

SLOPES OF EIGENCURVES OVER BOUNDARY DISKS

DAQING WAN, LIANG XIAO, AND JUN ZHANG

In memory of Professor Robert F. Coleman

ABSTRACT. Let p be a prime number. We study the slopes of U_p -eigenvalues on the subspace of modular forms that can be transferred to a definite quaternion algebra. We give a sharp lower bound of the corresponding Newton polygon. The computation happens over a definite quaternion algebra by Jacquet-Langlands correspondence; it generalizes a prior work of Daniel Jacobs [Ja04] who treated the case of $p = 3$ with a particular level.

In case when the modular forms have a finite character of conductor highly divisible by p , we improve the lower bound to show that the slopes of U_p -eigenvalues grow roughly like arithmetic progressions as the weight k increases. This is the first very positive evidence for Buzzard-Kilford's conjecture on the behavior of the eigencurve near the boundary of the weight space, that is proved for arbitrary p and general level. We give the exact formula of a fraction of the slope sequence.

CONTENTS

1. Introduction	1
2. Coleman-Mazur eigencurves	6
3. Automorphic forms for a definite quaternion algebra	9
4. Explicit computation of the U_p -operator	13
5. An example of explicit computation	18
6. Improving the lower bound	20
7. Techniques for separation by residual pseudo-representations	32
References	39

1. INTRODUCTION

Let p be a fixed prime number which we assume to be odd for simplicity in this introduction. For N a positive integer (the “tame level”) coprime to p , $k + 1 \geq 2$ an integer (the “weight”)¹, m a positive integer, and ψ a character of $(\mathbb{Z}/p^m\mathbb{Z})^\times$, we use $S_{k+1}(\Gamma_0(p^m N); \psi)$ to denote the space of modular cuspforms of weight $k + 1$, level $p^m N$, and nebentypus character ψ over some finite extension E of \mathbb{Q}_p . This space comes equipped with the action of Hecke operators, most importantly the action of the Atkin U_p -operator. It is a central question in the theory of p -adic modular forms to understand the distributions of the “slopes”, namely,

Date: June 13, 2014.

2010 Mathematics Subject Classification. 11F33 (primary), 11F85 (secondary).

Key words and phrases. Eigencurves, slope of U_p operators, quaternionic automorphic forms, overconvergent modular forms, Gouvêa-Mazur Conjecture, Gouvêa's conjecture on slopes.

¹For the subject we study, writing $k + 1$ for the weight will simplify the presentation.

the p -adic valuations of the eigenvalues of U_p acting on $S_{k+1}(\Gamma_0(p^m N); \psi)$, as the weight $k+1$ varies. All p -adic valuations or norms in this paper are normalized so that p has valuation 1 and norm p^{-1} .

One of the most interesting expectations concerns the case when the nebentypus character ψ has exact conductor p^m for $m \geq 2$, i.e. ψ does not factor through a character on $(\mathbb{Z}/p^{m-1}\mathbb{Z})^\times$. Let $\omega : (\mathbb{Z}/p\mathbb{Z})^\times \rightarrow \mathbb{Z}_p^\times$ denote the Teichmüller character.

The following question was asked by Coleman and Mazur [CM98] and later elaborated by Buzzard and Kilford [BK05].

Conjecture 1.1. *Fix an integer N coprime to p and a character ψ_0 of $(\mathbb{Z}/p\mathbb{Z})^\times$ such that $\psi_0(-1) = -1$. Then there exists a non-decreasing sequence of rational numbers a_1, a_2, \dots approaching to infinity such that*

- *for any integers $m \geq 2, k+1 \geq 2$ and any character ψ of $(\mathbb{Z}/p^m\mathbb{Z})^\times$ of exact conductor p^m such that $\psi|_{(\mathbb{Z}/p\mathbb{Z})^\times} \cdot \omega^k = \psi_0$, the slopes of U_p acting on $S_{k+1}(\Gamma_0(p^m N); \psi)$ is given by the first few terms of the sequence*

$$a_1/p^m, a_2/p^m, \dots$$

consisting of all numbers strictly less than k and some equal to k 's.

Moreover, the sequence a_1, a_2, \dots is a union of finitely many arithmetic progressions.

There has been many direct computations supporting this Conjecture in special cases, first by Buzzard and Kilford [BK05] (extending the work of Emerton [Em98]) in the case when $p = 2$ and $N = 1^2$, then in many similar particular cases with small primes p and small levels; see [Ro13, Kil08, KM12, Ja04]. Nonetheless, this Conjecture was never recorded in the literature for lack of theoretic or heuristic evidences. The goal of this paper is to provide some positive indications in the general case.

1.2. The geometry of the eigencurve. Before proceeding, we explain the meaning of Conjecture 1.1 in terms of the geometry of the eigencurve.

Eigencurves were introduced by Coleman and Mazur [CM98] to p -adically interpolate modular eigenforms of different weights. Here the notion of weights is generalized to mean a continuous character of \mathbb{Z}_p^\times ; for examples, $x \mapsto x^k \psi(x)$ corresponds a classical weight $k+1$ with nebentypus character ψ . In the loosest terms, the eigencurve is a rigid analytic closed subscheme of the product of the weight space and \mathbb{G}_m , defined as the Zariski closure of the set of pairs $(x^k \psi(x), a_p(f))$ for each eigenform f of weight $k+1$ and nebentypus character ψ with U_p -eigenvalue $a_p(f)$. In particular, its fiber over the point $x^k \psi(x)$ of the weight space parametrizes the U_p -eigenvalues on the space of modular forms $S_{k+1}(\Gamma_0(p^m N); \psi)$ and the overconvergent ones.

The eigencurve plays a crucial role and has many applications in the modern p -adic number theory; to name one: Kisin's proof of Fontaine-Mazur conjecture [Kis09]. Despite the many arithmetic applications, the geometry of the eigencurve was however poorly understood for a long time. For example, the properness of the eigencurve was not known until the very recent work of Diao and Liu [DL].

Conjecture 1.1 and this paper focus on another intriguing property: the behavior of the eigencurve near the boundary of the weight space. The striking computation of Buzzard

²We earlier excluded the case of $p = 2$ for simple presentation; but slight modification allows us to include this case, as we will do for the rest of the paper.

and Kilford [BK05] mentioned above shows that, when $p = 2$ and $N = 1$, the Coleman-Mazur eigencurve, when restricted over the boundary annulus of the weight space, is an infinite disjoint union of copies of this annulus. This is a family and a much stronger version of Conjecture 1.1; see Conjecture 2.5 for the precise expectation. Generalizing this result would have many number theoretical applications. For example, in [PX], the second author and Pottharst reduced the parity conjecture of Selmer rank for modular forms to this precise statement.

1.3. Main result of this paper. For the sake of presentation, we assume that there exists a prime number ℓ such that $\ell \nmid N$. We only consider the subspace of modular forms which are ℓ -new, denote by a superscript ℓ -new, e.g. $S_{k+1}(\Gamma_0(p^m N); \psi)^{\ell\text{-new}}$. This is the subspace of modular forms that can be identified by Jacquet-Langlands correspondence with the automorphic forms on a definite quaternion algebra D which ramifies at ℓ and ∞ .

The following lower bound of the Newton polygon of the U_p -action on $S_{k+1}(\Gamma_0(p^m N); \psi)^{\ell\text{-new}}$ might be known among some experts.³

Theorem A. *Assume that the conductor of ψ is exactly p^m . (By our later convention, this will include the case when ψ is trivial and $m = 1$.) Let t denote $\dim S_2(\Gamma_0(p^m N); \psi)^{\ell\text{-new}}$ so that $\dim S_{k+1}(\Gamma_0(p^m N); \psi)^{\ell\text{-new}} = kt$. Then the Newton polygon of the U_p -action on $S_{k+1}(\Gamma_0(p^m N); \psi)^{\ell\text{-new}}$ lies above the polygon with vertices*

$$(0, 0), (t, 0), (2t, t), \dots, (nt, \frac{n(n-1)}{2}t), \dots$$

The complete proof is given in Theorem 4.8. Note that the lower bound is independent of k , and thus uniform in k . A similar uniform quadratic lower bound of Newton polygon was obtained by the first named author in [Wa98] using a variant of Dwork's trace formula. We point out that our lower bound is much sharper. In fact, the distance between the end point of the Newton polygon of U_p acting on $S_{k+1}(\Gamma_0(p^m N); \psi)^{\ell\text{-new}}$ and our lower bound is *linear* in k (comparing to the quadratic difference in [Wa98]).

When the character ψ is trivial, Theorem A gives a heuristic explanation of a conjecture of Gouvêa on the distributions of slopes. Unfortunately, we cannot prove a family version of Theorem A while keeping the same bound in this case; therefore, it does not prove this conjecture of Gouvêa. We refer to Remarks 4.9 and 4.10 for related discussions.

The proof of Theorem A (and the proof of the subsequent theorems in this paper) uses Jacquet-Langlands correspondence to transfer all information into automorphic forms for a definite quaternion algebra. The advantage of working with definite quaternion algebra is its simpler geometry compared to the modular curves. The quadratic error term in [Wa98] is partly a result of lack of a good integral structure on the modular curves. In contrast, the theory of overconvergent automorphic forms on a definite quaternion algebra d'après Buzzard [Bu07] come equipped with a nice integral basis. Our computation essentially reproduces Jacobs' thesis [Ja04], except taking a more theoretical as opposed to computational approach.

The real improvement over Jacobs' work is that, when the conductor p^m of ψ is large (e.g. $m \geq 4$), we can improve the lower bound above so that it *agrees* with the Newton polygon (in the overconvergent setting) at infinitely many points which form an arithmetic progression. This gives the following

³We think that Buzzard probably has an unpublished note on certain version of this theorem; see [Bu05].

Theorem B. *Keep the notation as in Theorem A and assume that $m \geq 4$. Let $a_0(k) \leq a_1(k) \leq \dots \leq a_{kt-1}(k)$ denote the slopes of the U_p -action on $S_{k+1}(\Gamma_0(p^m N); \psi)^{\ell\text{-new}}$, in non-decreasing order (with multiplicity). Then we have*

$$\lfloor \frac{n}{t} \rfloor \leq a_n(k) \leq \lfloor \frac{n}{t} \rfloor + 1.$$

This is proved in Theorem 6.17. Note that the inequality of the slopes does not depend on the weight $k + 1$. In fact, we prove a family version of such inequality which gives rise to a decomposition (Theorem 6.22) of the eigencurve over the disks $\mathcal{W}(x\psi, p^{-1})$ of radius p^{-1} centered around the character $x\psi$, just as in Buzzard-Kilford [BK05]. Unfortunately, we cannot extend this result to the entire weight annulus of radius $p^{-1/p^{m-2}(p-1)}$ which contains $x\psi$.

The main idea of the proof consists of two major inputs: (1) We show that there is a natural isomorphism

$$(1.3.1) \quad S^{D,\dagger}(U; \kappa) \cong \widehat{\bigoplus}_{n=0}^{\infty} S_2^D(U; \psi\omega^{-2n}) \otimes (\omega^n \circ \det),$$

such that the U_p -action on the left hand side is ‘‘approximately’’ the action of $\bigoplus_{n \geq 0} (p^n \cdot U_p)$ on the right hand side. Here the letter U is the corresponding level structure which looks like $\Gamma_0(p^m)$ at p ; $S^{D,\dagger}(U; \kappa)$ stands for the space of overconvergent automorphic forms over a definite quaternion algebra D with weight character κ living in $\mathcal{W}(x\psi, p^{-1})$; the right hand side is the completed direct sum of *classical* automorphic forms over D of weight 2 with characters $\psi\omega^{-2n}$, *twisted by the character* $\omega^n \circ \det$. It thus follows that the U_p -slopes on $S^{D,\dagger}(U; \kappa)$ is approximately determined by the U_p -slopes on these space of classical forms of weight 2.

(2) To carry out the approximation in (1.3.1), it is important to show that the slopes of the *Hodge polygon* of the U_p -action on each $S_2^D(U; \psi\omega^{-2n})$ are between 0 and 1. Here the Hodge polygon of the matrix for the U_p -action refers to the convex hull of points given by the minimal p -adic valuation of the minors of the matrix. To prove this key result, we make use of (in the definite quaternion situation) the natural integration/summation pairing

$$\langle \cdot, \cdot \rangle : S_2^D(U; \psi) \times S_2^D(U; \psi^{-1}) \rightarrow E$$

and the fact that $\langle U_p(f), U_p(g) \rangle = p \langle S_p(f), g \rangle$, where S_p is the unramified central character action at p . In fact, we also need certain deformed version of this pairing in order to improve the result from the open disks of radius p^{-1} to the closed disks of the same radius. This small improvement is also essential to Theorem B. We refer to Section 6 for details.

We also point out that the condition $m \geq 4$ is currently an unfortunate technical condition. See Remark 6.18 for the discussion in the case when $m = 3$.

