $\mathbf{x}$ could belong to several boundary components. (For instance, if $\mathbf{x}$ is on the intersection of two boundary faces, it is on both faces and on the defined edge.) Choose the lowest dimensional boundary component. Because $\mathbf{x}$ is not a vertex, it *must* be in the interior of the convex hull defined by the gradient vectors in this component. The gradient vectors (and agents) on this component are the ones we analyze.

A slight modification of the above argument (where $\mathbf{x}$ is an interior point) proves that a change $\mathbf{v}^*$ in this boundary component, or a change with a nonzero coordinate in this component, in unacceptable to some voter. This is because, as $\mathbf{x}$ is in the interior of the hull defined by vectors on the boundary component, for some voter with a gradient in this component, the directional derivative $\frac{\partial u_k}{\partial \mathbf{v}^*}$ is negative. Consequently, the only potential admissible changes for this coalition are orthogonal to the boundary component, i.e., the changes must be in a direction $\mathbf{v}$ that is orthogonal to all of the gradient vectors on this component. By strict convexity, for any voter with a gradient in the plane, the only point of the set $M_k(\mathbf{x})$ that is in the tangent plane passing though $\mathbf{x}$ with normal $\nabla u_k(\mathbf{x})$ is $\mathbf{x}$. Namely, the level set of $u_k(\mathbf{x}) \backslash \{\mathbf{x}\}$ is strictly on the $\nabla u_k(\mathbf{x})$ side of the tangent plane. So, for each voter with a gradient vector in the boundary plane, any change in the $\mathbf{v}$ direction constitutes a change of lower utility. This means that $\mathbf{x}$ is an infinitesimal core point. □

Proposition 1, which is critical for what follows, can be viewed as an higher order extension of an application of Smale's [Sm] characterization of Pareto points to core points. The connection with Pareto points is that a core point is a Pareto point for every decisive coalition. The higher order derivative conditions are snuck in with the strict convexity arguments. As it will become clear, these strict convexity properties are critical for our analysis (when $\beta \geqq 0$) because they determine what happens when $\mathbf{x}$ is on the boundary of this hull. (This insight permits me to avoid the restrictive embedding argument of Banks, McKelvey, and Schofield.) It is important to stress that this assumption does not restrict our conclusions because level sets with flat spots belong to the nongeneric setting.

A similar argument shows why the differences between the necessary and sufficient conditions of Prop. 1 do not matter. (They differ only by whether a vertex must be a bliss point.) If $\mathbf{x}$ is a vertex of the hull for some q-voter coalition C, then $\nabla u_j(\mathbf{x}) = \mathbf{0}$ for a voter of this coalition. However, this need not mean that $\mathbf{x}$ is a bliss point because the level set might be sufficiently flat. Again, because the arguments for Sect. 6 show that such behavior is non-generic, we can assume that when $\mathbf{x}$ is a vertex, it is a bliss point.

Proof of Theorem 2d. The projection of a convex set to a lower dimension subspace is a convex set. As the projection of $\nabla u_j(\mathbf{x})$ is the gradient of u_j restricted to the lower dimensional submanifold, the proof follows directly from Prop. 1. □

5 Constructing examples

In Sect. 6, which could be read first, it is shown that you cannot do better than the bounds specified in Thm. 1. Also, it is outlined why, when singularity theory asserts that an event exists "generically," either it exists with open sets of preferences, or it never occurs. This section avoids the empty set option by constructing examples. These examples start with "boundary situations" where certain slight changes in preferences can cause the core to vanish, but my perturbations generate robust settings. For technical reasons (Sect. 6), the examples are restricted to bliss-core point examples where $k \leqq 2q - n$ and nonbliss core points where $k \leqq 2q - n - 1$.

With the exception of a bliss-core point, Prop. 1 uses directions, rather than lengths, of the vectors $\{\nabla u_j(\mathbf{x})\}_{j \in C}$. Thus, it suffices to find appropriate gradient directions at $\mathbf{x}$ that satisfy Prop. 1. Once directions are found, preferences are easy to construct. For instance, as a direction defines a ray from $\mathbf{x}$, place on each ray an agent's ideal point to define her Euclidean preferences. For examples without Euclidean preferences, use Taylor series. To illustrate by constructing examples with $\mathbf{x} = (1, 2)$ and the abbreviated expansion

$$u(x_1, x_2) = u(1,2) + \nabla u(1,2) \cdot (x_1 - 1, x_2 - 2) + \tfrac{1}{2} D^2 u_{(1,2)}((x_1 - 1, x_2 - 2), (x_1 - 1, x_2 - 2))$$

choose $\nabla u(1,2)$ and $D^2_{(1,2)} u$ consistent with the strict convexity assumption. For instance, for the direction $\frac{1}{5}(-4, 3)$, we could choose $\nabla u = (-4, 3)$ and $D^2 u$ as the diagonal matrix with entries $-1, -1$ to define the utility function $u(x_1, x_2) = -4(x_1 - 1) + 3(x_2 - 2) - (x_1 - 1)^2 - (x_2 - 2)^2$. A similar construction holds for all dimensions.

Thus, once gradient directions are found, examples easily follow. As "directions" are identified with unit vectors, treat them as points $\{\mathbf{x}_k\}$ on a unit sphere, S^{k-1}, with center $\mathbf{x}$. (So, $\mathbf{x}_j$ defines the "direction" $\mathbf{x}_j - \mathbf{x}$.) For bliss-core point settings, the bliss point is $\mathbf{x}$.

Gradient directions. To identify coalitions which have alternatives (in k-dimensional issue space) preferred to $\mathbf{x}$, pass a $(k-1)$-dimensional "dividing plane" through $\mathbf{x}$. If q points are on a side of this plane, they form a decisive coalition denying core status for $\mathbf{x}$. Thus, we need to analyze all dividing planes passing through $\mathbf{x}$ by counting the points on each side.

[1] If no $\{\mathbf{x}_j\}$ points are on the $(k-1)$-dimensional dividing plane passing through $\mathbf{x}$, then, according to Prop. 1, there are no more than $q-1$ points on a side.

Because an arbitrarily small change in the orientation of a plane defined by $k-1$ linearly independent vectors can force all $k-1$ points onto a specified side of its new position, [1] restricts the number of points on a side of its original orientation.

[2] A dividing plane with $k-1$ linearly independent vectors can have no more than

$$s^* = (q-1) - (k-1) = q - k \tag{5.1}$$

points on either side.

Conditions [1] and [2] are used repeatedly to construct examples and prove the theorems. The number of points on a side change only when a dividing plane rotates through a point, so a sufficient condition (Prop. 1) for $\mathbf{x}$ to be a core point in a k-dimensional space is if an example satisfies [1] for a dividing plane without gradient vectors and [2] for all $(k-1)$-dimensional dividing planes where a dividing plane has, other than a bliss point, $k-1$ of these vectors.

Because $\mathbf{x}$ is a core point for a q_1-rule if it is a core point for a q-rule, $q_1 > q$, it suffices to create examples supporting the minimal q value. From Eq. 2.5, this requires assigning points so that half of the points not on a dividing plane are on each side of the plane. (This obtains the specified bounds on k values.) A natural approach is to symmetrically position the n points on S^{k-1}. While it is trivial to symmetrically locate n points on a circle (place them $2\pi/n$ radians apart), it is difficult to do this for higher dimensional spheres. (For instance, what is the symmetric choice for $n = 5$ on S^2?) To sidestep this problem, a method is devised to capture the relevant geometry. Specific cases are considered before the general argument is described.

One-dimensional issue space. I use $k = 1$ to show how to use [1] and [2] to create examples with a specified $\mathbf{x}$ as a core point. Start by positioning m points according to the "alternating rule" where the first point is placed to the right of $\mathbf{x}$, the second to the left, the third to the right, etc. until all points are positioned and no two points occupy the same position. For even integer m, there are $\frac{m}{2}$ points on each side of $\mathbf{x}$. According to [1], this choice supports *all* $q \geqq \left[\frac{m}{2}\right] + 1$ rules. For $k = 1$, a "dividing plane" is just the point $\mathbf{x}$, so [2] is not applicable. Choosing $n = m$ as the number of voters, the example supports $\mathbf{x}$ as a nonbliss core point for all $q \geqq \left[\frac{n}{2}\right] + 1$ rules with an even number of voters. (The example is robust because the points can be altered without changing the argument; e.g., the set of nonbliss core points is the interval defined by the closest points on either side of $\mathbf{x}$.) If $n = m + 1$ (i.e., $\mathbf{x}$ is a bliss-core point for an odd number of voters where one voter's gradient is zero), this construction supports all $q \geqq \left[\frac{n-1}{2}\right] + 1 = \left[\frac{n}{2}\right] + 1$ rules. $\left(\text{Because } n \text{ is an odd integer, we have } \left[\frac{n}{2}\right] = \left[\frac{n-1}{2}\right].\right)$

Let $\lceil x \rceil$ be the "round-up" function with value x iff x is an integer, and $[x] + 1$ otherwise. For odd integer m, at most $\left\lceil \frac{m}{2} \right\rceil$ points are on a side of $\mathbf{x}$, so this construction supports all $q \geqq \left\lceil \frac{m}{2} \right\rceil + 1 = \left[\frac{m}{2}\right] + 2$ rules. If $n = m$, so $\mathbf{x}$ is a nonbliss core point, this example supports all $q \geqq \left\lceil \frac{n}{2} \right\rceil + 2$ rules. When $n - 1 = m$ (so $\mathbf{x}$ is a bliss core point with an even number of voters), the example supports all $q \geqq \left\lceil \frac{n-1}{2} \right\rceil + 2 = \left[\frac{n}{2}\right] + 1$ rules. Observe that when n is even, we have $\left\lceil \frac{n-1}{2} \right\rceil = \left[\frac{n}{2}\right] - 1$. Actual preferences are found by using one of the described methods. Because these examples evenly split the points not on a dividing plane, they support all q, n rules allowed by Theorem 1 for $k = 1$.

Two dimensions. The $k = 1$ "alternate side" construction is used for all $k \geqq 2$. This involves perturbing "degenerate" preferences where Plott's construction is used when possible. The following geometry simplifies the perturbation argument.

For $k = 2$, place m points on S^1, the circle with center $\mathbf{x}$. Whatever the choices, some line through $\mathbf{x}$ misses all of them; call it the x_2-axis and its orthogonal complement the x_1-axis. (See Fig. 3.) Identify each point on the circle with a point on either the "left line" (the line $x_1 = -1$) or the "right line"

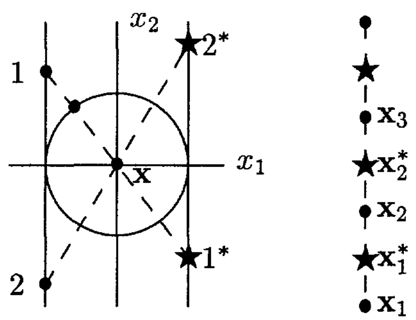

Figure 3. The two-dimensional case.

($x_1 = 1$). Namely, draw a ray from $\mathbf{x}$ through the circle point (e.g., the dot on the circle in Fig. 3) and determine where and on which line this ray intersects (e.g., the point "1" on the left line of Fig. 3). As a point on either line uniquely identifies a circle point, it suffices to consider only the point on the lines.

The "dual" for a left line point is where the line through this point and $\mathbf{x}$ intersects the right line. Denote each dual point by star. (See Fig. 3.) Because a dual point uniquely determines a regular point on the left side, a configuration is equivalent to points and dual points on the right line. Observe that a point and its dual lie on opposite sides of a dividing plane (e.g., in Fig. 3, 2 and 2* are on opposite sides of the 1–1* line). Also, if a point is on a dividing plane, then so is its dual.

For a two-dimensional example to span a two-dimensional space, $m \geqq 3$. Arbitrarily place $\left\lceil \frac{m}{2} \right\rceil$ of them, $\{\mathbf{x}_j\}_{j=1}^{\lceil m/2 \rceil}$, as regular points (dots) on the right line. Plott's configuration places each dual point, denoted by a star, on top of a regular point. Denote the dual point on $\mathbf{x}_j$ by $\mathbf{x}_j^*$ and the regular point on the left line defined by $\mathbf{x}_j^*$ by $\mathbf{x}_{\lceil m/2 \rceil + j}$. If m is even, each regular point is covered; if m is odd, the uncovered regular point is at the top. For even m values, a dividing line satisfying [2] is defined by one of these points. Because dual points are on regular points, the same number of starred and regular points are on each side of this dividing line. Thus, after translating dual to regular points, the same number of points are on either side.