A consequence of the proof of Theorem B is that we can in fact show that some of the slopes indeed form arithmetic progressions.

Theorem C. *Keep the notation as in Theorem B. Fix $r \in \{0, 1, \dots, \frac{p-3}{2}\}$. Let $\text{NP}_r(i)$ and $\text{HP}_r(i)$ denote the Newton polygon and Hodge polygon functions for the U_p -action on $S_2(\Gamma_0(p^m N); \psi\omega^{-2r})^{\ell\text{-new}}$. Suppose that $(s_0, \text{NP}_r(s_0))$ is a vertex of the Newton polygon NP_r and suppose that*

$$\text{NP}_r(s) < \text{HP}_r(s - 1) + 1 \text{ for all } s = 1, \dots, s_0.$$

Then for any $s = 0, 1, \dots, s_0$, the following subsequence

$$a_{s+rt}(k), a_{s+rt+\frac{p-1}{2}t}(k), \dots, a_{s+rt+i\frac{p-1}{2}t}(k), \dots$$

is independent of the positive integer k whenever it is congruent to $2r + 1$ modulo $p - 1$ (whenever it makes sense) and it forms an arithmetic progression with common difference $\frac{p-1}{2}$.

This is proved in Corollary 6.19. Note that the common difference for the arithmetic progression is $\frac{p-1}{2}$ but not 1. This is due to the periodic appearance of the powers of Teichmüller characters in (1.3.1). In fact, this (larger) common difference agrees with the computation of Kilford [Kil08] and Kilford-McMurdy [KM12] in the case $m = 2$, where the common difference is 2 when $p = 5$ and is $\frac{3}{2}$ (which can be further separated into two arithmetic progressions with common difference 3) when $p = 7$.⁴

The power of Theorem C is limited by how close the Hodge polygon is to the Newton polygon. In particular, as N and m get bigger, the gap between the Newton and Hodge polygons will be inevitably widened, and hence s_0 is relatively small compared to t .

One remedy we propose is to “decompose” the space of (overconvergent) modular forms according to residual Galois pseudo-representations.

Theorem D. *Let $\bar{\rho}_1, \dots, \bar{\rho}_d$ be the residual Galois pseudo-representations appearing as the pseudo-representations attached to the eigenforms in $S_2^D(U; \psi\omega^{-2r})$ for some $r = 0, 1, \dots, \frac{p-3}{2}$. Then we have a natural decomposition of (overconvergent) automorphic forms:*

$$S^{D,\dagger}(U; \kappa) = \bigoplus_{j=1}^d S^{D,\dagger}(U; \kappa)_{\bar{\rho}_j} \quad \text{and} \quad S_{k+1}^D(U; \psi) = \bigoplus_{j=1}^d S_{k+1}^D(U; \psi)_{\bar{\rho}_j}$$

for all weights $k + 1$. Moreover, Theorem C holds for each individual $S_2^D(U, \psi\omega^{-2r})_{\bar{\rho}_j}$.

This is proved in Theorem 7.12. The idea behind this theorem is that the isomorphism (1.3.1) is also approximately equivariant for tame Hecke actions. One can certainly decompose the right hand side of (1.3.1) according to the reductions of the associated Galois (pseudo-)representations; the isomorphism (1.3.1) allows us, to some extent, transfer the decomposition to the space of overconvergent automorphic forms. The error terms can be killed by taking the limit of repeated p -powers of the approximate projectors on the space of overconvergent automorphic forms.

We believe that the decomposition by Galois pseudo-representations has its own interest; for example, it gives a natural decomposition of the eigencurve according to the residual Galois pseudo-representations. Our decomposition is given in a reasonably explicit way on the Banach space of overconvergent automorphic forms and we have a good “model” of each factor. So the decomposition of the eigencurve over disks of radius p^{-1} centered around $x\psi(x)$ applies to the piece corresponding to each Galois pseudo-representation.

1.4. Structure of the paper. We first briefly recall the construction of eigencurves in Section 2 as well as the conjecture of Buzzard-Kilford. Section 3 sets up basic notations for classical and overconvergent automorphic forms for a definite quaternion algebra. Section 4 gives the most fundamental computation of the infinite matrix for the U_p -action on

⁴We believe that the assumption $m \geq 4$ is technical; Theorem B should hold for $m = 2$ or 3 if one can overcome all technical difficulties.

the space of overconvergent automorphic forms. In particular, Theorem A is proved here. The theoretical computation is complemented by a concrete example which we present in Section 5; this was previously studied by Jacobs [Ja04], but made much more accessible here as a by-hand computation. We hope this explicit example can inspire the readers to seek for new ideas. After this, we study the pairing between classical automorphic forms in Section 6 and prove Theorems B and C at the end of the section. Section 7 is devoted to separating the eigencurve according to residual Galois pseudo-representations. Theorem D is proved at the end of Section 7.

Acknowledgements. We are grateful to Frank Calegari, Matthew Emerton, Chan-Ho Kim, and Xinyi Yuan for many useful discussions. We thank Chris Davis and Hui June Zhu for their interests. We thank Sage notebook and `lmfdb.org` for providing numerical input in the course of this research. The first author is partially supported by Simons Fellowship. The second author is partially supported by a grant from the Simons Foundation #278433 and CORCL research grant from University of California, Irvine. The third author would like to thank University of California, Irvine for the hospitality during his visit.

Unconventional use of notations. We list a few unconventional use of notations.

- The conductor of a trivial character of \mathbb{Z}_p^\times is p as opposed to 1.
- We use $k + 1$, as opposed to k , for the weight of modular forms. Related to this, the right action appearing in the definition of automorphic forms on definite quaternion algebra uses a slightly different normalization; see (3.2.1).
- Although the Hecke actions seem to come from certain right action on the Tate algebras, we still view them as left action. Therefore, we exclusively work with column vectors. We will try to clarify this in the context (e.g. Proposition 4.6).
- All row and column indices of a matrix starts with 0 as opposed to 1; this will be extremely useful when considering infinite matrices later.

2. COLEMAN-MAZUR EIGENCURVES

2.1. Weight space. We fix a prime number p . We write $\Gamma = \mathbb{Z}_p^\times$ as $\Delta \times \Gamma_0$, where $\Gamma_0 = (1 + 2p\mathbb{Z}_p)^\times \cong \mathbb{Z}_p$ (identified via the map $x \mapsto \frac{1}{2p} \log(x)$) and $\Delta = (\mathbb{Z}_p/2p\mathbb{Z}_p)^\times$ is isomorphic to $\mathbb{Z}/(p-1)\mathbb{Z}$ if $p \geq 3$, and $\mathbb{Z}/2\mathbb{Z}$ if $p = 2$. We choose the topological generator γ_0 of Γ_0 to be the element $\exp(2p) \in \Gamma_0 \subseteq \mathbb{Z}_p^\times$.

We use $\Lambda = \mathbb{Z}_p[[\Gamma]]$ and $\Lambda_0 = \mathbb{Z}_p[[\Gamma_0]]$ to denote the Iwasawa algebras. In particular, we have $\Lambda \cong \Lambda_0 \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\Delta]]$. For an element $\gamma \in \Gamma$, we use $[\gamma]$ to denote its image in the Iwasawa algebra Λ . The chosen γ_0 defines an isomorphism $\mathbb{Z}_p[[T]] \simeq \Lambda_0$ given by $T \mapsto [\gamma_0] - 1$.

The weight space is defined to be $\mathcal{W} := \text{Max}(\Lambda[\frac{1}{p}])$, the rigid analytic space associated to the formal scheme $\text{Spf}(\Lambda)$; it is a disjoint union of $\#\Delta$ copies of the open unit disk. The natural projection

$$\mathcal{W} \cong \text{Max}(\Lambda_0 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p[[\Delta]]) \rightarrow \text{Max}(\Lambda_0[\frac{1}{p}]) \simeq \text{Max}(\mathbb{Z}_p[[T]][\frac{1}{p}])$$

gives each point on \mathcal{W} a T -coordinate.

The weight space \mathcal{W} may be viewed as the universal space for continuous characters of Γ . More precisely, a continuous character $\kappa : \Gamma \rightarrow \mathcal{O}_{\mathbb{C}_p}^\times$ gives rise to a continuous homomorphism $\kappa : \Lambda = \mathbb{Z}_p[[\Gamma]] \rightarrow \mathcal{O}_{\mathbb{C}_p}$ and hence defines a point, still denoted by κ , on the weight space \mathcal{W} . The T -coordinate of the point κ is $T_\kappa = \kappa(\gamma_0) - 1$. We point out that, the T -coordinate of

a point of \mathcal{W} depends on the choice of the topological generator γ_0 , but its p -adic valuation does not.

Example 2.2. For $k \in \mathbb{Z}$, the character $x^k : \Gamma \rightarrow \mathbb{Z}_p^\times$ sending a to a^k has T -coordinate $T_{x^k} = \exp(2kp) - 1$. We observe that $|\exp(2kp) - 1| = p^{-v_p(2kp)}$; in other words, these types of points are very closed to the centers of the weight disks.

Let $\psi_m : \Gamma \rightarrow (\mathbb{Z}_p/p^m\mathbb{Z}_p)^\times \rightarrow \mathcal{O}_{\mathbb{C}_p}^\times$ denote a finite continuous character which does not factor through smaller *positive* m ($m \geq 2$ if $p = 2$). We say that ψ_m has *conductor* p^m , ignoring the prime-to- p part of the conductor. In particular, a trivial character has conductor p (or 4 if $p = 2$); so $m = 1$ (or $m = 2$ if $p = 2$). When $m \geq 2$ and $p > 2$, $\psi_m(\gamma_0)$ is a primitive p^{m-1} -st root of unity $\zeta_{p^{m-1}}$. Thus, the point $x^k\psi_m$ has T -coordinate $\zeta_{p^{m-1}} \exp(2pk) - 1$, which has norm $p^{-1/p^{m-2}(p-1)}$ (independent of k). So these points move towards the boundary of the weight space as m grows; *but stay in the same “rim” as k varies, and accumulate as k becomes more congruent modulo powers of p .*

We call characters $x^k\psi_m$ with $k \geq 1$ *classical characters*. (Our weight will always be $k + 1$ from now on.)

We use $\omega : \Delta \rightarrow \mathbb{Z}_p^\times$ to denote the Teichmüller character. We use $\langle \cdot \rangle : \Gamma \rightarrow \mathbb{Z}_p^\times$ to denote the character $x\omega^{-1}$.

2.3. Coleman-Mazur eigencurve. Instead of working with the usual eigencurves, we shall work with the so-called “spectral curves”; the main Conjecture 2.5 is, for a large part, equivalent for these two curves.

We first recall the definition of spectral curves; for details, we refer to [Bu07, Section 2]. Suppose that we are given an affinoid algebra A^5 over \mathbb{Q}_p and a Banach A -module S which is *potentially orthonormalizable*, that is a Banach A -module isomorphic to a direct summand of a Banach A -module P which admits a countable orthonormal basis $(e_i)_{i \in \mathbb{N}}$. Moreover, suppose that we are given a *nuclear* operator U_p on S , that is, the uniform limit of a sequence of continuous A -linear operators on S whose images are finite A -modules. Then we can extend the action of U_p to the ambient space P by taking the zero action on the other direct summand of P . Write U_p as an infinite matrix M , respect to the basis (e_i) . Then the *characteristic power series* of U_p acting on S

$$\text{Char}(U_p; S) := \det(I - XM) = 1 + c_1X + c_2X^2 + \cdots \in A[[X]]$$

converges and is independent of the choices of the ambient space P and its basis (e_i) . Moreover, we have $\lim_{n \rightarrow \infty} |c_n|r^n = 0$ for any $r \in \mathbb{R}^+$. Consequently, it makes sense to talk about the zero locus of the characteristic power series $\text{Char}(U_p; S)$ in $\text{Max}(A) \times \mathbb{G}_{m,\text{rig}}$, where X is the coordinate of the second factor. We denote this zero locus by $\text{Spc} := \text{Spc}(U_p; S)$; it is called the *spectral variety* associated to the Banach module S and the U_p -operator. The natural projection $\text{wt} : \text{Spc} \rightarrow \text{Max}(A)$ is called the *weight map*; the map $a_p : \text{Spc} \rightarrow \mathbb{G}_{m,\text{rig}} \xrightarrow{x \rightarrow x^{-1}} \mathbb{G}_{m,\text{rig}}$ given by the composite of the other natural projection with an inverse

⁵Typically, $\text{Max}(A)$ is an affinoid subdomain of \mathcal{W} .

map is called the *slope map*.

$$\begin{array}{ccc} \mathrm{Spc} & \xrightarrow{a_p} & \mathbb{G}_{m,\mathrm{rig}} \\ \downarrow \mathrm{wt} & & \\ \mathrm{Max}(A) & & \end{array}$$

The weight map is known to be locally finite. For each closed point $z \in \mathrm{Spc}$, we use $|\mathrm{wt}(z)|$ to denote the absolute value of the T -coordinate of z and $|a_p(z)|$ to denote the absolute value of the corresponding point with respect to the natural coordinate on $\mathbb{G}_{m,\mathrm{rig}}$.