Only for $q = 3$, $n = 4$ is this a robust configuration. (See Sect. 6.) So, to perturb Plott's configuration (to generate a robust example), translate each $\mathbf{x}_j^*$ slightly upwards from $\mathbf{x}_j$. (See the figure on the right in Fig. 3.) For odd values of m, each bounded interval (defined by regular points) has a starred point. When m is even, each bounded interval has a starred point and one starred point is in the top unbounded interval.

Start with m odd. One side of the $x_1 = 0$ dividing plane has $\left\lceil \frac{m}{2} \right\rceil$ points, so from [1] we are restricted to $q - 1 \geqq \left\lceil \frac{m}{2} \right\rceil$, or $q \geqq \left[\frac{m}{2} \right] + 2$ rules. If a dividing plane passes through the top dot on the right line (see Fig. 3), then, by construction, precisely $\left[\frac{m}{2} \right]$ dots and $\left[\frac{m}{2} \right]$ stars are below this line. Consequently, there are precisely $\left[\frac{m}{2} \right]$ right points below and $\left[\frac{m}{2} \right]$ left points above the dividing plane. According to [2] and Eq. 5.1 where $k = 2$ and $s^* = \left[\frac{m}{2} \right]$, so far this example supports all $q \geqq \left[\frac{m}{2} \right] + 2$ rules.

Rotate the dividing plane downwards about $\mathbf{x}$. When the plane hits the next point, the dual point removed from the bottom side (it is on the new dividing plane) is balanced by the new regular point above the plane (the top point that defined the previous orientation of the dividing plane). Thus, s^*

keeps the same value. With the alternating assignment, each time the plane passes through a point, the point is removed from one side while a point of the opposite type (defining the previous orientation) now is on the upper side. After converting dual points to regular points, $\left[\frac{m}{2}\right]$ points are above and below this dividing plane. Thus this construction provides examples for $k=2$ core points for any $q \geqq \left[\frac{m}{2}\right]+2$ when m is odd. If $n=m$ (so n is odd and $\mathbf{x}$ is a nonbliss core point), this construction supports all $q \geqq \left[\frac{n}{2}\right]+2$ rules. For $n=m+1$ even and $\mathbf{x}$ a bliss core point, this construction supports all $q \geqq \left[\frac{m}{2}\right]+2=\left[\frac{n-1}{2}\right]+2=\left(\left[\frac{n}{2}\right]-1\right)+2=\left[\frac{n}{2}\right]+1$ rules.

For even values of m, this argument shows that when the dividing plane passes through a point, there are $\left[\frac{m-1}{2}\right]=\left[\frac{m}{2}\right]$ points on one side and $\left[\frac{m}{2}\right]-1$ points on the other. According to Eq. 5.1 and [2], the associated bounds are $q \geqq s^* + k = \left[\frac{m}{2}\right]+2$. In terms of n, when n is even and the core is not a bliss point (so $n=m$), this construction supports all $q \geqq \left[\frac{n}{2}\right]+2$ rules. When n is odd and $\mathbf{x}$ is a bliss point (so $n=m+1$) then, because $\left[\frac{m}{2}\right]=\left[\frac{n-1}{2}\right]=\left[\frac{n}{2}\right]$, this construction supports all $q \geqq \left[\frac{m}{2}\right]+2=\left[\frac{n}{2}\right]+2$ rules.

The traits from this construction used for all k is that all dual points are rotated off of Plott's diagram in the same direction. (Larger k values admit other choices.) Similarly, the four cases are when m is odd or even and $\mathbf{x}$ is or is not a bliss point (i.e., n equals m or $m+1$). A factor always changing the outcome is that $\left[\frac{n-1}{2}\right]=\left[\frac{n}{2}\right]-1$ iff n is even; this computation occurs with an even number of voters and a bliss-core point. A simple argument (starting with Eq. 2.5) shows this construction provides $k=2$ supporting examples for all of the n, q rules admitted by Thm. 1 (with $\beta<0$) with the exception of the $q=3$, $n=4$ rule. This exceptional case was handled earlier.

Three-dimensions. Let $\mathbf{x}$ be at the origin of an axis system. Skip the sphere S^2 by immediately placing points on left and right planes defined, respectively, by $x_1=-1$, $x_1=1$. Restrict the construction by using the unit circle, S^1, in the right plane with center $(1,0,0)$.

A true $k=3$ example requires the convex hull to have a positive three-dimensional volume; i.e., $m \geqq 4$. As indicated in Fig. 4, place $\left[\frac{m}{2}\right]$ regular points in an arbitrary fashion on S^1. For boundary (Plott) preferences, place

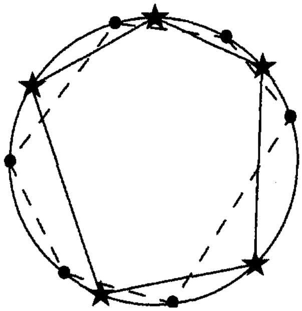

Figure 4. Points and dual points.

$\mathbf{x}_j^*$ on $\mathbf{x}_j$ where $\mathbf{x}_j^*$ is the dual of $\mathbf{x}_{[m/2]+j}$ from the left plane. If m is even, all regular points are covered; if m is odd, $\mathbf{x}_{[m/2]}$ is left uncovered. Using the x_1 axis as an axis of rotation (this axis is orthogonal to the right plane and it passes through the center of S^1), rotate all starred points a slight amount in a fixed direction. As the convex hull of the regular points has same number of edges as vertices, the shift defines an "alternating rule" where each dual point is in an arc of the circle defined by an edge. If m is even, there are as many arcs as dual points, so each arc has a dual point. If m is odd (as in Fig. 4), one arc has no dual point. Observe that even after slightly perturbing this new configuration in any manner (even off the circle), dual and regular points still alternate as determined by the connecting cords. Also notice that after converting to regular points, $m = 4$ defines a tetrahedron.

As [1] is satisfied for $x_1 = 0$, to analyze this configuration for even m values, consider a dividing plane defined by two of the positioned points, $\mathbf{p}_1, \mathbf{p}_2$ where $\mathbf{p}_1$ will serve as an axis of rotation. On each side adjacent to $\mathbf{p}_1$ are points; call them $\mathbf{n}_1, \mathbf{n}_2$. By the alternating rule, both points have the type opposite that of $\mathbf{p}_1$, say they are dual points. Either $\mathbf{n}_1$ or $\mathbf{n}_2$ along with $\mathbf{p}_1$ defines a face of the convex hull where all other points $\left(\frac{m}{2} - 1 \text{ of each kind}\right)$ are on one side of this plane. After converting dual points to regular points, there are $\frac{m}{2} - 1$ points on each side of the plane.

Let the "inner" or "lower" side of the $\mathbf{n}_1, \mathbf{p}_1$ plane be the side with the convex hull. Rotate the dividing plane (about $\mathbf{p}_1$) off of the chosen $\mathbf{n}_j$ inwards to the next point of the hull; by the alternating rule, it is a regular point. This new orientation removes a regular point from the "inner" side (it now is on the dividing plane), but adds a dual point to the "outer" side to keep the same number of regular points on each side of the new dividing plane. Because points of different type alternate on the circle, this argument continues. Consequently, with the final orientation of the dividing plane at $\mathbf{p}_2$, after dual points are converted back to regular points, there are $\frac{m}{2} - 1$ points on each side.

According to Eq. 5.1, this construction supports $q \geqq \left(\left[\frac{m}{2}\right]-1\right)+3=\frac{m}{2}+2$ rules. Therefore, if $n=m$ (so $\mathbf{x}$ is not a bliss point) it supports all $q \geqq \left[\frac{n}{2}\right]+2$ rules in a three-dimensional issue space. If $n=m+1$ (so n is odd and $\mathbf{x}$ is a bliss point), because $\left[\frac{n}{2}\right]=\left[\frac{n-1}{2}\right]$ this construction supports all $q \geqq \left[\frac{n}{2}\right]+2$ rules. Again, this configuration supports all q and n values which, according to Thm. 1, admit a three-dimensional issue space with $\beta<0$.

The only change for odd values of $m \geqq 5$ is to consider sectors without a dual point. By choosing the dividing plane to pass through these two regular points, all $\left[\frac{m}{2}\right]$ dual points and the remaining $\left[\frac{m}{2}\right]-2=\left[\frac{m}{2}\right]-1$ of the regular points are on one side. Consequently, one side of this dividing plane has $\left[\frac{m}{2}\right]$ points $\left(\text{and the other side has } \left[\frac{m}{2}\right]-1\right)$. Using the alternating point construction, it follows that this is true for any dividing plane defined by two points. Therefore, from [2] we have that $q \geqq \left[\frac{m}{2}\right]+3$. So, if $m=n$ (that is, when $\mathbf{x}$ is not a bliss point and n is odd), this example supports all $q \geqq \left[\frac{n}{2}\right]+3$ rules. Where $\mathbf{x}$ is to be a bliss-core point, n is even, and $m=n-1$, the relationship $\left[\frac{m}{2}\right]=\left[\frac{n-1}{2}\right]=\left[\frac{n}{2}\right]-1$ shows that this construction supports all $q \geqq \left[\frac{n}{2}\right]+2$ rules. Again, it is easy to show that these examples support all rules described in Thm. 1 with $\beta<0$.

General case. For $k \geqq 4$, the larger dimension of the unit sphere in the right plane, S^{k-2}, and added orientations make it more difficult to describe a configuration but easier to show that appropriate perturbations of $\mathbf{x}_j^*$ off of $\mathbf{x}_j$ are possible. This is because the argument now is identified with the algebraic comparison of the number of equations and a larger number of unknowns. I use a $m=2k$ construction to indicate the source of the equations and one of many possible solutions.

Place k regular points near the vertices of an equilateral k-gon in the right plane; e.g., for $k=4$, the points are near the vertices of an equilateral tetrahedron. The regular points are fixed, so the variables are $\mathbf{z}_j=\mathbf{x}_j^*-\mathbf{x}_j$, $j=1,\ldots,k$ with initial values $\mathbf{z}_j=\mathbf{0}$. These regular points are nearly on the orbit of the permutation $\tau=(1,2,\ldots,k)$ for an appropriate initial point $\mathbf{x}_1 \in S^{k-2}$. (Hence $\tau^j(\mathbf{x}_1)=\mathbf{x}_{j+1}$ where $\mathbf{x}_1$ is chosen so that the angle between $\mathbf{x}_1, \tau^j(\mathbf{x}_1)$ has the correct value; e.g., for $k=3$, select $\mathbf{x}_1$ so its angle with $\tau(\mathbf{x}_1)$ is

120°.) The placement of dual points is similar to how the τ orbit defines the Condorcet triplet from social choice ([Chap. 3.1, S1]).

With $k-1 \geqq 3$, replace the vector cross product with the wedge product of $(k-1)$ vectors; e.g., see [W]. Let $\mathbf{y}_j = \mathbf{x}_j - \mathbf{x}$, $\mathbf{y}_j + \frac{m}{2} = \mathbf{x} - \mathbf{x}_j^*, j = 1, \ldots, \frac{m}{2}$, (observe the difference in orientation), and let $\mathbf{n}$ be the wedge product of $k-1$ of these vectors. If $\mathbf{n}$ does not involve $(\mathbf{x}_j, \mathbf{x}_j^*)$ pairs, $\mathbf{n}$ is a non-zero $k-1$ form indicating a normal direction to the dividing plane spanned by these vectors. A $\mathbf{x}_j, \mathbf{x}_j^*$ pair used to define $\mathbf{n}$, however, adds a degree of degeneracy in $\mathbf{n}$'s functional form when $\mathbf{z}_j = \mathbf{0}$. This degeneracy represents the linear dependency of these vectors, so the space spanned by the vectors drops a dimension.

For $\mathbf{y}_j$ not used in $\mathbf{n}$, the sign of the wedge product $\mathbf{n} \wedge \mathbf{y}_j$ determines the orientation of $\mathbf{y}_j$ relative to $\mathbf{n}$. The orientation difference in defining $\mathbf{y}_j$ for a dual point requires the sign of $\mathbf{n} \wedge \mathbf{y}_j$ to differ from $\mathbf{n} \wedge \mathbf{y}_{j+m/2}$. (The orientation is that of $\mathbf{x}_{j+m/2}$.) This choice is equivalent to assigning -1 to dual points and 1 to regular points. As there are as many dual as regular points, the sum of these signs over all m points is zero. To satisfy [1] and [2], the sums of signs from the wedge product computation for points on each side of a dividing plane can differ by no more than unity. The choices of zero or unity depends on the parity of the k and m values.

For a configuration where $\mathbf{n}$ is non-degenerate and all $\mathbf{z}_j = \mathbf{0}$, the sum of signs on each size of the dividing plane is zero. (One side has no points; the other has a dual point on its mate.) We need to perturb $\mathbf{z}_j$ so that all $\mathbf{n}$ choices are non-zero and the counting condition is satisfied. In moving dual points, interpret "above" – for a plane containing a face of the hull defined by the regular points – as designating the side of the plane without the hull. To illustrate with an equilateral triangle, the "bottom" side of a line through two vertices is the side with the triangle. Thus, "above" and "below" are relative to the choice of points.

As the available degrees of freedom to move $\mathbf{x}_j^*$ correspond to the dimension of the tangent space $T_{\mathbf{x}_j} S^{k-2}$, perturbing all $\mathbf{z}_j$ terms defines a vector in $\prod_{j=1}^{k} T_{\mathbf{x}_j} S^{k-2}$ with dimension $k(k-2)$. (We could use $T_{\mathbf{x}_j} R^{k-1}$ to allow changes inside of the S^{k-2}. While such changes are admissible, add flexibility to the argument (corresponding to more variables), and occur after determining the basic location of the dual points, we don't need the extra dimensions.) For $\mathbf{n}$ defined by regular points, move the dual points so that $\left[\frac{k-1}{2}\right]$ of them are over each face of the k-gon. Over such a plane, the sum of signs is $-\left[\frac{k-1}{2}\right]$ (corresponding to the $\left[\frac{k-1}{2}\right]$ dual points) and below it the sum is $-\left(k - \left[\frac{k-1}{2}\right]\right) + 1 = -\left(1 + \left\lceil\frac{k-1}{2}\right\rceil\right) + 1 = -\left\lceil\frac{k-1}{2}\right\rceil$ corresponding to the $k - \left[\frac{k-1}{2}\right]$ dual points and one regular point. So, when k is even, both sums agree; when k is odd, they differ by unity. Because each plane has $k-1$

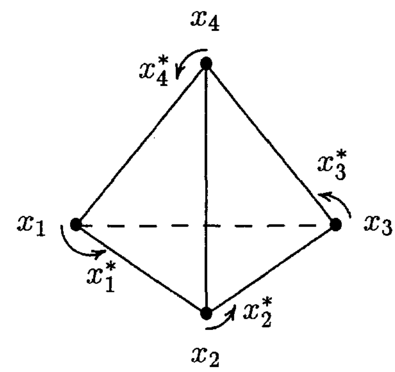

Figure 5. Placement over equilateral tetrahedron.

dual points, and because this is an open condition for each dual point, and because each dual point can be over j planes, $1 \leqq j \leqq k-2$, this construction is admissible if $k-2 \geqq \left\lceil \frac{k-1}{2} \right\rceil$, or if $k \geqq 3$.

One way to do this is to move $\mathbf{x}_1^*$ in direction $\mathbf{v}_1$ (the first order representation for $\mathbf{z}_1$) so that $\mathbf{x}_1^*$ is off the plane defined by $\mathbf{n}$ and over $\left\lceil \frac{k-1}{2} \right\rceil$ of the planes but below all others. As τ corresponds to a linear mapping, $\tau^j(\mathbf{v}_1)$ defines a tangent vector for $T^*_{\mathbf{x}+1}S^{k-2}, j = 1, \ldots, k$. By symmetry, $\tau^j(\mathbf{v}_1)$ moves $\mathbf{x}^*_{j+1}$ over and under the specified number of dividing planes defined by regular points. (I show below that the requisite number of dual points are above each plane.) Conversely, as this construction is equivalent to moving the regular points off of the dual points (i.e., move $\mathbf{x}_j$ off of $\mathbf{x}_j^*$ in direction $-\mathbf{v}_j$), the same condition holds for dividing planes defined solely by dual points.

To illustrate with $k = 4$ (Fig. 5), the regular point configuration is, essentially, an equilateral tetrahedron. Dual point $\mathbf{x}_1^*$ could be moved over any one of three faces $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, or $\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4$, or $\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4$. Choosing the first one places $\mathbf{x}_1^*$ below the other two. With this choice, τ moves $\mathbf{x}_2^*$ over $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$, point $\mathbf{x}_3^*$ over $\mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_4$, and $\mathbf{x}_4^*$ over face $\mathbf{x}_4, \mathbf{x}_1, \mathbf{x}_2$.

Instead of computing the orbit $\{\tau^j\}$, treat $\tau = (1, 2, \ldots, k)$ as having k equivalent forms where the jth form starts with j and retains the cyclic ordering; e.g., the second form is $(2, 3, \ldots, k, 1)$. Move $\mathbf{x}_1^*$ over $\left\lceil \frac{k-1}{2} \right\rceil$ faces in the following manner. Let the first two τ terms, 1 and 2, identify the $\mathbf{x}_j$ subscripts for a common edge of several faces. The remaining subscripts defining the first face are the first $k-3$ terms of the permutation $\sigma_1 = (3, 4, \ldots, k)$ obtained from τ by dropping the first two τ entries. To find the rest of the faces, use the same edge and let the remaining subscripts for the faces come from $\left\lceil \frac{k-1}{2} \right\rceil$ of the equivalent forms of σ_1 starting with σ_1, then $(4, 5, \ldots, k, 1)$, (for the second face), ... This places $\mathbf{x}_1^*$ over the specified faces and below all others.

To move $\mathbf{x}_j^*$, use the jth form of $\tau = (j, j+1, \ldots, 1, \ldots, j-1)$. The first two terms, $j, j+1$, define the common edge for the faces; the remaining subscripts are the first $k-3$ entries of the first $\left[\frac{k-1}{2}\right]$ forms of $\sigma_j = (j+2, j+3, \ldots, j-1)$, $(j+3, \ldots, j-1, j+2), \ldots$. Clearly, this permutation argument represents the $\mathbf{v}_j$ choices selected earlier and all faces of the k-gon are represented. Also, this procedure recovers the $k=4$ construction. To place regular points over the faces of the k-gon defined by dual points, use the inverse permutation $\tau^{-1} = (k, k-1, \ldots, 2, 1)$.

While a group theory exercise can be used to verify that $\left[\frac{k-1}{2}\right]$ dual points are above each face, the following geometric argument emphasizes the symmetry and the requirement that each $\mathbf{x}_j^*$ is above $\left[\frac{k-1}{2}\right]$ faces and below all others. (Note for $k=4$ (Fig. 5) that only $\left[\frac{4-1}{2}\right] = 1$ dual point is above each plane.) From symmetry, this condition only needs to be checked for one face, say, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1}$. Observe that $\mathbf{x}_j^*$ is above this face only if one of the $\left[\frac{k-1}{2}\right]$ admissible forms of σ_j has the index k at the end. This is true iff k is either the last or one of the first $\left[\frac{k-1}{2}\right] - 1$ entries of σ_j. (For instance, k is the last entry for σ_1, so $\mathbf{x}_1^*$ is above this plane.) The conclusion follows from a simple computation.

To illustrate with $k=6$ by positioning $\mathbf{x}_2^*$ with the σ_2 forms $(4, 5, 6, 1)$, $(5, 6, 1, 4)$, $(6, 1, 4, 5)$, and $(1, 4, 5, 6)$, observe that $\mathbf{x}_2^*$ is moved slightly off of $\mathbf{x}_2$ onto S^4 so it is above the $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6$ and $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_5, \mathbf{x}_6, \mathbf{x}_1$ planes, but below all others. To illustrate which $\left[\frac{6-1}{2}\right] = 2$ dual points are above which planes, observe that only $\mathbf{x}_1^*$ and $\mathbf{x}_4^*$ are above the $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_5$ plane. Point $\mathbf{x}_2^*$ is not above this plane because 6 is neither the first nor last entry of $\sigma_2 = (4, 5, 6, 1)$, and $\mathbf{x}_5^*$ is excluded because 6 is not in $\sigma_5 = (1, 2, 3, 4)$ ($\mathbf{x}_6$ is on the common edge for the faces). Similarly, only $\mathbf{x}_3^*, \mathbf{x}_6^*$ are above the $\mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6, \mathbf{x}_1$ plane.

Now consider a plane which, before the perturbation, coincides with a face of regular points, and after the perturbation it is defined by $\left[\frac{k-1}{2}\right]$ regular points and the $\left[\frac{k-1}{2}\right]$ dual points lifted above this face. By simple geometry using the facts that the dual points associated with the remaining regular points are below this plane but near the regular point and by symmetry (given by $\mathbf{v}_j = \tau^{j-1}(\mathbf{v}_1)$), this dividing plane has no points on one side of it. As this plane contains a face of the hull defined by the regular and perturbed dual

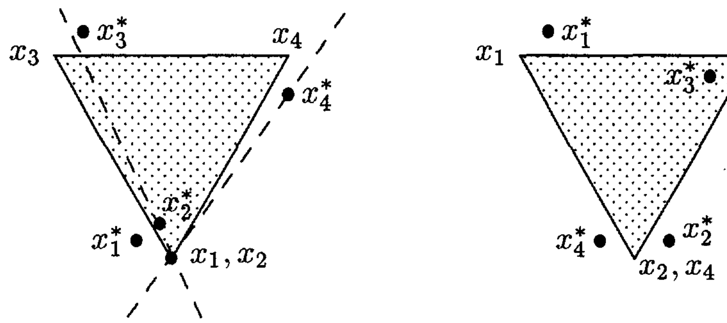

Figure 6. Viewing $k = 4$ from edges.

points, there are $k - \left[\frac{k-1}{2}\right]$ regular and $k - \left[\frac{k-1}{2}\right]$ dual points on the other side. This difference between the sums of signs is either zero or -1 depending on whether k is odd or even. Also, any dividing plane obtained by going from the face of regular points to the face of the hull of dual and regular points involves an alternating argument where each regular point is replaced with its dual mate. This adds $+1$ to the side with the new regular point, but removes a -1 from the side losing a dual point so the difference between sums of signs remains the same.

So far, only one condition is imposed on $\mathbf{v}_1$; other conditions (i.e., other equations) result from analyzing dividing planes with dual points below a particular face. Start with an "edge" or axis of rotation consisting of $k-2$ regular points. Adjoining one the remaining two regular points to this edge defines a face of the k-gon. There are two kinds of edges; a nonsingular edge is where the total number of dual points above both faces is $2\left[\frac{k-1}{2}\right]$ while a "singular edge" has a smaller number. It is not difficult to show that all edges are nonsingular for $k \leqq 5$ and that singular edges always occur for $k \geqq 6$. (Singular edges are created when the same dual points are above both planes sharing that axis.)