In the case of elliptic modular forms (with level $\Gamma_0(p)$), Coleman and Mazur [CM98] constructed, for each affinoid subdomain A of the weight space \mathcal{W} , a Banach module M consisting of overconvergent cuspidal modular forms of weight in A and of a fixed convergence radius; it carries a natural action of the U_p -operator. This construction was subsequently generalized by Buzzard [Bu07] to allow arbitrary tame level on the modular curve. Using the construction of the previous paragraph, one can define the spectral curve over $\mathrm{Max}(A)$, which patches together over \mathcal{W} as the subdomain $\mathrm{Max}(A)$ varies. We do not recall the precise definition here, but refer to [Bu07] for details. However, we shall later encounter a slightly different situation working with definite quaternion algebras; detailed construction of the corresponding Banach module will be given then.

2.4. The eigencurve near the boundary of the weight space. Recall that weight space \mathcal{W} has a natural coordinate T . For $r < 1$, we use $\mathcal{W}^{\geq r}$ to denote the sub-annulus of \mathcal{W} where $r \leq |T| < 1$. (Mazur prefers to call it the *rim* of the weight space.) We are mostly interested in the situation when $r \rightarrow 1^-$. As computed in Example 2.2, all powers x^k of the cyclotomic character are *not* in the rim of the weight space as soon as $r > p^{-1}$.

We put $\mathrm{Spc}^{\geq r} := \mathrm{wt}^{-1}(\mathcal{W}^{\geq r})$.

The following question was asked by Coleman and Mazur [CM98], and later elaborated by Buzzard and Kilford [BK05].

Conjecture 2.5. *When r is sufficiently close to 1, the following statements hold.*

- (1) *The space $\mathrm{Spc}^{\geq r}$ is a disjoint union of (countably infinitely many) connected components X_1, X_2, \dots such that the weight map $\mathrm{wt} : X_n \rightarrow \mathcal{W}^{\geq r}$ is finite and flat for each n .*
- (2) *There exist nonnegative rational numbers $\lambda_1, \lambda_2, \dots \in \mathbb{Q}$ in non-decreasing order and approaching to infinity such that, for each i and each point $z \in X_n$, we have*

$$|a_p(z)| = |\mathrm{wt}(z)|^{(p-1)\lambda_n}.$$

- (3) *The sequence $\lambda_1, \lambda_2, \dots$ is a disjoint union of finitely many arithmetic progressions, counted with multiplicity (at least when the indices are large enough).*

Clearly Conjecture 2.5 implies Conjecture 1.1 by specializing to classical weights using Coleman's classicality result [Co96, Co97].

Remark 2.6. Let us give a few evidences and remarks on Conjecture 2.5 (as well as Conjecture 1.1).

- (1) The novelty of our formulation lies in emphasizing statement (3) of Conjecture 2.5 as part of the general picture. In fact, the aim of this paper is to give strong evidence to support this expectation; see in particular, Corollary 6.19 and Theorem 7.12(3).

- (2) Similar properties near the center of the weight space are expected to be false; we refer to [Bu05, Cl05, Lo07] for more discussions. But see also Remarks 4.10.
- (3) One can reformulate this conjecture for eigencurves instead of spectral curves; the two statements would be essentially equivalent.
- (4) When $p = 2, 3$ and the modular curve is taken to be $X_0(p)$, Conjecture 2.5 is proved using direct computations by Buzzard-Kilford [BK05] and Roe [Ro13], extending the thesis of Emerton [Em98].
- (5) For $p = 5, 7$, the weaker version Conjecture 1.1 was verified in some cases by Kilford and McMurty [Kil08, KM12].
- (6) In an analogous situation where the eigencurve associated to Artin-Scheier-Witt tower of curves is considered, the analogous of Conjecture 2.5, in fact over the entire weight space,⁶ is proved by Davis and the first two authors [DWX13]. Our argument in Section 6 shares some similarities with this approach and is in part inspired by it.

Remark 2.7. We give our most optimistic expectation of the numerics in Conjecture 2.5. Suppose $p \geq 3$ for simplicity. First, we expect Conjecture 2.5 to hold for $r = p^{-1/(p-1)}$ (i.e. the radius for finite characters of conductor p^2).⁷ Moreover, we hope to make a guess about the sequence $\lambda_1, \lambda_2, \dots$ in Conjecture 2.5. Assume that the tame level structure is *neat*. Fix a connected component of the weight disk and fix a finite character ψ_2 of conductor p^2 so that the character $x\psi_2$ lies in that weight disk.

For $i = 0, \dots, \frac{p-3}{2}$, consider the action of U_p on the space of cusp forms $S_2(p^2; \psi_2\omega^{-2i})$ whose tame level is as given and the level at p is $\Gamma_0(p^2)$ with Nybentypus character $\psi_2\omega^{-2i}$. The dimension of such space is denoted by t (which does not depend on i). Let $\alpha_1^{(i)}, \dots, \alpha_t^{(i)}$ denote the p -adic valuations of the corresponding U_p -eigenvalues, counted with multiplicity.

Let d denote the number of cusps of the modular curve with only the tame level, or equivalently the dimension of the weight 2 Eisenstein series for the tame level.

Then the sequence $\lambda_1, \lambda_2, \dots$ is expected to be the union (rearranged into the non-decreasing order) of exactly the following list of numbers:

- the numbers $1, 2, 3, \dots$ with multiplicity d , and
- for $i = 0, \dots, \frac{p-1}{2}$ and $r = 1, \dots, t$, the numbers

$$\alpha_r^{(i)} + i, \alpha_r^{(i)} + i + \frac{p-1}{2}, \alpha_r^{(i)} + i + (p-1), \dots$$

The former part should be considered as “contributions from the Eisenstein series” although the overconvergent modular forms are cuspidal; and the latter part is the “contributions from the cuspidal part”, which is a union of arithmetic progressions with common difference $\frac{p-1}{2}$. (The number $\frac{p-1}{2}$ comes from the cyclic repetition of powers of the Teichmüller character.) Our guess is motivated by the main theorems of this paper and some computation of Kilford and McMurty [Kil08, KM12].

3. AUTOMORPHIC FORMS FOR A DEFINITE QUATERNION ALGEBRA

One of the major technical difficulties, among others, is the poor understanding of the geometry of the modular curves, in explicit coordinates. To bypass this difficulty, we consider

⁶The fact that the analogous statements hold over the entire weight space means that the situation is largely simplified; the method will probably not translate directly to the Coleman-Mazur eigencurve case.

⁷It is possible that Conjecture 2.5 holds for even smaller r , e.g. $r < p^{-1}$; but we do not have strong evidence either supporting or against this.

the eigencurve for a definite quaternion algebra; then a p -adic family version of Jacquet-Langlands correspondence [Ch05] allows us to recover a big part of Conjecture 2.5 from the corresponding statements for the quaternion algebra. We now recall the definition of the quaternionic eigencurves following [Bu04, Bu07].

3.1. Setup. Let \mathbb{A}_f denote the finite adèles of \mathbb{Q} and $\mathbb{A}_f^{(p)}$ its prime-to- p components. Let D be a definite quaternion algebra over \mathbb{Q} which splits at p ; in other words, $D \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to the Hamiltonian quaternion and $D \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq M_2(\mathbb{Q}_p)$. Put $D_f := D \otimes_{\mathbb{Q}} \mathbb{A}_f$. Let \mathcal{S} be a finite set of primes including p and all primes at which D ramifies. For each prime $l \neq p$, we fix an open compact subgroup of U_l of $(D \otimes_{\mathbb{Q}} \mathbb{Q}_l)^\times$. For $l \notin \mathcal{S}$, we fix an isomorphism $D \otimes_{\mathbb{Q}} \mathbb{Q}_l \simeq M_2(\mathbb{Q}_l)$ and require that $U_l \simeq GL_2(\mathbb{Z}_l)$ under this identification. We fix a positive integer $m \in \mathbb{N}$ and consider the Iwahori subgroup

$$U_0(p^m) = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^m \mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \subset GL_2(\mathbb{Q}_p) \simeq (D \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times.$$

We will also need the monoid

$$\Sigma_0(p^m) := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) \mid p^m | c, p \nmid d, \det(\gamma) \neq 0 \right\}.$$

Finally, we write $U = \prod_{l \neq p} U_l \times U_0(p^m)$ for the product, as an open compact subgroup of D_f^\times . We occasionally use U_1 to denote $\prod_{l \neq p} U_l \times \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^m \mathbb{Z}_p & 1 + p^m \mathbb{Z}_p \end{pmatrix}$.

We further assume that U is taken sufficiently small so that (see [Bu04, Section 4])

$$(3.1.1) \quad \text{for any } x \in D_f^\times, \text{ we have } x^{-1} D^\times x \cap U = \{1\}.$$

We fix a finite extension E of \mathbb{Q}_p as the coefficient field, which we will enlarge as needed in the argument. Let \mathcal{O} denote the valuation ring of E and ϖ an uniformizer. Write $\mathbb{F} = \mathcal{O}/(\varpi)$ for the residue field. Let $v(\cdot)$ denote the valuation on E normalized so that $v(p) = 1$.

We write $\mathcal{A}^\circ := \mathcal{O}\langle z \rangle$ and $\mathcal{A} = \mathcal{A}^\circ[\frac{1}{p}]$ for the Tate algebras.

Put $r_m = p^{-1/p^{m-1}(p-1)}$ if $p > 2$ and $r_m = 2^{-1/2^{m-2}}$ if $p = 2$. Let $\mathcal{W}^{<r_m}$ denote the open disks of \mathcal{W} where the T -coordinate has absolute value $< r_m$. Let $\text{Max}(A)$ be an affinoid space over $\mathcal{W}^{<r_m}$. (Typical examples of $\text{Max}(A)$ we consider are either a subdomain or a point.) Let $\kappa : \Gamma \rightarrow A^\times$ denote the universal character. Then κ extends to a continuous character

$$(3.1.2) \quad \begin{aligned} \kappa : (\mathbb{Z}_p + p^m \mathcal{A}^\circ)^\times &= \mathbb{Z}_p^\times \cdot (1 + p^m \mathcal{A}^\circ)^\times \longrightarrow (A \widehat{\otimes} \mathcal{A}^\circ)^\times \\ a \cdot x &\longmapsto \kappa(a) \cdot \kappa(\exp(2p))^{(\log x)/2p}. \end{aligned}$$

One checks easily that the condition $|\kappa(\exp(2p)) - 1| < r_m$ ensures the convergence and the independence of the factorization $a \cdot x$. See e.g. [Pi13, Section 2.1] for a more optimal convergence condition.

3.2. Overconvergent automorphic forms. Consider the right action of $\Sigma_0(p^m)$ on $A \widehat{\otimes} \mathcal{A}$ given by

$$(3.2.1) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma_0(p^m) \text{ and } h(z) \in A \widehat{\otimes} \mathcal{A}, \quad (h|_{\kappa\gamma})(z) := \frac{\kappa(cz + d)}{cz + d} h\left(\frac{az + b}{cz + d}\right).^8$$

⁸Our weight normalization is different from [Bu04, Bu07, Ja04] and most of the literature by using $cz + d$ in the denominator as opposed to $(cz + d)^2$; we will see a small benefit of our choice later in Proposition 4.6.

Note that it is crucial that $p^m|c$ and $d \in \mathbb{Z}_p^\times$ so that $\kappa(cz + d)$ and $(cz + d)^{-1}$ make sense.

We define the space of *overconvergent automorphic forms of weight κ and level U* to be

$$S^{D,\dagger}(U; \kappa) := \left\{ \varphi : D_f^\times \rightarrow A \widehat{\otimes} \mathcal{A} \mid \varphi(\delta gu) = \varphi(g) \|\kappa u_p, \text{ for any } \delta \in D^\times, g \in D_f^\times, u \in U \right\},$$

where u_p is the p -component of u .

Example 3.3. When $\kappa = x^k \psi_{m'} : \Gamma \rightarrow \mathbb{Q}_p(\zeta_{p^{m'-1}})^\times$ is the continuous character considered in Example 2.2, we can take the definition above for $A = E \supset \mathbb{Q}_p(\zeta_{p^{m'-1}})$ corresponding to the point κ on \mathcal{W} , which lies in $\mathcal{W}^{\leq r_m}$ if $m' \leq m$. In this case, the right action is given by

$$(3.3.1) \quad (h \|\kappa \gamma)(z) = (cz + d)^{k-1} \psi_{m'}(d) h\left(\frac{az + b}{cz + d}\right).$$

The space $S^{D,\dagger}(U; \kappa) = S_{k+1}^{D,\dagger}(U; \psi_{m'})$ is the space of *overconvergent automorphic forms of weight $k + 1$, Nybentypus character $\psi_{m'}$, and level U* .