A nonsingular edge, has $k - 2\left[\frac{k-1}{2}\right]$ dual points between (i.e., below) both faces. This value is 1 or 2 depending, respectively, on whether $k-1$ is even or odd. The former case, where the sum of signs on each size of a dividing plane agree, has only one dual point between faces. Thus the alternating structure and the "equality of sums" property hold. For the latter setting, where the sums of signs differ by unity, notice that the side of a dividing plane with the larger value of the two sums is "over" a face. Therefore, when the dividing plane rotates, at some position the side with the larger sum changes. This is the role played by the two dual points between the two faces; it preserves the summation property. The only problem (i.e., an extra equation to be satisfied) is to avoid the degeneracy where the plane passing through the axis of rotation and one dual point meets the second dual point. (In the left diagram in Fig. 6, for instance, there are $\mathbf{x}_2^*, \mathbf{x}_3^*$ choices where the plane passing through the $\mathbf{x}_1, \mathbf{x}_2$ axis and $\mathbf{x}_2^*$ contains $\mathbf{x}_3^*$. According to the assignment process, this degeneracy arises only with an axis containing an edge assigned to a dual point.) The equation (i.e., let $\mathbf{n}$ be the wedge product of the $k-2$ points on the axis and a dual point $\mathbf{x}_i - \mathbf{x} + \tau^{i-1}(\mathbf{v}_1)$ and then set $\mathbf{n} \wedge (\mathbf{x}_j - \mathbf{x} + \tau^{j-1}(\mathbf{v}_1))$ equal to zero) represents a zero determinant expressed in terms of the $k-1$ variables (coordinates) of $\mathbf{z}_1$. Thus this lower dimensional degeneracy is avoided by almost all $\mathbf{v}_1$ values. As a specific choice, move $\mathbf{x}_1^*$ closer to the $\mathbf{x}_1, \mathbf{x}_2$ edge than any other edge. (For instance, this requires $\mathbf{x}_2^*$ to be over the $\mathbf{x}_2, \mathbf{x}_3$ edge so that the line through $\mathbf{x}_2, \mathbf{x}_2^*$ is closer to $\mathbf{x}_3$ than $\mathbf{x}_3^*$.) By symmetry arguments, the rotating plane passes through each dual point separately and preserves the sum condition. This situation is displayed in Fig. 6 where the rotation is about the $\mathbf{x}_1, \mathbf{x}_2$ and about the $\mathbf{x}_2, \mathbf{x}_4$ axis.

To illustrate by rotating a plane about the $\mathbf{x}_2, \mathbf{x}_4$ edge (the bottom vertex of the shaded figure on the right in Fig. 6), the succession of points is $\mathbf{x}_2^*, \mathbf{x}_3, \mathbf{x}_3^*, \mathbf{x}_1^*, \mathbf{x}_1, \mathbf{x}_4^*$. (As the $\mathbf{x}_2, \mathbf{x}_4$ edge is not assigned to a dual point, the permutation argument does not allow a degeneracy.) The necessary transition between which side has the larger sum occurs when the dividing plane passes between $\mathbf{x}_3^*$ and $\mathbf{x}_1^*$. The figure on the left shows the transition about an axis corresponding to an edge in the assignment of faces for dual points. (This is where a degeneracy condition occurs because the "next" τ dual point is near the "next" τ edge.) The rotating dividing plane passes through $\mathbf{x}_4^*, \mathbf{x}_4, \mathbf{x}_3^*, \mathbf{x}_2^*, \mathbf{x}_3, \mathbf{x}_1^*$.

The restrictions (i.e., new equations) imposed by a singular edge occur because with only $2\left[\frac{k-1}{2}\right] - \alpha$ points above the two faces, there are $k - 2\left[\frac{k-1}{2}\right] + \beta$ dual points between them, $\alpha, \beta > 0$. But the extra dual points ($\beta > 0$) between the faces jeopardize the alternating count procedure. To rectify this problem, we first show that $\beta = \alpha$. This equality follows because to have α dual points above both faces, they are in a sector such as the one indicated in Fig. 7a. (The lower vertex of the shaded region is the singular edge.

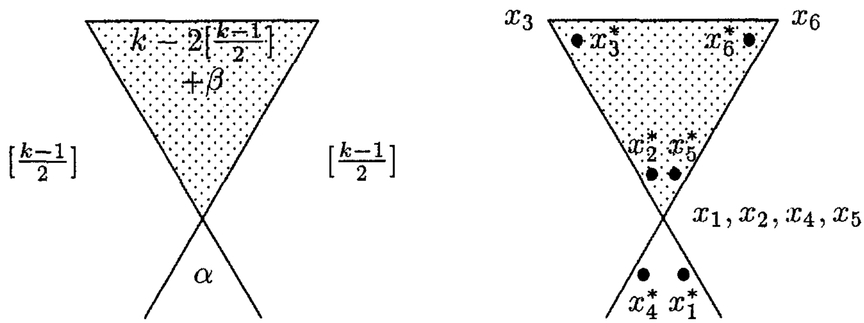

Figure 7. A $k = 6$ singular edge.

This construction requires $k-2 \geqq 2$ because lower dimensions would require a point in this sector to be outside S^{k-2}. This edge is the hypersurface connecting the indicated S^{k-2} points, and the sector is the region between the planes and the sphere – or the surface of the sphere.) The wedges on the left and right sides have $\left[\frac{k-1}{2}\right]-\alpha$ dual points. There are the $k-\left[\frac{k-1}{2}\right]$ dual points below the face represented by the downward slanting line. Adding the points in the two sectors on this side of the line, we obtain $\left\{k-2\left[\frac{k-1}{2}\right]+\beta\right\}+\left\{\left[\frac{k-1}{2}\right]-\alpha\right\}$. The $\alpha=\beta$ conclusion follows by comparing values.

As there are $k-2\left[\frac{k-1}{2}\right]$ more points (either one or two) between the faces than over both, it is clear how to preserve the alternating rule (i.e., solve the added equations); position the α dual points so that when the dividing plane rotates, it alternates passing through a point between the planes and one of the α points above both planes. Because dual points from the two different regions are moved onto different sides of the rotating dividing plane, the alternating structure is preserved. A dimensional (each $\mathbf{x}_j^*$ is over at most $\left[\frac{k-1}{2}\right]$ faces, so these one-dimensional restrictions occur at most $\left[\frac{k-1}{2}\right]-1$ times) and symmetry argument (each $\tau^j(\mathbf{v}_1)$ vector emphasizes different components of $\mathbf{v}_1$) shows that such a construction is possible. (Specific choices determine an ordering the faces that $\mathbf{x}_j^*$ is over in terms of the distance of this point to each plane; this positions the dual point in a sector.) When $k-1$ is odd, there are two more dual points between the faces which, as indicated in Fig. 6 for $k=4$, are needed to change the side with the larger sum.

To illustrate with the figure to the right in Fig. 7, the plane with indicated axis passes through the dual points in the order $\mathbf{x}_6, \mathbf{x}_5^*, \mathbf{x}_4^*, \mathbf{x}_6^*, \mathbf{x}_3^*, \mathbf{x}_1^*, \mathbf{x}_2^*, \mathbf{x}_3$. (The $\mathbf{x}_5^*$ point comes before $\mathbf{x}_6^*$ because the line between $\mathbf{x}_5, \mathbf{x}_5^*$ is closer to $\mathbf{x}_6$ than $\mathbf{x}_6^*$. This construction, then, imposes restrictions on other dimensions – distance from planes – between $\mathbf{x}_4^*$ and $\mathbf{x}_5^*$.) Because dual points are chosen from opposite sides of the axis of rotation, this preserves the alternating strcture needed for the construction. This argument concerns only those dual points above both faces, there are more variables than conditions to satisfy.

This argument easily is modified to handle an axis of rotation with dual points. If the axis has $\mathbf{x}_j^*$, start with $\mathbf{x}_j$ and obtain the alternating rule established above. By construction, before moving the dividing plane from $\mathbf{x}_j$ to $\mathbf{x}_j^*$, the difference between sums of signs satisfies the desired property. One side of this plane has $\mathbf{x}_j^*$; it adds -1 to the summation. By moving the plane off of $\mathbf{x}_j$ and toward $\mathbf{x}_j^*$, $+1$ is added to side that now has $\mathbf{x}_j$. When the plane reaches $\mathbf{x}_j^*$, the -1 value of -1 is removed. Thus, the sum on both sides increase by $+1$, so the difference in sums remains the same. Observe that this argument only works for $\left[\frac{k-1}{2}\right]$ dual points. However, using the symmetry between dual and regular points, the same argument applies should we start with an axis consisting of all dual points, rather than regular points. Thus, for any number of dual and regular points on the dividing plane, the conclusion about the difference between the sums of signs remains.

A **n** degeneracy created by a $\mathbf{x}_j, \mathbf{x}_j^*$ pair already is partially resolved because they occur for various orientations of the rotating dividing plane. (For instance, in Fig. 6, one dividing plane has the $\mathbf{x}_2, \mathbf{x}_2^*$ pair.) To resolve the remaining situations, use the fact that $\mathbf{x}_j^*$ is near $\mathbf{x}_j$ and the line defined by these two points passes closer to $\mathbf{x}_{j+1}$ than $\mathbf{x}_{j+1}^*$ but over the $\mathbf{x}_j, \mathbf{x}_{j+1}, \mathbf{x}_{j+2}, \ldots$ plane. The first assumption means that only a small perturbation is made in changing from the regular to the partner dual point. The second, which provides one dimension for the orientation of a dividing plane, allows $\mathbf{x}_j, \mathbf{x}_j^*$ lines to be replaced, initially, by $\mathbf{x}_j, \mathbf{x}_{j+1}$ directions and then perturbed. The only issue is the side containing $\mathbf{x}_{j+1}^*$, and this is decided by the initial assumption. This completes the construction.

From this construction, $s^* = \left[\frac{k+1}{2}\right]$, so this configuration supports all $q \geqq k + \left[\frac{k+1}{2}\right] = \left[\frac{3k+1}{2}\right]$ rules for $n = 2k$ where $\mathbf{x}$ is a nonbliss core point and $n = 2k+1$ where $\mathbf{x}$ is a bliss core point. To illustrate with $k = 4$, these are all $q \geqq 7$ rules for $n = 8$ and $\mathbf{x}$ a nonbliss core point, and $n = 9$ and $\mathbf{x}$ a bliss core point. For $k = 50$, this supports all $q \geqq 71$ rules for $n = 100$ and $\mathbf{x}$ a nonbliss core point.

The same construction and reasoning holds for all remaining cases, so I emphasize only differences. The first step, moving a specified number of dual points above each plane, is an open condition. The purpose of the remaining equations is to avoid degeneracies; they are related to singular edges. As no more than $\left[\frac{k-1}{2}\right] - 1$ equations are assigned to each dual point, and each equation involves one of the $k-1$ available coordinates, standard dimension arguments (of the n-equations, m unknowns, $m \geqq n$ type) about algebraic degeneracy conditions ensure the conclusion. In fact, because not all of the available variables are used, $k \geqq 4$ admit alternative constructions with the desired behavior. (Indeed, this argument essentially uses only $k-2$ of the available $k(k-2)$ dimensions. More important for the alternating behavior are the regions created by the planes and by placing points closer to one plane than another as used for singular edges.) Other settings use some of these extra sectors defined by the various planes.

To start with $m = 2k-1$, remove $\mathbf{x}_k^*$ which is over the $\left[\frac{k-1}{2}\right]$ faces with $\mathbf{x}_j$ subscripts $(k, 1, 2, 3, \ldots, k-2)$, $(k, 1, 3, 4, \ldots, k-1), \ldots, \left(k, 1, \left[\frac{k-1}{2}\right] + 1, \ldots, \left[\frac{k-1}{2}\right] - 1\right)$. As we need $\left[\frac{k-1}{2}\right]$ dual points over each face, compensate for the loss of $\mathbf{x}_k^*$ by moving $\mathbf{x}_1^*$ so it also is over the first $\mathbf{x}_k^*$ face, $\mathbf{x}_3^*$ so also it is over second $\mathbf{x}_k^*$ face,..., and $\mathbf{x}_{[k-1/2]+1}^*$ so it also is over the last $\mathbf{x}_k^*$ face. Standard linear algebra arguments ensure that this construction is possible. The analysis differs from $m = 2k$ only in that singular edges start with $k = 4$ (because $\mathbf{x}_1^*$ is over two faces sharing the $\mathbf{x}_1, \mathbf{x}_2$ edge). This singular edge construction just involves choosing positions (how close to what plane) for dual points that are above both faces so they alternate with the dual points between the surfaces. (This singular edge manifests the loss of the $\mathbf{z}_k$ variables.)

The analysis remains essentially the same; e.g., a plane passing through all regular points has $\left[\frac{k-1}{2}\right]$ dual points above, and $(k-1) - \left[\frac{k-1}{2}\right]$ dual points and a regular point below it. The same dimensional arguments for the singular edges apply to preserve the alternating sign convention. By comparing sums we have, again, that the difference between sums is zero or unity, depending, respectively, on whether $k-1$ is odd or even. As this construction defines $s^* = \left[\frac{k-1}{2}\right] + 1 = \left\lceil\frac{k}{2}\right\rceil$, it supports all $q \geq k + \left\lceil\frac{k}{2}\right\rceil = \left\lceil\frac{3k}{2}\right\rceil$ rules for $n = 2k - 1$ and $\mathbf{x}$ a nonbliss core point, or $n = 2k$ and $\mathbf{x}$ a bliss core point.

To illustrate with $k = 4$ and $m = 7$, to replace $\mathbf{x}_4^*$ move $\mathbf{x}_1^*$ above the $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4$ plane so that the $\mathbf{x}_1^*, \mathbf{x}_1, \mathbf{x}_2$ dividing plane separates $\mathbf{x}_2^*$ and $\mathbf{x}_3^*$. (This singular edge construction places $\mathbf{x}_1^*$ close to the $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ plane. In Fig. 5, $\mathbf{x}_1^*$ is in the wedge defined by extending the bottom and left side planes with $\mathbf{x}_1, \mathbf{x}_2$ as a singular edge. In left figure of Fig. 6, $\mathbf{x}_1^*$ is in the small wedge with dashed lines.) The same "alternating side" argument, or sum of signs, shows that (after dual points are converted) $s^* = 2$, so this configuration supports all $q \geq \left\lceil\frac{m-3}{2}\right\rceil + 4 = 6$ rules when $n = m = 7$ and $\mathbf{x}$ a nonbliss core point, and $n = m + 1 = 8$ where $\mathbf{x}$ is a bliss core point.

The argument for $m > 2k$ an even integer mimics the k-gon construction. In the two-dimensional subspace of the right plane defined by $y_1 = 1, y_2, y_3$, place $c = \frac{m}{2} - (k-3)$ points (roughly equally spaced) on the unit circle with center $(1, 0, 0)$; this two-dimensional configuration, $\mathscr{C}(3)$, has c vertices and c faces. To define a three-dimensional configuration $\mathscr{C}(4)$ by adding the dimension y_4 (so the configuration is in the three-dimensional subspace $y_1 = 1$ of a four-dimensional space), in the unit sphere in this subspace, place a (rescaled) copy of $\mathscr{C}(3)$ on the intersection of the $y_4 = t < 0$ plane with the sphere, and place an added point near where the positive y_4 axis passes through the sphere. This figure has $c + 1$ vertices and faces. The number of vertices is obvious; c faces come from connecting each lower dimensional $\mathscr{C}(3)$ face with the new vertex; the last face is the hull $\mathscr{C}(3)$ which becomes a face in the higher dimensional setting. Notice that $c = 3$ and an appropriate t choice returns the equilateral tetrahedron used with $k = 4$.

To extend $\mathscr{C}(\alpha - 1)$ to $\mathscr{C}(\alpha)$, increase the subspace dimension by including coordinate y_α. Place a rescaled copy of $\mathscr{C}(\alpha - 1)$ on the sphere defined by the intersection of the $y_\alpha = t < 0$ plane with the unit sphere. Add a point near where the positive y_α axis passes through the sphere. Again, the number of faces and vertices of $\mathscr{C}(\alpha)$ is $c + \alpha - 3$ because the new vertex and each lower dimensional $\mathscr{C}(\alpha - 1)$ face defines a $\mathscr{C}(\alpha)$ face, and the hull $\mathscr{C}(\alpha - 1)$ becomes a $\mathscr{C}(\alpha)$ face. Thus $\mathscr{C}(k)$ has $c + (k - 3) = \frac{m}{2}$ faces and vertices. When $c = 3$, the t values at each stage of the construction can be selected to obtain the equilateral k-gon.

A degeneracy, created by placing $c > 3$ points on a two-dimensional subspace, needs to be broken to ensure robust examples and to introduce additional regions (generated by the intersections of the various planes) for dual points. If t_j is the y_k coordinate of $\mathbf{x}_j$ in $\mathscr{C}(k)$, we have $t_j = t$ for $j < \frac{m}{2}$. For a small $\varepsilon > 0$ value, change the t_j values so that $t = t_1 < t_2 < \cdots < t_{m/2-1} < t + \varepsilon < 0$, and denote this configuration by $\mathscr{D}(k)$. Denote by $\mathscr{D}(3)$ the hull defined by the perturbation of the first c points. To show that the dual points can be perturbed off of the vertices of $\mathscr{D}(k)$ in the desired manner, dimensional arguments are used (but illustrated with examples). While $\mathscr{C}(k)$ has $\frac{m}{2}$ faces and vertices, $\mathscr{D}(k)$ has $\frac{m}{2}$ vertices and $c + (c - 2)(k - 3)$ faces. (At each stage in the argument computing the number of faces for $\mathscr{C}(k)$, $\mathscr{C}(\alpha - 1)$ becomes a face for $\mathscr{C}(\alpha)$. The stated number arises by computing how this face divides after the perturbation.) More important than the number is that with a small ε value $\mathscr{D}(k)$ is essentially the same as $\mathscr{C}(k)$.

An alignment with $\left\lceil \frac{k-1}{2} \right\rceil$ dual points above each $\mathscr{D}(k)$ face has a $-\left[\frac{k-1}{2}\right]$ sum for these points. Below the face are the $(k - 1) - \left\lceil \frac{k-1}{2} \right\rceil = \left[\frac{k-1}{2}\right]$ dual points that are moved off of a vertex of the face, and $\frac{m}{2} - (k - 1)$ pairs of dual and regular points. The sum of signs from each pair is zero, so the total sum of signs is $-\left\lceil \frac{k-1}{2} \right\rceil$. Thus, the difference between sums is zero when $k - 1$ is even, and unity when $k - 1$ is odd.

The argument about moving a dividing plane by replacing regular points with dual points above this face is as given earlier. Thus we only need consider dual points that are below faces. So, consider an axis defined by $k - 1$ regular points. Adding another regular point defines a plane; for two choices of an added regular point the plane contains a $\mathscr{D}(k)$ face. Between these extreme faces are $\frac{m}{2} - k$ regular points and their dual mates. As these points cancel in the sum of signs, the major concern involves dual points with a regular point mate on one of the extreme planes. The earlier singular edge argument shows that with less than $2\left[\frac{k-1}{2}\right]$ dual points over both faces, $\alpha = \beta$. Consequently, ignoring pairs of regular and dual points, the difference between the number of dual points between and above both planes is either one or two. Therefore, the earlier alternating structure argument applies. Thus, the construction is possible if $\left\lceil \frac{k-1}{2} \right\rceil$ points can be placed over each face.

Notice that each face is defined by $k-1$ points and each $\mathbf{x}_j, j \neq 1$, is a vertex for at least $k-1$ faces. Point $\mathbf{x}_1$ is a vertex for $k-1$ $\mathscr{C}(k)$ faces and all the new $\mathscr{D}(k)$ faces obtained by breaking the degeneracy. By choosing a small ε value, the dual points above both $\mathscr{D}(3)$ is essentially flat) while remaining near the regular point mate. The fact that it is possible to move the dual points so that over each $\mathscr{D}(k)$ face there are precisely $\left\lceil \frac{k-1}{2} \right\rceil$ dual points is an immediate consequence of the observation that if $\mathbf{x}_j$ is a vertex for b faces, then $\mathbf{x}_j^*$ can be moved over any number of faces between one and $b-1$. It cannot be over all b faces as this would place the dual point outside of S^{k-2}. This is reflected by the earlier construction where $(k-1)-1$ faces were identified by considering only those faces passing through the $\mathbf{x}_j, \mathbf{x}_{j+1}$ edge.

With this construction, $s^* = \left\lceil \frac{m-(k-1)}{2} \right\rceil$, so it supports all

$$q \geqq k + \left\lceil \frac{m-k-1}{2} \right\rceil \tag{5.2}$$

rules when $n = m$ and $\mathbf{x}$ is a nonbliss core point, or $n = m+1$ and $\mathbf{x}$ is a bliss core point.

To illustrate with $k = 4$ and $m \geqq 10$, place $\mathbf{x}_1^*$ above $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ face, $\mathbf{x}_j^*$ above $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_{m/2}$ face, $\mathbf{x}_j^*$ above both $\mathbf{x}_j, \mathbf{x}_{j+1}, \mathbf{x}_{m/2}$ and $\mathbf{x}_1, \mathbf{x}_j, \mathbf{x}_{j+1}$ faces, $j = 3, \ldots, c-1, \mathbf{x}_c^*$ over $\mathbf{x}_c, \mathbf{x}_1, \mathbf{x}_{m/2}$, and $\mathbf{x}_{m/2}$ over the $\mathbf{x}_{m/2}, \mathbf{x}_1, \mathbf{x}_2$ face. Each face has $\left\lceil \frac{4-1}{2} \right\rceil = 1$ dual point over it. The only singular edge constructions involve the $\mathbf{x}_j, \mathbf{x}_{j+1}$ edges for $j = 3, \ldots c-1$. It is easy to see that this condition is satisfied should $\mathbf{x}_j^*$ also be "above" *relative to* $\mathscr{D}(3)$ the $\mathbf{x}_c, \mathbf{x}_j, \mathbf{x}_{j+1}$ plane; that this can be done is obvious from the geometry. This is illustrated in Fig. 8 with $c = 5$ where the shaded region is $\mathscr{D}(3)$ from the perspective of the side facing $\mathbf{x}_{m/2}$. The dagger, representing $\mathbf{x}_3^*$, is placed in the narrow (because ε has a small value) sector between the extensions of the two planes so it is above both of them. (As the differences between $\mathscr{D}(k)$ and $\mathscr{C}(k)$ arise in $\mathscr{D}(3)$, the singular edge arguments involve $\mathscr{D}(3)$.)

The case where $m > 2k$ is odd is done by dropping $\mathbf{x}_{m/2}$ and using the same counting argument. Again, Eq. 5.2 holds, but with the smaller m value.

Finally, we need to consider the special cases where m satisfies $k+1 \leqq m < 2k$. These special constructions are in R^k rather than in the right plane, and only regular points are used. For $m = k+1$, place the points near the vertices of an equilateral $(k+1)$-gon on S^{k-1} where $\mathbf{x}$ is at the center of the sphere. A dividing plane consists of $k-1$ of these points, so $s^* = 1$. Thus, this construction supports all $q \geqq k+1$ rules for $n = m$ (i.e., only the unanimity

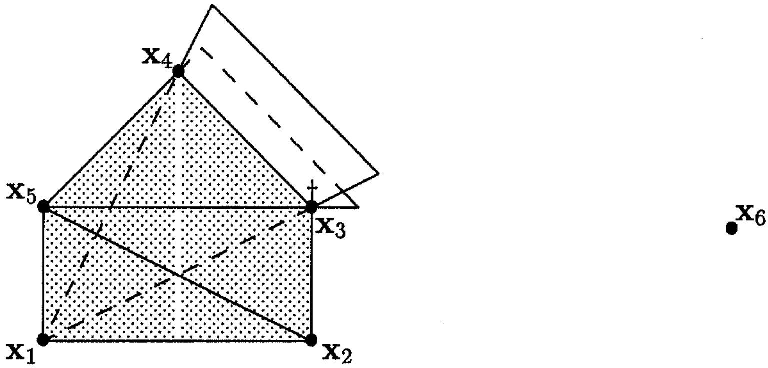

Figure 8. Position of $\mathbf{x}_3^*$ in $\mathscr{D}(4)$ for $m = 12$.

rule) and $\mathbf{x}$ a non-bliss core point, or $n = m + 1 = k + 2$ (so $q \geqq n - 1$) and $\mathbf{x}$ a bliss core point.