Moreover, when $k \geq 1$ is a positive integer, we observe that the subspace L_{k-1} of \mathcal{A} consisting of polynomials in z with degree $\leq k - 1$ is stable under the action (3.3.1); so we can define the space of *classical automorphic forms of weight $k + 1$, character $\psi_{m'}$, and level U* to be the subspace $S_{k+1}^D(U; \psi_{m'})$ of $S_{k+1}^{D,\dagger}(U; \psi_{m'})$ consisting of functions φ with value in L_{k-1} . In particular, when $k = 1$,

$$(3.3.2) \quad S_2^D(U; \psi_m) := \left\{ \varphi : D_f^\times \rightarrow E \mid \varphi(\delta gu) = \psi_m(d) \varphi(g) \text{ for any } \delta \in D^\times, g \in D_f^\times, \right. \\ \left. \text{and } u \in U \text{ with } u_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}^9$$

We occasionally write $S_2^D(U; \psi_m; \mathcal{O})$ for the subspace of functions that take integral values.

3.4. Hecke actions. The space $S^{D,\dagger}(U; \kappa)$ carries actions of Hecke operators, which preserves the subspace of classical automorphic forms $S_{k+1}^D(U; \psi_m)$ when $\kappa = x^k \psi_m$ is given as in Example 3.3.

Let l be a prime not in \mathcal{S} ; then $U_l \simeq \mathrm{GL}_2(\mathbb{Z}_l)$. We write $U_l \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} U_l = \coprod_{i=0}^l U_l w_i$, with $w_i = \begin{pmatrix} l & 0 \\ i & 1 \end{pmatrix}$ for $i < l$ and $w_l = \begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix}$, viewed as elements in $\mathrm{GL}_2(\mathbb{Q}_l) \simeq D \otimes_{\mathbb{Q}} \mathbb{Q}_l$. We define the action of the operator T_l on $S^{D,\dagger}(U; \kappa)$ by

$$T_l(\varphi) = \sum_{i=0}^l \varphi|_{\kappa} w_i, \quad \text{with } (\varphi|_{\kappa} w_i)(g) := \varphi(g w_i^{-1}).^{10}$$

Similarly, we write (note $m \geq 1$)

$$U_0(p^m) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} U_0(p^m) = \coprod_{i=0}^{p-1} U_0(p^m) v_i, \quad \text{with } v_i = \begin{pmatrix} p & 0 \\ i p^m & 1 \end{pmatrix}.$$

Then the action of the operator U_p on $S^{D,\dagger}(U; \kappa)$ is defined to be

$$(3.4.1) \quad U_p(\varphi) = \sum_{i=0}^{p-1} \varphi|_{\kappa} v_i, \quad \text{with } (\varphi|_{\kappa} v_i)(g) := \varphi(g v_i^{-1}) \|\kappa v_i.$$

We point out that the definition of U_p - and T_l -operators do not depend on the choices of the double coset representatives w_i and v_i . But our choices may ease the computation.

¹⁰This looks slightly different from (3.4.1) below because $\|\kappa w_i$ is trivial as w_i is not in the p -component.

These U_p - and T_l -operators are viewed as acting on the space on the left (although the expression seems to suggest a right action); they are pairwise commutative.

Notation 3.5. If an (overconvergent) automorphic form φ is a (generalized) eigenvector for the U_p -operator, we call the p -adic valuation of its (generalized) U_p -eigenvalue the U_p -slope or simply the slope of φ . By U_p -slopes on a space of (overconvergent) automorphic forms, we mean the set of slopes of all generalized U_p -eigenforms in this space, counted with multiplicity.

3.6. Classicality of automorphic forms. The relation between the classical and the overconvergent automorphic forms in weight $k+1 \geq 2$ can be summarized by the following exact sequence

$$0 \rightarrow S_{k+1}^D(U; \psi_m) \rightarrow S_{k+1}^{D,\dagger}(U; \psi_m) \xrightarrow{\left(\frac{d}{dz}\right)^k} S_{1-k}^{D,\dagger}(U; \psi_m) \rightarrow 0,$$

where the first map is the natural embedding and the second map is given by

$$\left(\left(\frac{d}{dz}\right)^k(\varphi)\right)(g) := \left(\frac{d}{dz}\right)^k(\varphi(g)).$$

One checks that $\left(\frac{d}{dz}\right)^k \circ U_p = p^k \cdot U_p \circ \left(\frac{d}{dz}\right)^k$ (see [Bu04, §7]). As a corollary, all U_p -eigenforms of $S_{k+1}^{D,\dagger}(U; \psi_m)$ with slope strictly less than k are classical. It is also well known that the U_p -slopes on $S_{k+1}^D(U; \psi_m)$ are always less than or equal to k by the admissibility of the associated Galois representation. It follows that the U_p -slopes on $S_{k+1}^D(U; \psi_m)$ are exactly the smallest $\dim S_{k+1}^D(U; \psi_m)$ ¹¹ numbers (counted with multiplicity) in the set of U_p -slopes on $S_{k+1}^{D,\dagger}(U; \psi_m)$.

3.7. Jacquet-Langlands correspondence. We recall a very special case of the classical Jacquet-Langlands correspondence, which was used in the introduction. Let N be a positive integer coprime to p . Assume that there exists a prime number ℓ such that $\ell \parallel N$. Let $D_{\ell\infty}$ denote the definite quaternion algebra over \mathbb{Q} which ramifies at exactly ℓ and ∞ . If we take the level structure so that \mathcal{S} is the set of prime factors of $p\ell N$, U_ℓ is the maximal open compact subgroup of $(D_{\ell\infty} \otimes \mathbb{Q}_\ell)^\times$, and $U_q = \begin{pmatrix} \mathbb{Z}_q^\times & \mathbb{Z}_q \\ N\mathbb{Z}_q & \mathbb{Z}_q^\times \end{pmatrix} \subset \mathrm{GL}_2(\mathbb{Q}_q) \simeq (D_{\ell\infty} \otimes \mathbb{Q}_q)^\times$ for a prime $q|N$ but $q \neq \ell, p$, then the Jacquet-Langlands correspondence says that there exists an isomorphism of modules of U_p - and all T_q -operators for $q \nmid Np$:

$$(3.7.1) \quad S_{k+1}(\Gamma_0(p^m N); \psi_m)^{\ell\text{-new}} \cong S_{k+1}^{D_{\ell\infty}}(U; \psi_m)$$

for all weights $k+1 \geq 2$. This allows us to translate our results about automorphic forms on definite quaternion algebras to results about modular forms. One can certainly make variants of this; but we do not further discuss.

3.8. Eigencurve for D . It is clear that $S^{D,\dagger}(U; \kappa)$ is (potentially) orthonormalizable (see [Bu07, §10], or imitate Lemma 4.2). The action of the U_p -operator on $S^{D,\dagger}(U; \kappa)$ is nuclear by [Bu07, Lemma 12.2]. So the construction in Subsection 2.3 applies with $S = S^{D,\dagger}(U; \kappa)$ to give a spectral curve over $\mathrm{Max}(A)$. The construction is clearly functorial in A and hence defines a spectral curve Spc_D over $\mathcal{W}^{<r_m}$. As explained in [Bu07, Section 13], the construction for different m also glues over small weight disks and hence gives rise to a spectral curve Spc_D over the entire weight space \mathcal{W} .

¹¹This number can be expressed in a simple way as in Corollary 4.3.

The Jacquet-Langlands correspondence above can be made into p -adic families. By [Ch05], there is a closed immersion $\mathrm{Spc}_D^{\mathrm{red}} \hookrightarrow \mathrm{Spc}^{\mathrm{red}}$, where the superscript means to take the reduced subscheme structure.¹² Therefore, it is natural to expect that Conjecture 2.5 holds for Spc_D in place of Spc . Conversely, knowing Conjecture 2.5 for Spc_D , it is quite possible to infer a lot of information regarding Spc via the comparison [Ch05].

4. EXPLICIT COMPUTATION OF THE U_p -OPERATOR

We now make the first attempt to prove certain weak version of Conjectures 1.1 and 2.5, ending with a proof of Theorem A. To our best knowledge, the only known approach to any form of these conjectures is via “brutal force” computation, that is to compute directly the characteristic power series of the operator U_p to the extent that one can determine its slopes. Our approach is derived from a computation made by Jacobs [Ja04] of the infinite matrix for U_p in terms of concrete numbers. The novelty of our improvement is to make “brutal but formal computation” as opposed to using numbers. We include his example in the next section with some simplification. It serves as a toy model of our computation presented in this section.

Notation 4.1. We decompose D_f^\times into (a disjoint union of) double cosets $\coprod_{i=0}^{t-1} D^\times \gamma_i U$, for some elements $\gamma_0, \gamma_1, \dots, \gamma_{t-1} \in D_f^\times$. By our smallness hypothesis on U in Subsection 3.1, the natural map $D^\times \times U \rightarrow D^\times \gamma_i U$ for each i sending (δ, u) to $\delta \gamma_i u$ is bijective. We say that the double coset decomposition above is *honest*.

Since the norm map $\mathrm{Nm} : D^\times \rightarrow \mathbb{Q}_{>0}^\times$ is surjective, we may modify the representatives γ_i so that $\mathrm{Nm}(\gamma_i) \in \widehat{\mathbb{Z}}^\times$. Moreover, since $\mathrm{Nm}(U_0(p^m)) = \mathbb{Z}_p^\times$, we can further modify the p -component of each γ_i so that its norm is 1. Finally, using the fact that $(D^\times)^{\mathrm{Nm}=1}$ is dense in $(D \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\times, \mathrm{Nm}=1}$, we may assume that the p -component of each γ_i is trivial, still keeping the property that $\mathrm{Nm}(\gamma_i) \in \widehat{\mathbb{Z}}^\times$.

Let $\mathrm{Max}(A)$ be an affinoid space over $\mathcal{W}^{<r_m}$ and let $\kappa : \Gamma \rightarrow A^\times$ be the universal character.

Lemma 4.2. *We have an A -linear isomorphism of Banach spaces*

$$\begin{aligned} S^{D, \dagger}(U; \kappa) &\xrightarrow{\cong} \bigoplus_{i=0}^{t-1} A \widehat{\otimes} \mathcal{A} \\ \varphi &\longmapsto (\varphi(\gamma_i))_{i=0, \dots, t-1}. \end{aligned}$$

Proof. This is clear as the function φ is uniquely determined by its value at the chosen representatives γ_i . There is no further restriction on the value of $\varphi(\gamma_i)$ because the double coset decomposition in Notation 4.1 is honest. \square

Corollary 4.3. *We have $\dim S_{k+1}^D(U; \psi_m) = kt$, for the number t in Notation 4.1.*

Proposition 4.4. *In terms of the explicit description of the space of overconvergent automorphic forms, the U_p - and T_l - (for $l \notin \mathcal{S}$) operators can be described by the following*

¹²Rigorously speaking, [Ch05] proves the result for eigencurves; but the spectral curves, when taking the reduced scheme structure, are exactly the images of the eigencurves after forgetting the tame Hecke actions.

commutative diagram.

$$\begin{array}{ccc}
S^{D,\dagger}(U; \kappa) & \xrightarrow{\varphi \mapsto (\varphi(\gamma_i))} & \bigoplus_{i=0}^{t-1} A \widehat{\otimes} \mathcal{A} \\
\begin{array}{c} \varphi \mapsto U_p \varphi \\ \varphi \mapsto T_l \varphi \end{array} \downarrow & & \begin{array}{c} \mathfrak{U}_p \downarrow \text{Map of} \\ \mathfrak{T}_l \downarrow \text{interest} \end{array} \\
S^{D,\dagger}(U; \kappa) & \xrightarrow{\varphi \mapsto (\varphi(\gamma_i))} & \bigoplus_{i=0}^{t-1} A \widehat{\otimes} \mathcal{A}.
\end{array}$$

Here the right vertical arrow \mathfrak{U}_p (resp. \mathfrak{T}_l) is given by a matrix with the following description.

- (1) The entries of \mathfrak{U}_p (resp. \mathfrak{T}_l) are sums of operators of the form $\|_{\kappa} \delta_p$, where δ_p is the p -component of a global element $\delta \in D^\times$ of norm p (resp. norm l).
- (2) There are exactly p (resp. $l+1$) such operators appearing in each row and each column of \mathfrak{U}_p (resp. \mathfrak{T}_l).
- (3) We have $\delta_p \in \begin{pmatrix} p\mathbb{Z}_p & \mathbb{Z}_p \\ p^m\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$ (resp. $\delta_p \in U_0(p^m) = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^m\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$).