For $m = (k+1) + j$ where $1 \leqq j \leqq k - 2$. On j of the faces of the $(k+1)$-gon, attach another $(k+1)$-gon. (So, for $k = 3$ and $j = 1$, this becomes two tetrahedrons glued along one of the equilateral triangle faces.) Thus, j pairs of vertices are opposite each other. Adjust the length of the distances so all points are on S^{k-1} and the distances between opposing vertices is symmetric. Let $\mathbf{x}$ be near the center. (So, for the $k = 3$, $j = 1$ illustration, $\mathbf{x}$ is near the center of the equilateral triangle used to glue the two tetrahedrons together.) If $\mathbf{x}$ is not on the plane defined by k points, then it follows that at most $\left\lceil \frac{m-(k-1)}{2} \right\rceil$ points are on a side of a dividing plane. Thus, this configurations supports all $q \geqq k + \left\lceil \frac{m-k+1}{2} \right\rceil$ rules for $n = m$ and $\mathbf{x}$ a nonbliss core point, or $n = m + 1$ and $\mathbf{x}$ a bliss core point.

6 Upper bounds

The examples of Sect. 5, examples support the assertions of Thm. 1 for $\beta = -1$. It remains to show that these examples are robust, to construct $\beta \geqq 0$ examples, to show that the bounds of Thm. 1 are tight (i.e., that one cannot do better), and to prove Thm. 2a, c. All are done with singularity theory.

The reader unfamiliar with singularity theory (e.g., [GG] or [SS]) can view it as a sophisticated implicit function theorem. Let $f: R^n \to R^m$, be a smooth mapping and Σ a smooth b dimensional manifold in R^m (so, the codimension of Σ is $m - b$). According to the implicit function theorem, if f satisfies appropriate conditions, then, at least locally, $f^{-1}(\Sigma)$ is a codimension $m - b$ (or dimension $n - (m - b)$) submanifold of R^n. The needed *transversality condition* for $\mathbf{x}$ where $f(\mathbf{x}) \in \Sigma$ is that the span of the tangent spaces $D_{\mathbf{x}} f(R^n)$ and $T_{f(\mathbf{x})}\Sigma$ is R^m. (The usual inverse function theorem where Σ is a point only requires the $D_{\mathbf{x}} f(R^n)$ to span R^m – to have rank m – as the tangent space of a point is the zero vector.)

The use this tool to analyze first and second derivative conditions imposed upon a function f, use the "jet" map. This is the mapping $j^2f(\mathbf{x}) = (\mathbf{x}, f(\mathbf{x}), Df_{\mathbf{x}}, D^2_{\mathbf{x}}f)$. The domain of this mapping is R^n, and the range is $J^2 = R^n \times R^m \times L(R^n, R^m) \times B(R^n, R^m)$ where $L(R^n, R^m)$ are the linear maps from R^n to R^m and $B(R^n, R^m)$ are the bilinear symmetric maps. If the derivative conditions define a manifold Σ in J^2 for $f = (u_1, \ldots, u_n)$ where j^2f meets Σ transversely, then the implicit function theorem ensures that the set of points satisfying these conditions is (locally) smooth submanifold with codimension equal to the codimension of Σ. The difficulty is to verify the transversality condition.

This problem is resolved by extending the notion of a transverse intersection to allow j^2f to miss Σ. Then the Thom Transversality Theorem ensures for a generic set of functions, the intersection is transverse. (When issue space is restricted to a compact subset, this assertion holds for an open-dense set of functions [SS].) In words, the general situation is that the preferences either fail to satisfy, or do satisfy the core conditions robustly. Thanks to this important assertion, the proof of the theorem reduces to (1) representing the core derivative conditions of Prop. 1 in terms of a manifold Σ in J^2, (2) finding the codimension of Σ, and (3) showing that we are not discussing the empty set.

Task (3) requires showing that there exists a f where $j^2f \in \Sigma$. This is important because for many examples, j^2f never meets Σ; this is where the alternatives force the voters' preferences to cluster in ways that do not satisfy Prop. 1. In other words, rather than asserting that the core always exists, Theorem 1 only ensures that there exist robust examples in the indicated issue spaces. It is important to note that we do not need to verify that j^2f meets Σ transversely. The assertion that there exist functions that do so, hence robust examples exist (not necessarily this particular f) is a gift from Thom. The idea is that if $j^2f \in \Sigma$ is a boundary point, then, as Thom's result has the conclusion holding for an open set about f, near-by functions satisfy the transversality condition. To illustrate, the examples of Sect. 5 satisfy the conditions, so from singularity theory, we know there is an open set of preferences with this property for the specified dimensions of issue space.

Proof of Theorem 1. For specified n and q, to construct a manifold Σ in $L(R^k, R^n)$ (the space of linear maps from R^k to R^n), use the identification

$$L(R^k, R^n) \approx \{\mathbf{A} = (\mathbf{A}_1 \mathbf{A}_2, \ldots, \mathbf{A}_n) \mid \mathbf{A}_j \in R^k\}.$$

Namely, think of $\mathbf{A}_j$ as a dummy variable where its range of R^k represents all possible choices for $\nabla u_k(\mathbf{x})$. An advantage of using this identification, rather than discussing properties of matrices, is that it is a simple parametric representation of a manifold in $L(R^k, R^n)$ in terms of the $\mathbf{A}_j$ vectors. Since $\mathbf{A}_j$ is meant to capture conditions on $\nabla u_j(\mathbf{x})$, these Σ manifold definition follows directly from the core conditions on the gradients. (This representation is another way my method differs significantly from the approach of [B, Sc, MS1].)

Bliss-core points. To start with bliss-core points, assume that the core point is the bliss point for the first agent. This means we must examine all points

$\mathbf{x}$ where $\nabla u_1(\mathbf{x}) = \mathbf{0}$. All such situations are captured by the manifold $\Sigma_1 = \{\mathbf{A} | \mathbf{A}_1 = \mathbf{0}\}$. Specify the core conditions of Prop. 1 as $\mathscr{CC}_m = \{\mathbf{A}|$ there are m nonzero vectors $\mathbf{A}_j$ and the plane in R^K defined by any choice of $k-1$ of them has no more than $\left\lceil \frac{m-k+1}{2} \right\rceil$ vectors on either side$\}$. This is an open condition, and the examples of Sect. 5 show it is nonempty. Because there are no restrictions on any other J^2 coordinates, if there is any $\mathbf{x} \in R^k$ where $\nabla u_1(\mathbf{x}) = \mathbf{0}$, then $j^2(u_1, \ldots, u_n) \in \Sigma_1$. The codimension of $\Sigma_1 \cap \mathscr{CC}_{n-1}$ is k (reflecting that all k components of $\mathbf{A}_1$ are completely specified), so, generically, the set of $\mathbf{x}$'s satisfying this condition is $k-k$, or zero-dimensional; it is a set of isolated points.

Imposing any other condition on Σ_1 which increases the codimension (making it larger than k), corresponds to where the set of points satisfying the conditions is generically empty. For instance, if the core point is the bliss point for two or more agents where one is the jth, $j \neq 1$, then the manifold capturing this situation is $\Sigma_2 = \{\mathbf{A} | \mathbf{A}_1 = \mathbf{A}_j = \mathbf{0}\}$ with codimension $2k$. Thus, the set of points satisfying this condition generically form a union of $k-2k$ dimensional manifolds. As this dimension is negative, this behavior is generically impossible. Generically, a point is a bliss point for at most one agent.

A condition motivated by Plott's construction is to suppose that at the bliss-core point the gradients of two other agents lie along the same line. If these agents are, say 2 and 3, then the manifold in $L(R^n, R^m)$ capturing these conditions is $\Sigma_3 = \{\mathbf{A} | \mathbf{A}_1 = \mathbf{0}, \mathbf{A}_2 = \lambda \mathbf{A}_3, \lambda \in R\}$ with codimension $2k-1$. (k codimensions come by specifying $\mathbf{A}_1 = \mathbf{0}$ and $k-1$ from the fact that the direction of $\mathbf{A}_2$ is specified.) As the codimension of Σ_3 is larger than the dimension of issue space when $k \geqq 2$, such behavior is generically impossible outside of a one-dimensional issue space.

To generalize Σ_2, Σ_3, consider where $\nabla u_1(\mathbf{x}) = \mathbf{0}$ and there are b agents, $1 \leqq b \leqq k$, with index set D, so that $\{\nabla u_j(\mathbf{x})\}_{j \in D}$ is in a $b-1$ dimensional subspace. To see that this is generically impossible, consider the manifold $\Sigma_{1,D} = \{\mathbf{A} | \mathbf{A}_1 = \mathbf{0}$, there exists scalars λ_j, not all zero, so that $\Sigma_{j \in D} \lambda_j \mathbf{A}_j = \mathbf{0}\}$. As the sum ensures that one $\mathbf{A}_j, j \in D$, is determined by the others, another value is added to the codimension; as $\Sigma_{1,D}$ has codimension $k+1$, this defines a generically empty situation.

To use $\Sigma_{1,D} \cap \mathscr{CC}_{n-1}$ to prove Theorem 1, recall from [2] and Prop. 1 that a plane defined by $k-1$ linearly independent vectors can have no more than $(q-1)-(k-1) = q-k$ gradient vectors on either side. If no other gradient vectors are on this plane, then by counting the maximum number of vectors on both sides, the number on the dividing plane and the bliss point, we have $2(q-k)+k-1+1 \geqq n$ or $k \leqq 2q-n$. That is, if $k > 2q-n$, then another gradient is on this plane and these core points are in $\Sigma_{1,D}$ for some D with k indices.

All possible situations are given by the $\binom{n-1}{k}$ distinct sets of k indices from $\{2, 3, \ldots, n\}$. If D denotes such a set, $j^2 f$ must be in some $\Sigma_{1,D}$. Thus, the set of points satisfying such situations is generically empty. This condition includes the bliss-core points for $k > 2q - n$. As all bliss core points are obtained by changing the choice of D and the identity of the voter whose bliss point is the core point, this is a finite condition. Namely, it corresponds to a finite union of submanifolds, all with the same codimension. Each submanifold is obtained from the first by a permutation of the indices. Thus, the assertion follows. □

Nonbliss core points. The argument showing that Eqs. 2.3, 2.4 determine upper bounds for the generic existence of nonbliss core points resembles the bliss-core point setting but the counting argument differs. To help the reader, the ideas are introduced with special cases.

The $q = 3, n = 4$ case. For $k \geqq 1$, if the four gradients are on the same line, then the line is defined by one vector, so $\Sigma = \{\mathbf{A} | \mathbf{A}_j = \lambda_j \mathbf{A}_1, j = 2, 3, 4\}$. Here, the codimension is $k - 1$ for each vector $k = 2, 3, 4$ for a total codimension of $3(k - 1)$. Such a situation exists generically only if $3(k - 1) \leqq k$, or if $k = 1$.

Suppose all points are not on a line. Suppose all points are not on a two-dimensional plane passing through $\mathbf{x}$. As three points define a two-dimensional plane, some two-dimensional plane passes through three points and excludes $\mathbf{x}$. If follows from [2] that $\mathbf{x}$ is not a core point. Therefore, we can assume all points are in a two-dimensional plane. If three of these points are on the same line, then two are on the same side of $\mathbf{x}$. Thus, these two points and the point off of the line form a decisive coalition with points preferred to $\mathbf{x}$. This means that the four points are on two lines passing through $\mathbf{x}$. Here, one vector defines each line, so $\Sigma = \{\mathbf{A} | \mathbf{A}_2 = \lambda \mathbf{A}_1, \mathbf{A}_4 = \lambda \mathbf{A}_3\}$. This defines a codimension $2(k - 1)$ setting, so it exists generically as long as $2(k - 1) \leqq k$, or for $k \leqq 2$.

The $q = 4$, $n = 5$ rule. The first choice allowing $\beta \geqq 0$ is the $q = 4$, $n = 5$ rule; Theorem 1 asserts that issue space can have dimension up to $k = 3$. To show this is an upper bound, let $\mathbf{x}$ be a nonbliss ideal point and consider the plane defined by two gradient vectors $\nabla u_j(\mathbf{x})$. Using the arguments leading to [2], no more than one gradient vector can be on either side of this dividing plane. Counting one for each side, and the two that define the plane, we have accounted for four of the five gradient vectors. Thus the dividing plane also must contain the last gradient vector. Moreover, $\mathbf{x}$ is in the convex hull defined by these three gradient vectors. If not, then there is a line passing through $\mathbf{x}$ where all three gradient vectors are on the same side. If the dividing plane is rotated about this line, all three gradient vectors will end up on the same side as one of the remaining gradient vectors; this violates the assumption that $\mathbf{x}$ is a core point.

This condition, where a plane defined by two gradient vectors has a third and their convex hull contains $\mathbf{x}$, imposes strict conditions on all five points! To see why, suppose $\{\mathbf{x}_j\}_{j=1}^{3}$ are on the plane, $\mathbf{x}$ is in the interior of this hull, and $\mathbf{x}_5$ is off of the plane. (See Fig. 9.) The pairs of points $\{\mathbf{x}_1, \mathbf{x}_5\}$, $\{\mathbf{x}_2, \mathbf{x}_5\}$, and $\{\mathbf{x}_3, \mathbf{x}_5\}$ (along with $\mathbf{x}$) define three more planes which intersect along the line connecting $\mathbf{x}$ and $\mathbf{x}_5$. With this condition, another point must be on each plane. This extra point, $\mathbf{x}_4$, cannot be $\mathbf{x}$ as $\mathbf{x}$ is not a bliss point. If $\mathbf{x}_4$ is not

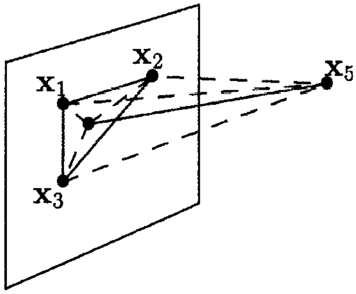

Figure 9. New planes.

on the $\mathbf{x} - \mathbf{x}_5$ line, then $\mathbf{x}_4$ is on one of the three planes, so we need two more points. With $n = 5$, this is impossible. Consequently, $\mathbf{x}_4$ is on the $\mathbf{x} - \mathbf{x}_5$ line. Whatever the orientation of the $\mathbf{x} - \mathbf{x}_5$ line, there is a plane though $\mathbf{x}$ with two of $\{\mathbf{x}_j\}_{j=1}^3$ and $\mathbf{x}_5$ on the same side. If $\mathbf{x}_4$ were between $\mathbf{x}_5$ and $\mathbf{x}$, four points would be on the same side of a dividing plane; this violates [1]. Thus, $\mathbf{x}_4$ is on the other side of the plane.

In J^2, these conditions are captured by $\Sigma = \{\mathbf{A} | \{\mathbf{A}_j\}_{j=1}^3$ are linearly dependent; there exists $\lambda < 0$ so that $\mathbf{A}_4 = \lambda \mathbf{A}_5\} \cap \mathscr{C}\mathscr{C}_5$. To find the codimension, notice that $k - 2$ dimensions are imposed by the dependency condition on the first three vectors, and $k - 1$ dimensions are obtained from the second. Thus, as the codimension is $2k - 3$, such examples are generically possible only when $2k - 3 \leqq k$ or $k \leqq 3$. If $\mathbf{x}$ is on the boundary of the hull, same argument holds except that an extra codimension is added to reflect that two vectors are along the same line. This situation, then, holds generically only for $k \leqq 2$. As the construction requires $k \geqq 3$, it is generically empty.

The final possibility is if all five vectors are on the plane. Here, $\Sigma = \{\mathbf{A} | \mathbf{A}_j, j = 3, 4, 5$, is a linear combination of $\mathbf{A}_1$ and $\mathbf{A}_2\}$. There are no restrictions on the choice of $\mathbf{A}_1, \mathbf{A}_2$, but a codimension $k - 2$ imposed on the choice of all remaining vectors. Thus, the total codimension is $3k - 6$. Such behavior is generically possible only if $3k - 6 \leqq k$, or if $k \leqq 3$.

To obtain all possible cases, use all possible permutations of the indices. As this number is finite, the conclusion holds. To create examples, use the gradient directions defined by the construction. Notice, for Euclidean preferences, both cases restrict the positioning of ideal points so, here, the conclusion is not generic.

The case of $q = n - 1$ **and** $k = n - 2$. In general, the construction for $q = n - 1$ where all of the gradients are not on the same $2q - n - 1$ dimensional subspace follows the lead of $q = 4$, $n = 5$. Namely, place $n - 2$ points in a $(n - 3)$-dimensional subspace so that no point is in the interior of the hull. (For instance, place the points near the vertices of the equilateral $(n - 2)$-gon.) Place each of the remaining points on each side of this hyperplane so that the line connecting them passes through the interior of the hull. This point of intersection is a nonbliss core point.

The proof that one cannot do better is essentially the same as for $n = 5$. Instead of two-dimensional planes, $k - 1$ dimensional planes are used. This shows that if one point is off of the plane, then, to ensure **x** is a core point, another point must be off the plane; both points off the plane must be on the line passing through **x**. Thus, in the codimension count of $\Sigma = \{\mathbf{A} \mid \{\mathbf{A}_j\}_{j=1}^{n-2}$ are linearly dependent and there exists $\lambda < 0$ so that $\mathbf{A}_{n-1} = \lambda \mathbf{A}_n\}$, the first condition imposes a codimension of one while the second condition adds $k - 1$ for a total of k. This means that, generically, such a nonbliss core point exists. In general, they form a zero dimensional manifold; e.g., an isolated point. Any other construction imposes a higher codimension, so it is generically unlikely.

General case. The proof for the general case extends the above construction. Let **x** be a nonbliss core point, and choose any $k - 1$ linearly independent gradient vectors. If no other gradient vectors are on this plane, then, according to Prop. 1 and [2], no more than $q - k$ gradient vectors can be on either side of this plane. Using this maximum as the number on each side and counting all gradient vectors on the plane, we have that $2(q - k) + (k - 1) \geqq n$, or $2q - n - 1 \geqq k$. Thus, if $k > 2q - n - 1$, another gradient, vector must be on this dividing plane. The underscore this assertion, denote the dimension of issue space as $k = \beta + (2q - n)$ where the admissible values for the "excess dimension" are $\beta = 0, 1, \ldots$. First we consider what happens should all points be on this hyperplane, then we analyze what happens when some points are off of it.

If all gradients are in the same $2q - n - 1$ dimensional space, then this space is defined by a basis; assume it is given by the first $2q - n - 1$ vectors. As all other vectors are in this linear space, the J^2 representation is $\Sigma = \{\mathbf{A} \mid \mathbf{A}_i \in Span(\{\mathbf{A}_j\}_{j=1}^{2q-n-1}) \forall i \geqq 2q - n$, the plane defined by any $k - 1$ vectors from $(\{\mathbf{A}_j\}_{j=1}^{2q-n-1})$ has no more than $q - k$ vectors on either side$\}$. The dependency condition imposes a $\beta + 1$ dimensional constraint on the choice of $\mathbf{A}_j, j \geqq 2q - n - 1$. (The dividing plane condition is an open one that, from Sect. 5 can occur, so it does not contribute to the codimension.) Thus, the codimension of Σ is $(1 + \beta)(2n - 2q + 1)$. Consequently, such a condition exists generically only if the inequality

$$(\beta + 1)(2n - 2q + 1) \leqq k = 2q - n + \beta \tag{6.1}$$

is satisfied. Collecting terms and solving for q we obtain Eq. 2.3; solving for k leads to Eq. 2.4. The difference between the left and right hand sides of Eq. 6.1 is the dimension on the core points, so it is the proof of Thm. 2c. Considering all possible permutations of indices completes the proof.

(This construction does not explicitly consider where no set of $2q - n - 1$ vectors are linearly independent. But, this condition increases the codimension for each $\mathbf{A}_j$ so the total codimension of the new version of Σ is larger. Therefore, such situations become generically unlikely with smaller k values.)

Vectors out of the plane. It remains to consider where not all vectors are in the same $2q - n - 1$ dimensional subspace. This analysis uses the next statement.

Lemma 2. *Suppose* $\mathbf{x}$ *is a nonbliss core point and a set of* $2q-n-1$ *linearly independent gradient vectors define a plane* $\mathscr{L}$ *passing through* $\mathbf{x}$. *The convex hull of the gradient vectors on* $\mathscr{L}$ *contains* $\mathbf{x}$ *but not as a vertex. If* $\mathbf{x}$ *is a boundary point of this convex hull, and if there are* $2q-n-1+e$ *vectors in* $\mathscr{L}$, *then at least* $e+1$ *of the vectors lie on the same boundary surface of this hull and the hull they define has* $\mathbf{x}$ *as an interior point.*

Proof. Replace gradients with points $\mathbf{x}_j$ where directions are $\mathbf{x}_j-\mathbf{x}$. As n is finite, we can extend $\mathscr{L}$ to a codimensional one plane $\mathscr{L}_1$ which contains no point off of $\mathscr{L}$. If $\mathbf{x}$ is not in the convex hull, then, because it is separated from this hull, a hyperplane $\mathscr{H}$ passes through $\mathbf{x}$ where all points from $\mathscr{L}$ are strictly on one side. With $2q-n-1+e$ points on $\mathscr{L}$, and, hence, on $\mathscr{L}_1$, $e \geqq 1$. A slight change in the orientation of $\mathscr{L}_1$, using $\mathscr{H} \cap \mathscr{L}_1$ as an axis of rotation, can force all points on $\mathscr{L}_1$ to one side of the new dividing plane. As $\mathbf{x}$ is a core point, no more than $(q-1)-(2q-n-1+e)=n-q-e$ points are on either side of $\mathscr{L}_1$, so, at most, $2(n-q-e)$ points can be off of $\mathscr{L}_1$. The number of points not on $\mathscr{L}$ is $n-(2q-n-1+e)=2n-2q+1-e$, so $2(n-q-e) \geqq 2n-2q-e+1$ or $-1 \geqq e$. This contradiction proves the assertion.

If $\mathbf{x}$ were a vertex, it would be a bliss point which violates the assumption. If $\mathbf{x}$ is a boundary point of the hull, it may be on one or several boundary components of the hull; choose the one, B, with minimal dimension γ. Because $\mathbf{p}$ is not a vertex, the number of points, r, on B satisfies $r \geqq \gamma+1$ where (because B has minimal dimension) $\mathbf{x}$ is in the interior of the hull defined by these points. To find a bound on r in terms of e, notice that (n is finite) there is a plane $\mathscr{P}$ passing through $\mathbf{x}$ and intersecting $B \subset \mathscr{L}_1$ so that at least $\frac{r}{2}$ of the B points are on the same side as the $2q-n-1+e-r$ remaining points of $\mathscr{L}_1$. By using the dividing plane argument, where the plane comes from rotating $\mathscr{L}_1$ about the $\mathscr{P} \cap \mathscr{L}_1$ axis, and the fact that one side of $\mathscr{L}_1$ has at least half, $n-q-\frac{e-1}{2}$, of the remaining points, it must be that $\left(n-q-\frac{e-1}{2}\right)+(2q-n-1+e-r)+\left(\frac{r}{2}\right) \leqq q-1$ or $e+1 \leqq r$. □

While there exist configurations allowing core points where all points are not on the $2q-n-1$ dimensional space, we show how the rapid growth of the binomial coefficient forces the codimension to grow so fast that this is generically possible only in very special cases. Assume that $\mathbf{x}_n$ (representing $\nabla u_n(\mathbf{x})$) is not in $\mathscr{L}$ and that $\mathbf{x}$ is in the interior of the hull of the $2q-n$ points $\{\mathbf{x}_j\}_{j=1}^{2q-n}$ on $\mathscr{L}$. Because $\{\mathbf{x}_j\}_{j=1}^{2q-n}$ defines a convex hull with interior, any of $\mathbf{x}_j$ is removed, the remaining directions (i.e., $\mathbf{x}_i-\mathbf{x}$) are linearly independent. Moreover, because $\mathbf{x}_n \notin \mathscr{L}$, when $\mathbf{x}_n$ replaces any two points from $\{\mathbf{x}_j\}_{j=1}^{2q-n}$, the new set defines a $2q-n-1$ dimensional plane. As there are $\binom{2q-n}{2}$ ways to choose pairs from $\{\mathbf{x}_j\}_{j=1}^{2q-n}$, there are $\binom{2q-n}{2}$ planes. From the linear independence statement, each plane does not include the pair dropped to define the basis and these planes meet only along the $\mathbf{x}-\mathbf{x}_n$ line.