Proof. We only prove this for the U_p -operator and the proof for the T_l -operator ($l \notin \mathcal{S}$) is exactly the same. For each γ_i , we have

$$(U_p \varphi)(\gamma_i) = \sum_{j=0}^{p-1} \varphi(\gamma_i v_j^{-1}) \|_{\kappa} v_j.$$

Now we can write each $\gamma_i v_j^{-1}$ uniquely as $\delta_{i,j}^{-1} \gamma_{\lambda_{i,j}} u_{i,j}$ for $\delta_{i,j} \in D^\times$, $\lambda_{i,j} \in \{0, \dots, t-1\}$, and $u_{i,j} \in U$. Then we have

$$(U_p \varphi)(\gamma_i) = \sum_{j=0}^{p-1} \varphi(\delta_{i,j}^{-1} \gamma_{\lambda_{i,j}} u_{i,j}) \|_{\kappa} v_j = \sum_{j=0}^{p-1} \varphi(\gamma_{\lambda_{i,j}}) \|_{\kappa} (u_{i,j,p} v_j),$$

where $u_{i,j,p}$ is the p -component of $u_{i,j}$. Substitute back in $u_{i,j} v_j = \gamma_{\lambda_{i,j}}^{-1} \delta_{i,j} \gamma_i$ and note the fact that both γ_i and $\gamma_{\lambda_{i,j}}$ have trivial p -component by our choice in Notation 4.1. We have

$$(U_p \varphi)(\gamma_i) = \sum_{j=0}^{p-1} \varphi(\gamma_{\lambda_{i,j}}) \|_{\kappa} \delta_{i,j,p},$$

where $\delta_{i,j,p}$ is the p -component of the global element $\delta_{i,j} \in D^\times$. We now check the description of each $\delta_{i,j}$:

$$\delta_{i,j} = \gamma_{\lambda_{i,j}} u_{i,j} v_j \gamma_i^{-1} \in \gamma_{\lambda_{i,j}} U \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} U \gamma_i^{-1}.$$

From this, we see that the p -component of $\delta_{i,j}$ lies in $\begin{pmatrix} p\mathbb{Z}_p & \mathbb{Z}_p \\ p^m\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$. Moreover, the norm of $\gamma_{\lambda_{i,j}} U \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} U \gamma_i^{-1}$ lands in $p\widehat{\mathbb{Z}}^\times$, because our choice of the representatives satisfies $\text{Nm}(\gamma_i) \in \widehat{\mathbb{Z}}^\times$ by Notation 4.1. Therefore, $\text{Nm}(\delta_{i,j}) \in \mathbb{Q}_{>0}^\times \cap p\widehat{\mathbb{Z}}^\times = \{p\}$. This concludes the proof of the proposition. \square

4.5. Infinite matrices and generating functions. For an infinite matrix (where the row and column indices start with 0 as opposed to 1)

$$(4.5.1) \quad M = \begin{pmatrix} m_{0,0} & m_{0,1} & m_{0,2} & \cdots \\ m_{1,0} & m_{1,1} & m_{1,2} & \cdots \\ m_{2,0} & m_{2,1} & m_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with coefficients in an affinoid E -algebra A , we consider the following formal power series:

$$H_M(x, y) = \sum_{i, j \in \mathbb{Z}_{\geq 0}} m_{i,j} x^i y^j \in A[[x, y]].$$

It is called the *generating series* of the matrix M . When M is the matrix for an operator T acting on the Tate algebra $A \widehat{\otimes} \mathcal{A} = A \langle z \rangle$ over A with respect to the basis $1, z, z^2, \dots$, we call $H_M(x, y)$ the *generating series* of T .

For $u \in E$, we write $\text{Diag}(u)$ for the infinite diagonal matrix with diagonal elements $1, u, u^2, \dots$. Then we have

$$H_{\text{Diag}(u)M \text{Diag}(v)}(x, y) = H_M(ux, vy).$$

For $t \in \mathbb{N}$, we write $\text{Diag}(u; t)$ for the infinite diagonal matrix with diagonal elements $1, \dots, 1, u, \dots, u, u^2, \dots$ where each number appears repeatedly t times.

The following key calculation is due to Jacobs [Ja04, Proposition 2.6].

Proposition 4.6. *Let $\kappa : \Gamma \rightarrow A^\times$ be the universal character for an affinoid space $\text{Max}(A)$ over $\mathcal{W}^{< r_m}$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in $\Sigma_0(p^m)$. The generating series of the operator $\|_{\kappa} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acting on $A \widehat{\otimes} \mathcal{A}$ (with respect to the basis $1, z, z^2, \dots$) is given by*

$$\frac{\kappa(cx + d)}{cx + d - axy - by} \text{.}^{13}$$

Here we point out that, although the operator $\|_{\kappa} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ when viewed as the action of the monoid $\Sigma_0(p^m)$ is a right action, we only use one particular operator and will not discuss the composition; so we still use the column vector convention (pretending it as a left operator).

Proof. This is straightforward. By definition,

$$\begin{aligned} H_{\|_{\kappa} \begin{pmatrix} a & b \\ c & d \end{pmatrix}}(x, y) &= \sum_{i \in \mathbb{Z}_{\geq 0}} y^i \frac{\kappa(cx + d)}{cx + d} \cdot \left(\frac{ax + b}{cx + d} \right)^i \\ &= \frac{\kappa(cx + d)}{cx + d} \cdot \frac{1}{1 - y \cdot \frac{ax+b}{cx+d}} = \frac{\kappa(cx + d)}{cx + d - axy - by}. \end{aligned}$$

□

Combining Proposition 4.6 with Proposition 4.4, we can give a good description of the infinite matrices for \mathfrak{U}_p and \mathfrak{T}_l (for $l \notin \mathcal{S}$).

¹³Comparing to the convention in [Ja04], we lose an extra factor of $cx + d$ in the denominator due to our normalization (3.2.1). There is no real improvement in our formula except that it is shorter.

4.7. Hodge polygon and Newton polygon of a matrix. Before proceeding, we remind the readers some basic facts about p -adic analysis; we will use them later freely without referencing back here. Let $M \in M_n(E)$ be a matrix.

- (1) The *Newton polygon* of M is the convex polygon starting at $(0, 0)$ whose slopes are exactly the p -adic valuations of the eigenvalues of M , counted with multiplicity.
- (2) The *Hodge polygon* of M is the convex hull of the vertices
 $(i, \text{the minimal } p\text{-adic valuation of the determinants of all } i \times i\text{-minors})$.
- (3) The Hodge polygon is invariant when conjugating M by elements in $\text{GL}_n(\mathcal{O})$; the Newton polygon is invariant when conjugating M by elements in $\text{GL}_n(E)$.
- (4) If the slopes of the Hodge polygon of M are $a_1 \leq \dots \leq a_n$, then there exist matrices $A, B \in \text{GL}_n(\mathcal{O})$ such that AMB is a diagonal matrix whose diagonal elements have valuation exactly a_1, \dots, a_n . Conversely, if such A and B exist, the Hodge polygon of M has the described slopes.
- (5) If the slopes of the Hodge polygon of M are $a_1 \leq \dots \leq a_n$, then there exists a matrix $A \in \text{GL}_n(\mathcal{O})$ such that the valuations of all entries in the i -th row of AMA^{-1} are at least a_i for all i . Conversely, when such A exists, the Hodge polygon of M lies above the polygon with slopes a_1, \dots, a_n .
- (6) When $\mathcal{O}^{\oplus n}$ can be written as $V \oplus V'$ for two M -stable \mathcal{O} -submodules. Then the Newton slopes for M is the union of the Newton slopes of M acting on V and V' . The same holds for Hodge slopes.
- (7) It is always true that the Newton polygon lies above the Hodge polygon. This also holds for an infinite matrix associated to a nuclear operator.

We now prove Theorem A from the introduction.

Theorem 4.8. *Let ψ_m be a finite character of \mathbb{Z}_p^\times of exact conductor p^m with the same m that defines the level structure U .¹⁴ Recall that $\dim S_2^D(U; \psi_m) = t$. Then the Newton polygon for the slopes of U_p acting on $S^{D, \dagger}(U, x\langle x \rangle^k \psi_m)$ for $k \in \mathbb{Z}_p$ lies above the polygon with vertices*

$$(4.8.1) \quad (0, 0), (t, 0), (2t, t), \dots, (nt, \frac{n(n-1)}{2}t), \dots$$

Proof. By Lemma 4.2 and Proposition 4.4 (in our case $A = E$), it suffices to understand the matrix for the operator \mathfrak{U}_p . We first give $\bigoplus_{i=0}^{t-1} \mathcal{A}$ a basis:

$$1_0, z_0, z_0^2, \dots, 1_1, z_1, z_1^2, \dots, 1_{t-1}, z_{t-1}, \dots,$$

where the subscripts indicate which copy of \mathcal{A} the element comes from. Then the matrix for \mathfrak{U}_p is a $t \times t$ -block matrix such that each block is an infinite matrix. By Proposition 4.6, the generating series of each block is the sum of power series of the form

$$\frac{d\psi_m(d)\langle d \rangle^k (1 + \frac{c}{d}x)^{k+1}}{cx + d - axy - by}, \quad \text{with } p|a \text{ and } p^m|c \text{ by Proposition 4.4.}$$

When $k \in \mathbb{Z}_p$, the expression above lands in $\mathcal{O}[[x, pxy, p^m y]] \subseteq \mathcal{O}[[x, py]]$; in particular, the i th row of the corresponding infinite matrix is divisible by p^i .

We can then rewrite the matrix of \mathfrak{U}_p under the following basis of $\bigoplus_{i=0}^{t-1} \mathcal{A}$:

$$1_0, 1_1, \dots, 1_{t-1}, z_0, \dots, z_{t-1}, z_0^2, \dots$$

¹⁴Again, we allow ψ to be trivial, in which case $m = 1$ if $p > 2$ and $m = 2$ if $p = 2$.

Then the matrix of \mathfrak{U}_p becomes an infinite block matrix, where each block is $t \times t$. Moreover, the discussion above implies that the i th block row is entirely divisible by p^i . In other words, the Hodge polygon of this matrix lies above the polygon with vertices given by (4.8.1). So the Newton polygon of \mathfrak{U}_p also lies above it. \square

Remark 4.9. We discuss how one can improve the lower bound of the Newton polygon of the U_p -action on the classical automorphic forms $S_{k+1}^D(U; \psi)$ when ψ has conductor p and $m = 1$ (or 4 and $m = 2$). (The case when $m \geq 2$ for $p > 2$ and $m \geq 3$ for $p = 2$ will be studied in length in Section 6.) Note that this includes the case when ψ is trivial. For simplicity, we assume that the condition (3.1.1) holds for U replaced by $\prod_{l \neq p} U_l \times \mathrm{GL}_2(\mathbb{Z}_p)$. In particular, $(p+1)|t$ if $p > 2$ and $6|t$ if $p = 2$.

- (1) When ψ is non-trivial of conductor p or 4, we know that the U_p -slopes on $S_{k+1}^D(U; \psi)$ are exactly given by k minus the U_p -slopes on $S_{k+1}^D(U; \psi^{-1})$. (See Proposition 6.5 for the proof in the case of $k = 1$; and the general case is similar.) Thus, applying Theorem 4.8 to $S_{k+1}^D(U; \psi^{-1})$ and using the fact above, we can improve the lower bound in Theorem 4.8 over the interval $[\frac{kt}{2}, kt]$. Hence the Newton polygon for the U_p -action on $S_{k+1}^D(U; \psi)$ lies above the polygon with slopes
 - (if k is even) $0, 1, \dots, \frac{k}{2} - 1, \frac{k}{2} + 1, \frac{k}{2} + 2, \dots, k$, each with multiplicity t ;
 - (if k is odd) $0, 1, \dots, \frac{k-1}{2}, \frac{k+1}{2}, \frac{k+3}{2}, \dots, k$, each with multiplicity t , except the slopes $\frac{k-1}{2}$ and $\frac{k+1}{2}$ each has multiplicity $\frac{t}{2}$.¹⁵
- (2) When ψ is the trivial character, $S_{k+1}^D(U; \mathrm{triv})$ is the direct sum of the p -old part $S_{k+1}^D(U; \mathrm{triv})^{p\text{-old}}$ and the p -new part $S_{k+1}^D(U; \mathrm{triv})^{p\text{-new}}$. Given our earlier hypothesis on U , we have

$$\dim S_{k+1}^D(U; \mathrm{triv})^{p\text{-old}} = \frac{2}{p+1}kt, \quad \text{and} \quad \dim S_{k+1}^D(U; \mathrm{triv})^{p\text{-new}} = \frac{p-1}{p+1}kt.$$

The eigenvalues of U_p -action on $S_{k+1}^D(U; \mathrm{triv})^{p\text{-new}}$ all have valuation $(k-1)/2$; whereas the eigenvalues of U_p -action on $S_{k+1}^D(U; \mathrm{triv})^{p\text{-old}}$ can be paired so that the product of each pair has valuation k . Then the lower bound in Theorem 4.8 applies to the lesser of the pair of eigenvalues on the p -old space. Using this piece of information and the knowledge of the slopes on p -new forms, we conclude that the Newton polygon for the U_p -action on $S_{k+1}^D(U; \mathrm{triv})$ lies above the polygon with slopes

- $0, 1, \dots, [\frac{k}{p+1}] - 1$, each with multiplicity t ,
- $[\frac{k}{p+1}]$ with multiplicity $\frac{kt}{p+1} - [\frac{k}{p+1}]t$,
- $\frac{k-1}{2}$ with multiplicity $\frac{(p-1)kt}{p+1}$,
- $k - [\frac{k}{p+1}]$ with multiplicity $\frac{kt}{p+1} - [\frac{k}{p+1}]t$, and
- $k - [\frac{k}{p+1}] + 1, k - [\frac{k}{p+1}] + 2, \dots, k$, each with multiplicity t .

We point out that the bounds in both cases share the same end point with the actual Newton polygon of the U_p -action. Moreover, the distance of this end point and the vertex $(kt, \frac{k(k-1)}{2}t)$ given by Theorem 4.8 is linear in k ; so Theorem 4.8 is already a quite sharp bound in this sense.

¹⁵Here we temporarily allow rational multiplicity to mean to horizontal span of the corresponding segment of the polygon, which may not be an integer.

Remark 4.10. Keep the setup as in Remark 4.9(2) and consider the case of trivial character now. Gouvêa [Go01] has computed many numerical examples¹⁶ to support his expectation of the distribution of U_p -slopes on $S_{k+1}^D(U; \text{triv})$. If one uses $a_1(k) \leq \dots \leq a_{kt/(p+1)}(k)$ to denote the lesser slopes on the space of p -old forms, Gouvêa conjectured that the distribution given by the numbers

$$a_1(k)/k, a_2(k)/k, \dots, a_{kt/(p+1)}(k)/k, \text{ as } k \rightarrow \infty,$$

converges to a uniform distribution on $[0, \frac{1}{p+1}]$.

In view of the discussion above, this conjecture can be reinterpreted as: the Newton polygon of U_p -action on $S_{k+1}^D(U; \text{triv})$ “stays close” to the lower bound given in Remark 4.9. At least, the polygon lower bound provides an inequality for the distribution conjectured by Gouvêa.

5. AN EXAMPLE OF EXPLICIT COMPUTATION

In this section, we give an example of by-hand computation of the U_p -slopes for a particular definite quaternion algebra, a prime number p , and a level structure. This case was considered earlier by Jacobs [Ja04], a former student of Buzzard, in his thesis. Unfortunately, Jacobs relied too much on the computer and hence made the computation unaccessible to people who are interested in checking for patterns. We reproduce a variant of this computation to serve as a key toy model of our various proofs. We hope that this hand-on computation can inspire the readers to further develop this technique.

5.1. The quaternion algebra. In this section, we consider the quaternion algebra D which ramifies at only 2 and ∞ . Explicitly, it is

$$D = \mathbb{Q}\langle \mathbf{i}, \mathbf{j} \rangle / (\mathbf{i}\mathbf{j} = -\mathbf{j}\mathbf{i}, \mathbf{i}^2 = \mathbf{j}^2 = -1).$$

Here we use angled bracket to signify the non-commutativity of the algebra. It is conventional to put $\mathbf{k} = \mathbf{i}\mathbf{j}$. The maximal order of D is given by

$$\mathcal{O}_D = \mathbb{Z}\langle \mathbf{i}, \mathbf{j}, \frac{1}{2}(1 + \mathbf{i} + \mathbf{j} + \mathbf{k}) \rangle.$$

The unit group consists of 24 elements; they are

$$\mathcal{O}_D^\times = \{ \pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}, \frac{1}{2}(\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k}) \}.$$

5.2. Level structure. Our distinguished prime p is 3. Put $D_f = D \otimes \mathbb{A}_f$. For each $l \neq 2$, we identify $D \otimes \mathbb{Q}_l$ with $M_2(\mathbb{Q}_l)$. For $l = 2$, we use $D^\times(\mathbb{Z}_2)$ to denote the maximal compact subgroup of $(D \otimes \mathbb{Q}_2)^\times$. We consider the following open compact subgroup of D_f^\times :

$$(5.2.1) \quad U = D^\times(\mathbb{Z}_2) \times \prod_{l \neq 2,3} \text{GL}_2(\mathbb{Z}_l) \times \begin{pmatrix} \mathbb{Z}_3^\times & \mathbb{Z}_3 \\ 9\mathbb{Z}_3 & 1 + 3\mathbb{Z}_3 \end{pmatrix}.^{17}$$

We point out that for our choice of $p = 3$, this corresponds to $m = 2$ in Theorem B; so it is not literally covered by it.

¹⁶Rigorously speaking, Gouvêa [Go01] worked with actual modular forms, but we expect the analogous of his conjecture applies in this case.

¹⁷Our choice of the level structure is slightly different from [Ja04], who uses the $\Gamma_1(9)$ -level structure. Here $\Gamma_1(9)$ is defined in the same way as (5.2.1) but with the lower right entry of the last factor replaced by $1 + 9\mathbb{Z}_3$. As a result, Jacobs had to go through an additional factorization to get the same answer.

Notation 5.3. Let ν_3 denote the square root of -2 that is congruent to 1 modulo 3. We have a 3-adic expansion

$$\nu_3 = 1 + 3 + 2 \cdot 3^2 + 2 \cdot 3^5 + 3^7 + \dots .$$

We choose the isomorphism between $D \otimes \mathbb{Q}_3$ and $M_2(\mathbb{Q}_3)$ so that

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} \leftrightarrow \begin{pmatrix} \nu_3 & 1 \\ 1 & -\nu_3 \end{pmatrix}, \quad \mathbf{j} \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and } \mathbf{k} \leftrightarrow \begin{pmatrix} 1 & -\nu_3 \\ -\nu_3 & -1 \end{pmatrix}.$$

Lemma 5.4. *The following natural map is bijective.*¹⁸

$$\begin{aligned} D^\times \times U &\longrightarrow D_f^\times \\ (\delta, u) &\longmapsto \delta u. \end{aligned}$$

Proof. This is of course coincidental for our choices of D , p and U . We first observe that $D_f^\times = D^\times \cdot U_{\max}$ (see [Ja04, Lemma 1.22]), where U_{\max} is a maximal open compact subgroup of $(D \otimes \mathbb{A}_f)^\times$, defined using the same equation as in (5.2.1) except the factor at 3 is replaced by $\mathrm{GL}_2(\mathbb{Z}_3)$. Taking into account of the duplication, we have

$$D_f^\times = D^\times \times_{\mathcal{O}_D^\times} U_{\max}.$$

So it suffices to check that the image of \mathcal{O}_D^\times in $\mathrm{GL}_2(\mathbb{Z}_3)$ turns out to form a coset representative of U_{\max}/U . This can be checked easily by hand. (See the proof of [Ja04, Theorem 2.1] for the list of residues of \mathcal{O}_D^\times when taking modulo 9.) \square

Corollary 5.5. *Let ψ be a continuous character of \mathbb{Z}_3^\times of conductor 9 such that $\psi(-1) = 1$ and let $\kappa = x \langle x \rangle^w \psi$ with $w \in \mathcal{O}_{\mathbb{C}_3}$ be a character considered in Example 2.2. Then evaluation at 1 induces an isomorphism $S^{D,\dagger}(U; \kappa) \cong \mathcal{A}$.*

Lemma 5.6. *For the case considered in this section, the map \mathfrak{U}_3 in Proposition 4.4 is given by $\mathfrak{U}_3 = \|\kappa\delta_1 + \|\kappa\delta_2 + \|\kappa\delta_3$, where*

$$\delta_1 = -1 + \mathbf{i} - \mathbf{j}, \quad \delta_2 = \frac{1}{2}(1 + \mathbf{i} + 3\mathbf{j} + \mathbf{k}), \quad \text{and } \delta_3 = \frac{1}{2}(1 - 3\mathbf{i} - \mathbf{j} - \mathbf{k}).$$

The images of $\delta_1, \delta_2, \delta_3$ in $\mathrm{GL}_2(\mathbb{Z}_3)$ are given by

$$\begin{pmatrix} \nu_3 - 1 & 2 \\ 0 & -1 - \nu_3 \end{pmatrix}, \quad \begin{pmatrix} 1 + \frac{\nu_3}{2} & -1 - \frac{\nu_3}{2} \\ 2 - \frac{\nu_3}{2} & -\frac{\nu_3}{2} \end{pmatrix}, \quad \text{and } \begin{pmatrix} -\frac{3\nu_3}{2} & -1 + \frac{\nu_3}{2} \\ -2 + \frac{\nu_3}{2} & 1 + \frac{3\nu_3}{2} \end{pmatrix}.^{19}$$

Modulo 9, they are

$$\begin{pmatrix} 3 & 2 \\ 0 & 4 \end{pmatrix}, \quad \begin{pmatrix} 3 & 6 \\ 0 & 7 \end{pmatrix}, \quad \text{and } \begin{pmatrix} 3 & 1 \\ 0 & 7 \end{pmatrix}.$$

Proof. We follow the computation in Proposition 4.4. We need to compute

$$U_3(\varphi)(1) = \sum_{j=1}^3 \varphi(v_j^{-1}) \|\kappa v_j, \quad \text{for } v_j = \begin{pmatrix} 3 & 0 \\ 9 & 1 \end{pmatrix}$$

¹⁸In [Ja04], D_f^\times is written as the disjoint union of three double cosets, which in fact corresponds to the double coset decomposition of U over $\Gamma_1(9)$.

¹⁹One compares these matrices with the ones appearing after [Ja04, Lemma 2.5]. Jacobs has a different normalizations which could be removed if one wishes. Also, we think his matrices involving v_1^{-1} are not correct; this error is however fixed on the next page of *loc. cit.*

By Lemma 5.4, we can write each v_j^{-1} uniquely as $\delta_j^{-1}u_j$ for $\delta_j \in D^\times$ and $u_j \in U$. Then

$$\varphi(v_j^{-1})||_\kappa v_j = \varphi(1)||_\kappa(u_{j,3}v_j) = \varphi(1)||_\kappa \delta_{j,3},$$

where $u_{j,3}$ and $\delta_{j,3}$ denote the 3-components of u_j and δ_j , respectively. On the other hand, we have

$$\delta_j = u_j v_j \in D^\times \cap U v_j \subseteq D^\times \cap U \left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix} \right) U = D^\times \cap U_1(3) \left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix} \right),$$

where $U_1(3)$ is defined as U in (5.2.1) except the last factor is replaced by $\left(\begin{smallmatrix} \mathbb{Z}_3^\times & \mathbb{Z}_3 \\ 3\mathbb{Z}_3 & 1+3\mathbb{Z}_3 \end{smallmatrix} \right)$. If we put $\delta_j = \delta'_j(1 - \mathbf{i} + \mathbf{j})$, then we have

$$\begin{aligned} \delta'_j &\in D^\times \cap U_1(3) \left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix} \right) (1 - \mathbf{i} + \mathbf{j})^{-1} = D^\times \cap U_1(3) \left(\begin{smallmatrix} 1+\nu_3 & 2 \\ 0 & (1-\nu_3)/3 \end{smallmatrix} \right) \\ &= D^\times \cap U_1(3) \left(\begin{smallmatrix} 5 & 2 \\ 0 & 2 \end{smallmatrix} \right) = \left\{ -1, \frac{1}{2}(1 + \mathbf{i} + \mathbf{j} - \mathbf{k}), \frac{1}{2}(1 - \mathbf{i} - \mathbf{j} + \mathbf{k}) \right\} \end{aligned}$$

The last equality follows from looking at the list of \mathcal{O}_D^\times modulo 3. (In the notation of Jacobs's thesis [Ja04], this set is $\{-1, u_5, -u_8\}$.)