This dimension requires each plane to have one more point. One possibility is to place a point, $\mathbf{x}_{n-1}$, on the $\mathbf{x}-\mathbf{x}_n$ line to simultaneously satisfy this condition for all planes. (Again, $\mathbf{x}$ must be between $\mathbf{x}_{n-1}$ and $\mathbf{x}_n$.) The J^2 manifold containing this condition is $\Sigma=\{\mathbf{A}\,|\,\text{vectors } \mathbf{A}_j, j=2q-n,\ldots,n-2$ are in the space spanned by $\{\mathbf{A}_j\}_{j=1}^{2q-n-1}$; there exists a scalar $\lambda<0$ so that $\lambda\mathbf{A}_n=\mathbf{A}_{n-1}\}$. Each $\mathbf{A}_j, j=2q-n,\ldots,n-2$ vector adds $(1+\beta)$ to the codimension and only the length of $\mathbf{A}_{n-1}$ can be chosen, so the codimension of Σ is $(1+\beta)(2n-2q-1)+(k-1)$. To be generically possible, this value must be bounded above by k which means that $(1+\beta)(2(n-q)-1)\leqq 1$. In turn, this inequality is satisfied only for $n=q+1, \beta=0$, or $q=n$. Notice, a larger β value is obtained when all points remain on the plane.

When the extra point for each plane is not chosen to satisfy all planes simultaneously, the codimension increases more rapidly. To minimize the number of extra points is to choose points that satisfy as many planes as possible. As $\gamma=2q-n\geqq 3$, this occurs by choosing a point that is in the intersection of $\gamma-2$ planes; in $\mathscr{L}$, this intersection is the $\mathbf{x}_j-\mathbf{x}$ line for some j. As there are $\binom{\gamma}{2}$ planes, we need at least $\delta=\left[\binom{\gamma}{2}\Big/(\gamma-2)\right]\geqq 3$ points chosen in this way. First assume that all these points are in $\mathscr{L}$. The J^2 manifold including this situation is $\Sigma=\{\mathbf{A}\,|\,\text{any } \gamma-1$ vectors from $\Gamma=\{\mathbf{A}_j\}_{j=1}^{\gamma}$ are linearly independent, each vectors $\{\mathbf{A}_{\gamma+j}\}_{j=1}^{\delta}$ is a scalar multiple of a vector from $\Gamma\}$. The codimension of Σ is bounded above by $(1+\beta)+\delta(k-1)$. As $\delta\geqq 3$, $k\geqq\gamma\geqq 3$, this value always is larger than k, so this setting never is generically possible.

More generally, if points are chosen to be on $\gamma-s$ planes, where $k=\gamma+\beta\geqq\gamma\geqq s+1$, then, as there are $\binom{\gamma}{s+1}$ ways to choose these planes, there must be at least $\delta_s=\left[\binom{\gamma}{s+1}\Big/(\gamma-(s+1))\right]$ points. Thus, the J^2 set containing this condition is $\Sigma_s=\{\mathbf{A}\,|\,\text{any } \gamma-1$ vectors from $\Gamma_s=\{\mathbf{A}_j\}_{j=1}^{\gamma}$ are linearly independent, each vectors $\{\mathbf{A}_{\gamma+j}\}_{j=1}^{\delta_2}$ can be expressed as a linear combination of s vectors from $\Gamma_s\}$. The codimension of Σ_s is bounded above by $(1+\beta)+\delta_s(k-s)$. Therefore, to be generically possible, this number must be bounded above by k, which, by using $k=\gamma+\beta$, requires satisfying $1+(\delta_s-1)\gamma+\delta_s\beta\leqq\delta_s s$. As $\delta_s>2$, $\gamma>s$, this never is satisfied. (I leave it as a simple exercise to show that if points are chosen to be on different number of planes, then the same conclusion holds. This is because the more planes a point is on, the higher its codimension. The other extreme, of choosing more points with lower codimension requires so many more points that the codimension grows faster.)

The remaining possibility is to choose extra points off of $\mathscr{L}$. This modifies Σ by allowing the $\mathbf{A}_j$ points to include $\mathbf{A}_n$ its representation. Thus the new codimension is $(1+\beta)+\delta_s(k-s-1)$. To be generically possible, this number must be less than k, or $1 + (\delta_s - 1)\gamma + \delta_s\beta \leqq \delta_s(s+1)$. Again, because $\delta_s \geqq 3$ and $\gamma \geqq s + 1$, this never can be satisfied.

Finally, if $\mathbf{x}$ is on the boundary of the convex hull of the points on $\mathscr{L}$, then the same argument applies, but the codimension escalates more rapidly because, as Lemma 2 asserts, most of the points are on a lower dimensional subset of $\mathscr{L}$ and all added points also are on this set. Thus, the same computations show that this setting is not generically possible.

Constructing examples. The construction of examples is simple. Choose any point $\mathbf{x}$. Using the construction of Sect. 5, find gradient directions for points where these gradient directions define a $2q - n - 1$ dimensional space and where they satisfy the alternating rules for the specified n, q values. Now, use these directions to define gradients. The only difference in choosing second derivative terms is that they must include all variables.

To illustrate with $n = 4$, $q = 5$ and $k = 3$, use Fig. 2. Translate and rotate the figure in R^3 so that a specified $\mathbf{x}$ from the shaded region is at the desired location of a nonbliss core point, and the orientation of the plane is consistent with the desired plane of gradient vectors. Define the gradient directions by the directions from $\mathbf{x}$ to the vertices of the star. This construction may suggest that a two-dimensional core results; this would be in conflict with Theorem 2c which asserts it should be zero-dimensional – isolated points. The explanation is that in R^2, the chosen gradient directions have only one degree of freedom when the base point is varied. Consequently, the same general star figure is defined by the gradients at neighboring points. With the added degrees of freedom from R^3, when the base point is varied, the gradient need not lie in the same plane. Once this happens, [1] and [2] are violated, so a core does not exist. Therefore, this construction does, in fact, define the core to be a collection of isolated, zero-dimensional points.

Dimensions. The dimensional and codimension statements follow directly from Eq. 6.1. This is because, generically, the sets are the union of isolated smooth submanifolds with the indicated codimension of appropriate Σ. Another approach is to note that the construction developed in Sect. 5 allows for freedom in the choice of the gradient directions (an open set about each direction), so the construction is robust. The same comment applies to when the construction of Sect. 5 applies to the $\beta \geqq 0$ settings.

I stated that, generally, the core has a stratified structure. As this statement is a direct consequences of singularity theory, it is not formally asserted nor proved. Yet, related ideas are in the proof of Thm. 2a. First, however, I need that the core is closed. This is a consequence of continuity of the gradients and the fact that Prop. 1 defines a closed condition. The only difference in the J^2 representation is that some of the $\mathbf{A}_j$ vectors are on the same plane. This adds to the dodimension which, in turn, reflects that the boundary is a lower dimensional object.

Theorem 2a. Only the proof of Theorem 2a remains. To show that if $\mathbf{x}$ is a q core point then it is a $q + 1$ core point, notice that the convex hull defined by

$q + 1$ gradient vectors includes the convex hull defined by a subset of them. The conclusion follows from Prop. 1.

Next, consider n, k, q values where, generically, the core for the q rule must be in a lower dimensional submanifold. If the generic situation for the $q + 1$ rule allows the core to have a nonempty interior (e.g., $q = n - 1$ or where $q + 1$ admits $\beta = -1$), then, generically, the two cores cannot agree as they have different dimensions.

Suppose the core for the $q + 1$ rule is, generically, in a lower dimensional set. The dimension for the q rule is $d_q = k - (\beta_1 + 1)(2q - n + 1)$ where $\beta_1 = k - (2q - n - 1$. The excess dimension for the $q + 1$ rule is $\beta = k - (2(q + 1) - n - 1) = \beta_1 - 2$. The above argument handles $\beta_1 < 2$, so let $\beta_1 \geqq 2$. The core dimension for the $q + 1$ rule is $d_{q+1} = k - (\beta + 1)(2q - n + 3) = k - (\beta_1 - 1)(2q - n + 1) + 2)$. To see that $d_{q+1} > d_q$, so the conclusion will follow because the dimensions disagree, compute $d_{q+1} - d_q = 2((2q - n + 1) - \beta_1) + 2$. As $q \leqq n - 2$, the value of β_1 is bounded above by $q - \frac{3}{4}n$, (see the expression for β found between Eqs. 2.3, 2.4) so $d_{q+1} - d_q > 0$.

The remaining case is where the core for the q rule has a nonempty interior. Assume that the core for the q and $q + 1$ rules are the same. Generically, the core is a closed set where if $\mathbf{x}$ is a boundary point, then $\mathbf{x}$ satisfies Eq. 4.1 by being on the boundary of the convex hull for some coalition. Suppose not; suppose $\mathbf{x}$ is in the interior of each $Co_{\mathbf{x}}(\{\nabla u_j(\mathbf{x})\}_{j \in C}$ for each decisive coalition C and that $\mathbf{v}$ is such that $\mathbf{x} + t\mathbf{v}$ is not a core point for any $t > 0$. By continuity and the fact that $\mathbf{x}$ is an interior point, there is an open neighborhood of $\mathbf{x}$ so that any point in this neighborhood also is in the convex hull of the gradients defined at that point. Any such point is a core point. This contradiction proves the assertion.

So, assume both cores agree and they have a nonempty interior. We consider points on the $k - 1$-dimensional boundary. This corresponds to a q-rule core point $\mathbf{x}$ and a coalition C, say $C = \{1, 2, \ldots, q\}$, where $\mathbf{x}$ is on the boundary of $Co_{\mathbf{x}}(\{\nabla u_j(\mathbf{x})\}_{j \in C}$. This boundary is $k - 1$ dimensional and it contains k of the gradient vectors. (If it contained more, then this would add to the codimension violating the fact that $\mathbf{x}$ is on a $k - 1$ dimensional component.) To represent this as a J^2 condition, let $\Sigma_p = \{\mathbf{A} \mid k$ vectors from $\{\mathbf{A}_j\}_{j=1}^{q}$ are linearly dependent$\}$. As this has codimension one, the set of points satisfying this condition is, generically, a collection of $k - 1$ dimensional manifolds. Indeed, it includes the boundary of the core. (To make it explicitly the boundary of the core, add the convexity conditions of Prop. 1.)

Because the cores agree, $\mathbf{x}$ is on the $k - 1$ dimensional boundary of the $q + 1$ rule core. In particular, this means that there is some agent i so that when C is augmented to C_i by adding agent i, $i > q$, then, $\mathbf{x}$ is a boundary point of $Co_{\mathbf{x}}(\{\nabla u_j(\mathbf{x})\}_{j \in C_i}$. From $C_i, q + 1$ different coalitions of q voters can be constructed; q of them are formed by replacing an agent from C with i. When $\nabla u_i(\mathbf{x})$ replaces one of the gradient vectors from C, then either there are $k - 1$ gradient vectors on this plane, or $\nabla u_i(\mathbf{x})$ is on this plane. The first case means that a plane can be passed through **x** using this new q voter coalition where **x** is not in its convex hull. As this means **x** is not a core point, it must be that $\nabla u_i(\mathbf{x})$ is in this plane. In J^2, this is captured by the manifold $\Sigma_{q,i} = \{\mathbf{A} \in \Sigma_q | \mathbf{A}_i$ is in the span of the k linearly dependent vectors$\}$. This adds another codimension, so, generically, **x** belongs to a $k-2$ dimensional manifold. The dimension contradiction proves the theorem. Indeed, with slight extra care, we have that, in general, the q-core is in the interior of the $q+1$ core.

Removing strict convexity. If the convexity assumption is removed, the portions of Prop. 1 where **x** is on the boundary of the convex hull need not hold. These conditions are needed primarily for the $\beta \geqq 0$ analysis. Second, an infinitesimal core point need not be a core point. So, in the more general setting, all infinitesimal core points that are not core points must be removed. The main change in the conclusion is that the core may be the union of several submanifolds of the indicated dimensions.

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