It is then clear that all δ_j 's are among the collections of the above right-multiplied by $1 - \mathbf{i} + \mathbf{j}$. The rest of the lemma is straightforward. \square

Theorem 5.7. *Let ψ be a character of \mathbb{Z}_3^\times of conductor 9 such that $\psi(-1) = 1$. We consider the characters $\kappa = x\langle x \rangle^w \psi$ ($w \in \mathcal{O}_{\mathbb{C}_3}$) as in Example 3.3. The slopes of the U_3 -operator acting on $S^{D,\dagger}(U; \kappa)$ are $\frac{1}{2}, 1 + \frac{1}{2}, 2 + \frac{1}{2}, 3 + \frac{1}{2}, \dots$*

Proof. Put $\xi = \psi(4)$; it is a primitive third root of unity. Then $\psi(7) = \xi^2$. Put $\pi = \xi - 1$ so that $v(\pi) = \frac{1}{2}$. Let $H_{\mathfrak{U}_3}(x, y)$ denote the generating series of the Hecke operator acting on $S^{D,\dagger}(U; \kappa) \cong \mathcal{A}$. By Lemma 5.6, the map \mathfrak{U}_3 is given as $\mathfrak{U}_3 = ||_\kappa \delta_1 + ||_\kappa \delta_2 + ||_\kappa \delta_3$ for the elements $\delta_1, \delta_2, \delta_3$ given therein. By Lemma 5.6, we have

$$\begin{aligned} H_{\mathfrak{U}_3} \left(\frac{1}{3\pi} x, \pi y \right) &\equiv \frac{\psi(4)4^w}{4 - 3\frac{1}{3\pi}x \cdot \pi x - 2\pi y} + \frac{\psi(7)7^w}{7 - 3\frac{1}{3\pi}x \cdot \pi y - 6\pi y} + \frac{\psi(7)7^w}{7 - 3\frac{1}{3\pi}x \cdot \pi y - \pi y} \\ &\equiv \frac{\xi}{4 - xy - 2\pi y} + \frac{\xi^2}{7 - xy - 6\pi y} + \frac{\xi^2}{7 - xy - \pi y} \\ &\equiv \frac{1 + \pi}{1 - xy + \pi y} + \frac{(1 + \pi)^2}{1 - xy} + \frac{(1 + \pi)^2}{1 - xy - \pi y} \pmod{3} \end{aligned}$$

It is now straightforward to check that this is congruent to $\frac{2\pi}{1 - xy}$ modulo 3. In other words, the matrix $\text{Diag}(\frac{1}{3\pi}) \cdot \mathfrak{U}_3 \cdot \text{Diag}(\pi)$ is congruent modulo 3 to $2\pi \cdot I_\infty$, where I_∞ is the infinite identity matrix. It follows from this easily that the slopes of the U_3 -operator acting on $S^{D,\dagger}(U; \kappa)$ are $\frac{1}{2}, 1 + \frac{1}{2}, 2 + \frac{1}{2}, 3 + \frac{1}{2}, \dots$ \square

6. IMPROVING THE LOWER BOUND

The key to obtain a strong result on U_p -slopes is to improve the lower bound in Theorem 4.8 so that it agrees with the Newton polygon for sufficiently many points.

Hypothesis 6.1. In this and the next section, we retain the notation from Section 4 to work with a general definite quaternion algebra D (which splits at p). We fix an integer $m \geq 4$. By writing ψ_m , we always mean a finite continuous character of \mathbb{Z}_p^\times of conductor p^m . The level structure at p is always taken to be $U_0(p^m)$ with the same number m . Some of the results (perhaps after modification) may hold for smaller m ; see Remark 6.18.

We assume that $\psi(-1) = 1$.

6.2. Facts about classical automorphic forms. To avoid future confusion, we must clarify how twisting an automorphic representation π (of weight 2) by a central Teichmüller character $\omega^r : \Delta = (\mathbb{Z}_p/2p\mathbb{Z}_p)^\times \rightarrow E^\times$ works, in an explicit way. For $r \in \mathbb{Z}$, we consider the following space of classical automorphic forms

$$(6.2.1) \quad S_2^D(U; \psi_m; \omega^r) := \left\{ \varphi : D_f^\times \rightarrow E \mid \varphi(\delta gu) = \omega^r(ad)\psi_m(d)\varphi(g) \text{ for any } \delta \in D^\times, \right. \\ \left. g \in D_f^\times, \text{ and } u \in U \text{ with } u_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}.$$

It carries an action of Hecke operators T_l (for $l \notin \mathcal{S}$) and U_p just as defined in Subsection 3.4, except multiplied by $\omega^r(l)$ and 1, respectively.

Recall that (after making a finite extension of the coefficient field E), we have a decomposition of automorphic representations under the actions of all Hecke operators T_l ($l \notin \mathcal{S}$):

$$(6.2.2) \quad S_2^D(U; \psi_m; \omega^r) = \bigoplus_{\pi} V(\pi),$$

where the sum is taken over all automorphic representations π of $\mathrm{GL}_2(\mathbb{A}^\infty)$ of weight 2. We say that π *appears* in $S_2^D(U; \psi_m; \omega^r)$ if the corresponding space $V(\pi) \neq 0$.

We may view ω^r as a Hecke character of \mathbb{A}^\times and write $\pi \otimes (\omega^r \circ \det)$ for the tensor product of the automorphic representations of $\mathrm{GL}_2(\mathbb{A})$; it has central character $\omega_\pi \omega^{2r}$, where ω_π is the central character of π . For each $l \notin \mathcal{S}$, the T_l -eigenvalue on the spherical vector at l for $\pi \otimes (\omega^r \circ \det)$ is $\omega^r(l)$ times that for π .

It is clear from this construction that π appears in $S_2^D(U; \psi_m)$ if and only if $\pi \otimes (\omega^r \circ \det)$ appears in $S_2^D(U; \psi_m; \omega^r)$. In fact we have a canonical isomorphism of modules of Hecke operators T_l for $l \notin \mathcal{S}$ and U_p

$$(6.2.3) \quad S_2^D(U; \psi_m) \otimes (\omega^r \circ \det) \cong S_2^D(U; \psi_m; \omega^r),$$

where T_l for $l \notin \mathcal{S}$ acts on the factor $(\omega^r \circ \det)$ by multiplication by $\omega^r(l)$ and U_p acts trivially. We must point out that $S_2^D(U; \psi_m; \omega^r)$ is *genuinely different* from $S_2^D(U; \psi_m \omega^{2r})$ (not even up to twists).

Lemma 6.3. *Assume Hypothesis 6.1. Then each Hecke eigenform in $S_2^D(U; \psi_m)$ is p -new, and the action of U_p on each of $V(\pi)$ in (6.2.2) is just the scalar multiplication by some $a_p(\pi) \in E$. Moreover, $v(a_p(\pi)) \in [0, 1]$.*

Proof. Equation (6.2.3) allows us to assume $r = 0$ in (6.2.2). The condition on ψ_m and the level structure ensures that the p -component π_p of π is forced to be a principal series, and has only one-dimensional fixed vector under the action of the group $U_1(p^m) = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^m \mathbb{Z}_p & 1+p^m \mathbb{Z}_p \end{pmatrix}$. So U_p acts on $V(\pi)$ in the same way as U_p acts on this one-dimensional fixed vector, by multiplication of some $a_p(\pi) \in E$. The norm bound on $v(a_p(\pi))$ follows from the admissibility at p of the Galois representation attached to π . \square

6.4. A pairing on the space of automorphic forms. Similar to the Petersson inner product for modular forms, the space of automorphic forms over a definite quaternion algebra also admits an inner product structure.

Recall that weight 2 automorphic forms are simply functions on D_f^\times . Consider the following pairing:

$$(6.4.1) \quad \langle \cdot, \cdot \rangle : S_2^D(U; \psi_m) \times S_2^D(U; \psi_m^{-1}) \longrightarrow E$$

$$\langle \varphi, \varphi' \rangle = \int_{D^\times \backslash D_f^\times / U} \varphi(g) \varphi'(g) := \sum_{i=0}^{t-1} \varphi(\gamma_i) \varphi'(\gamma_i).$$

Note that the choices of the characters above ensures that the pairing does not depend on the choice of the representatives γ_i 's.

Proposition 6.5. *Keep the notation as above. Then we have*

$$(6.5.1) \quad \langle U_p(\varphi), U_p(\varphi') \rangle = p \langle S_p(\varphi), \varphi' \rangle,$$

where S_p is the action $\varphi \mapsto \varphi(\bullet \begin{pmatrix} p^{-1} & 0 \\ 0 & p^{-1} \end{pmatrix})$ given by shifting the variable by an idèles at p .

Proof. In fact, before giving a proof, we convince ourselves by the numerical evidence described in Example 6.21.

We start the proof by pointing out a basic fact about the pairing (6.4.1). Recall the definition of U_1 from Subsection 3.1. Suppose that $\alpha \in D_f^\times$ is an element for which there exist coset representatives $\alpha_1, \dots, \alpha_s \in D_f^\times$ such that $U_1 \alpha U_1 = \coprod_{j=1}^s U_1 \alpha_j$ and $U \alpha U = \coprod_{j=1}^s U \alpha_j$. Then for $\varphi \in S_2^D(U; \psi_m)$, the expression

$$\varphi|[U \alpha U](g) := \sum_j \varphi(g \alpha_j^{-1}) \in S_2^D(U; \psi_m)$$

is independent of the choices of the representatives α_j 's. It is straightforward to check that the following equality holds (see [DS05, Proposition 5.5.2(2)] for a similar argument)

$$\langle \varphi|[U \alpha U], \varphi' \rangle = \langle \varphi, \varphi'|[U \alpha^* U] \rangle,$$

where $\alpha^* = \det(\alpha) \alpha^{-1}$.

We now compute the left hand side of (6.5.1) as follows:

$$\langle U_p(\varphi), U_p(\varphi') \rangle = \langle \varphi|[U \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} U], \varphi'|[U \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} U] \rangle = \langle \varphi|[U \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} U] |[U \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} U], \varphi' \rangle.$$

We take coset decompositions²⁰

$$U \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} U = \prod_{a=0}^{p-1} U \begin{pmatrix} p & 0 \\ ap^m & 1 \end{pmatrix}, \text{ and } U \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} U = \prod_{b=0}^{p-1} U \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}.$$

It then suffices to understand

$$(6.5.2) \quad \varphi|[U \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} U] |[U \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} U] = \sum_{a,b=0}^{p-1} \varphi(g \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}^{-1} \begin{pmatrix} p & 0 \\ ap^m & 1 \end{pmatrix}^{-1})$$

We observe that

$$\begin{pmatrix} p & 0 \\ ap^m & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 - abp^{m-1} & b \\ -a^2bp^{2m-2} & 1 + abp^{m-1} \end{pmatrix} \begin{pmatrix} p & 0 \\ ap^m & p \end{pmatrix}$$

²⁰We point out a subtlety here: we cannot pick the coset representatives $\begin{pmatrix} p & 0 \\ ap^m & 1 \end{pmatrix}$ for the U_p -action on φ' and take the adjugate to apply on the φ ; this is because that the chosen set of representatives is not a set of representatives for both left and right U -coset decomposition. It is therefore important to first work with the double cosets and then take the left coset decomposition for $U \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} U$.

and hence

$$\begin{aligned}\varphi\left(g\begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}^{-1}\begin{pmatrix} p & 0 \\ ap^m & 1 \end{pmatrix}^{-1}\right) &= \varphi\left(g\begin{pmatrix} p & 0 \\ ap^m & p \end{pmatrix}^{-1}\begin{pmatrix} 1-abp^{m-1} & b \\ -a^2bp^{2m-2} & 1+abp^{m-1} \end{pmatrix}^{-1}\right) \\ &= \psi_m(1+abp^{m-1})^{-1} \cdot \varphi\left(g\begin{pmatrix} p & 0 \\ ap^m & p \end{pmatrix}^{-1}\right).\end{aligned}$$

Since ψ_m has conductor exactly p^m , as we sum up (6.5.2) over b all terms cancel to zero except for those where $a = 0$, the corresponding terms exactly give p copies of $\varphi(g\begin{pmatrix} p^{-1} & 0 \\ 0 & p^{-1} \end{pmatrix})$. Thus we have

$$\langle U_p(\varphi), U_p(\varphi') \rangle = \langle pS_p(\varphi), \varphi' \rangle.$$

This concludes the proof of the Proposition. \square

Remark 6.6. One can also define a pairing analogous to (6.4.1) for the space of automorphic forms of higher weights $k + 1$. But we do not need it in this paper.

Notation 6.7. We identify the space of weight two classical automorphic forms $S_2^D(U; \psi_m)$ with $\oplus_{i=0}^{t-1} E$ by evaluating at $\gamma_0, \dots, \gamma_{t-1}$. We use $\mathfrak{U}_p^{\text{cl}}(\psi_m)$ and $\mathfrak{T}_l^{\text{cl}}(\psi_m)$ to denote the matrices for the Hecke actions of U_p and T_l (for $l \notin \mathcal{S}$) under the standard basis.

Let $\alpha_0(\psi_m) \leq \dots \leq \alpha_{t-1}(\psi_m)$ denote the slopes of the Hodge polygon of $\mathfrak{U}_p^{\text{cl}}(\psi_m)$, in non-decreasing order. For simplicity, we assume that E contains all powers $p^{\alpha_i(\psi_m)}$.

Corollary 6.8. *The numbers $\alpha_i(\psi_m)$ belong to $[0, 1]$. In particular, there exists a basis $e_0(\psi_m), \dots, e_{t-1}(\psi_m)$ of $S_2^D(U; \psi_m, \mathcal{O}) \cong \oplus_{i=0}^{t-1} \mathcal{O}$ such that the matrix of U_p -action is given by a matrix $\mathfrak{U}_p^{\text{cl}, e}(\psi_m)$ whose i th row is divisible by $p^{\alpha_i(\psi_m)}$.*

Proof. It is clear that $\mathfrak{U}_p^{\text{cl}}(\psi_m)$ has entries in the integral ring \mathcal{O} . By Proposition 6.5, we have

$$(6.8.1) \quad \mathfrak{U}_p^{\text{cl}}(\psi_m)^T \cdot \mathfrak{U}_p^{\text{cl}}(\psi_m^{-1}) = pA^T,$$

where $A \in \text{GL}_t(\mathcal{O})$ is the matrix for the action of the central character S_p . Write $\mathfrak{U}_p^{\text{cl}}(\psi_m) = BDC$ for $B, C \in \text{GL}_t(\mathcal{O})$ and D diagonal; so that the valuations of the diagonal entries of D are exactly $\alpha_0(\psi_m), \dots, \alpha_{t-1}(\psi_m)$ by Subsection 4.7(4). We rewrite (6.8.1) as

$$B^T \mathfrak{U}_p^{\text{cl}}(\psi_m^{-1})(A^T)^{-1} C^T = pD^{-1}.$$

By Subsection 4.7(4), this means that the slopes of the Hodge polygon of $\mathfrak{U}_p^{\text{cl}}(\psi_m^{-1})$ are given by $1 - \alpha_i(\psi_m)$; more precisely, $\alpha_i(\psi_m) + \alpha_{t-i-1}(\psi_m^{-1}) = 1$. Since both $\alpha_i(\psi_m)$ and $\alpha_i(\psi_m^{-1})$ are non-negative, they belong to $[0, 1]$. The existence of the basis $e_1(\psi_m), \dots, e_t(\psi_m)$ follows from Subsection 4.7(5). \square

6.9. A variant of the pairing (6.4.1). For a purely technical reason, we need a pairing for certain “deformed” classical automorphic forms of weight 2.

Let w be an indeterminant. Note that we have a character

$$\begin{aligned}\psi_{m,w} : \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^m \mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} &\longrightarrow (\mathcal{O}/p^2\mathcal{O}[w])^\times \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \psi_m(d) \langle d \rangle^w,\end{aligned}$$

where $\langle d \rangle := d\omega^{-1}(d)$ is defined as before and $\langle d \rangle^w = 1 + (\langle d \rangle - 1)w \in \mathcal{O}/p^2\mathcal{O}[w]$.²¹ We point out that the image of $\psi_{m,w}$ in fact lands in

$$(6.9.1) \quad (\mathcal{O}/p^2\mathcal{O})^\times + p\mathcal{O}/p^2\mathcal{O} \cdot w.$$

We think of $\psi_{m,w}$ as certain deformation of the character ψ_m .

We introduce the following deformed version of classical automorphic forms:

$$(6.9.2) \quad S_2^D(U; \psi_{m,w}) := \left\{ \varphi : D_f^\times \rightarrow \mathcal{O}/p^2\mathcal{O}[w] \mid \begin{array}{l} \varphi(\delta gu) = \psi_m(d)\langle d \rangle^w \varphi(g) \text{ for any} \\ \delta \in D^\times, g \in D_f^\times, \text{ and } u \in U \text{ with } u_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array} \right\}.$$

This space carries actions of T_l (for $l \notin \mathcal{S}$) and U_p in the natural way. (Note that we do not have a natural lift of this space over \mathcal{O} , as $\psi_{m,w}$ cannot be extended to a character with value in \mathcal{O} .)

Abstractly, we can identify $S_2^D(U; \psi_{m,w})$ with $S_2^D(U; \psi_m; \mathcal{O}) \otimes_{\mathcal{O}} (\mathcal{O}/p^2\mathcal{O}[w])$ by identifying the evaluations at γ_i 's. Then the elements $e_0(\psi_m), \dots, e_{t-1}(\psi_m)$ in Corollary 6.8 gives rise to a basis of $S_2^D(U; \psi_{m,w})$ over $\mathcal{O}/p^2\mathcal{O}[w]$. Let $\mathfrak{Y}_p^{\text{cl,e}}(\psi_{m,w}) \in \text{M}_t(\mathcal{O}/p^2\mathcal{O}[w])$ denote the matrix for the U_p -action on $S_2^D(U; \psi_{m,w})$ with respect to this basis. Since $\psi_{m,w}$ takes value in (6.9.1), all entries of $\mathfrak{Y}_p^{\text{cl,e}}(\psi_{m,w})$ in fact land in $\mathcal{O}/p^2\mathcal{O} + p\mathcal{O}/p^2\mathcal{O} \cdot w$. It follows that the i th row of $\mathfrak{Y}_p^{\text{cl,e}}(\psi_{m,w})$ is divisible by $p^{\alpha_i(\psi_m)}$ (the constants and the coefficients on w). This is true because all $\alpha_i(\psi_m) \in [0, 1]$; so the extra variable w which always comes together with p does not matter.

The following technical lemma will be important for us later.

Lemma 6.10. *Let $\overline{\mathfrak{Y}}_p^{\text{cl,e}}(\psi_{m,w})$ denote the matrix given by dividing the i th row of $\mathfrak{Y}_p^{\text{cl,e}}(\psi_{m,w})$ by $p^{\alpha_i(\psi_m)}$, viewed as a matrix with entries in $\mathcal{O}/p\mathcal{O}[w]$. Then*

$$\det(\overline{\mathfrak{Y}}_p^{\text{cl,e}}(\psi_{m,w})) \pmod{\varpi} \in \mathbb{F}^\times, \quad \text{i.e. there is no } w\text{-terms.}$$

Proof. Similar to the case of duality for classical automorphic forms, we have a natural pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle_{m,w} : S_2^D(U; \psi_{m,w}) \times S_2^D(U; \psi_{m,w}^{-1}) &\rightarrow \mathcal{O}/p^2\mathcal{O}[w] \\ \langle \varphi, \varphi' \rangle_{m,w} &:= \sum_{i=0}^{t-1} \varphi(\gamma_i) \varphi'(\gamma_i). \end{aligned}$$

Since the proof of Proposition 6.5 is tautological, we have

$$(6.10.1) \quad \langle U_p(\varphi), U_p(\varphi') \rangle_{m,w} = p \langle S_p(\varphi), \varphi' \rangle_{m,w}, \quad \text{in } \mathcal{O}/p^2\mathcal{O}[w].$$

Let $B \in \text{GL}_t(\mathcal{O})$ denote the change of basis matrix from the basis given by evaluation at γ_i 's to the basis $e_0(\psi_m), \dots, e_{t-1}(\psi_m)$. Then (6.10.1) gives

$$B^T (\mathfrak{Y}_p^{\text{cl,e}}(\psi_{m,w}))^T (B^T)^{-1} \mathfrak{Y}_p^{\text{cl}}(\psi_{m,w}^{-1}) = p A_w^T,$$

where $A_w \in \text{GL}_t(\mathcal{O}/p^2\mathcal{O}[w])$ is the matrix for the action of S_p on $S_2^D(U; \psi_{m,w})$. It follows that

$$(6.10.2) \quad (\mathfrak{Y}_p^{\text{cl,e}}(\psi_{m,w}))^T \cdot (B^T)^{-1} \mathfrak{Y}_p^{\text{cl}}(\psi_{m,w}^{-1}) (A_w^T)^{-1} B^T = pI.$$

²¹It is important here to consider torsion coefficients, otherwise, $\psi_{m,w}$ may not be a homomorphism of groups.

We claim that the matrix $M = (\psi_{m,w})^T \cdot (B^T)^{-1} \mathfrak{U}_p^{\text{cl}}(\psi_{m,w}^{-1})(A_w^T)^{-1} B^T$ has i th row divisible by $p^{1-\alpha_i(\psi_m)}$. Indeed, we write $(\mathfrak{U}_p^{\text{cl,e}}(\psi_{m,w}))^T = (\overline{\mathfrak{U}}_p^{\text{cl,e}}(\psi_{m,w}))^T \cdot \text{Diag}\{p^{\alpha_0(\psi_m)}, \dots, p^{\alpha_{t-1}(\psi_m)}\}$; so that

$$(\overline{\mathfrak{U}}_p^{\text{cl,e}}(\psi_{m,w})|_{w=0})^T \cdot \text{Diag}\{p^{\alpha_0(\psi_m)}, \dots, p^{\alpha_{t-1}(\psi_m)}\} \cdot M|_{w=0} = pI.$$

Taking determinants, one finds that $\overline{\mathfrak{U}}_p^{\text{cl,e}}(\psi_{m,w})$ is invertible when $w = 0$. We deduce

$$M|_{w=0} = \text{Diag}\{p^{1-\alpha_0(\psi_m)}, \dots, p^{1-\alpha_{t-1}(\psi_m)}\} \cdot ((\overline{\mathfrak{U}}_p^{\text{cl,e}}(\psi_{m,w})|_{w=0})^T)^{-1}.$$

So the claim holds when $w = 0$. Moreover, the matrix M is a product of matrices with entries in $\mathcal{O}/p^2\mathcal{O} + p\mathcal{O}/p^2\mathcal{O} \cdot w$, so its i th row is divisible by $p^{1-\alpha_i(\psi_m)}$ without evaluating w .

We use \overline{M} to denote the matrix given by dividing the i th row of M by $p^{1-\alpha_i(\psi_m)}$, viewed as a matrix in $M_t(\mathcal{O}/p\mathcal{O}[w])$. It then follows that

$$(\overline{\mathfrak{U}}_p^{\text{cl,e}}(\psi_{m,w}))^T \cdot \overline{M} = I.$$

Taking determinants modulo ϖ shows that $\det(\overline{\mathfrak{U}}_p^{\text{cl,e}}(\psi_{m,w})) \pmod{\varpi}$ is invertible in $\mathbb{F}[w]$ and hence lies in \mathbb{F}^\times . \square

Notation 6.11. Let ψ_m be as in Hypothesis 6.1. We use $\mathcal{W}(x\psi_m; p^{-1})$ to denote the closed disk of radius p^{-1} ($\frac{1}{4}$ in case $p = 2$)²² centered at $x\psi_m$ in the weight space. This disk corresponds to all characters of the form $x\psi_m \langle \cdot \rangle^w$ for $w \in \mathcal{O}_{\mathbb{C}_p}$, in particular, including classical characters $x^k \psi_m \omega^{1-k}$ for $k \geq 1$.

We take A° to be the Tate algebra $\mathcal{O}\langle w \rangle$ and A to be $E\langle w \rangle$. We identify $\text{Max}(A) = \text{Max}(E\langle w \rangle)$ with the disk $\mathcal{W}(x\psi_m; p^{-1})$ so that the universal character $\kappa : \Gamma \rightarrow E\langle w \rangle^\times$ is given by

$$\kappa(a) := a\psi_m(a)\langle a \rangle^w.$$

Here the expression $\langle a \rangle^w$ is understood as $(1 + 2pb)^w = \sum_{n \geq 0} (2pb)^n \binom{w}{n} \in 1 + 2pw\mathbb{Z}_p\langle w \rangle$, if $\langle a \rangle = 1 + 2pb$.

6.12. A variant of the space of overconvergent automorphic forms. For a technical reason, it is more convenient to consider a variant of the space of overconvergent automorphic forms, with coefficients in $\mathcal{B} := A\langle pz \rangle = E\langle w, pz \rangle \subset A \widehat{\otimes} \mathcal{A}$.

Recall that the right action $\|_{\kappa} \gamma$ of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma_0(p^m)$ on $A \widehat{\otimes} \mathcal{A}$ is given by

$$(6.12.1) \quad (h|_{\kappa} \gamma)(z) := \frac{\kappa(cz + d)}{cz + d} h\left(\frac{az + b}{cz + d}\right) = \psi_m(d) \langle d \rangle^w \left(1 + \frac{c}{d}z\right)^w h\left(\frac{az + b}{cz + d}\right) \text{ for } h(z) \in \mathcal{A}.$$

Since $p \nmid d$ and $p^m | c$, the expansion of the exponential $(1 + \frac{c}{d}z)^w$ lands in $\mathcal{O}\langle w, p^{m-1}z \rangle \subset \mathcal{B}$.²³ So (6.12.1) can be applied to an element $h(z) \in \mathcal{B}$ and gives rise to a right action of $\Sigma_0(p^m)$ on \mathcal{B} . Therefore, we can define the space of overconvergent automorphic forms with coefficients in \mathcal{B} (instead of $A \widehat{\otimes} \mathcal{A}$):

$$S_{\mathcal{B}}^{D,\dagger}(U; \kappa) := \left\{ \varphi : D_f^\times \rightarrow \mathcal{B} \mid \varphi(\delta gu) = \varphi(g) \|_{\kappa} u_p, \text{ for any } \delta \in D^\times, g \in D_f^\times, u \in U \right\};$$

²²We apologize for the confusing notation when $p = 2$.

²³This follows from the standard estimate $(1 + x)^w = 1 + \sum_{n \geq 1} \binom{w}{n} x^n \in \mathcal{O}\langle w, p^{-1}x \rangle$ (note that the binomial coefficients are not integral for a free variable w .) We will use this estimate freely later in the paper.

it is a subspace of $S^{D,\dagger}(U; \kappa)$ (with coefficients in $A \widehat{\otimes} \mathcal{A}$). In explicit forms, we have an isomorphism of Banach spaces $S_{\mathcal{B}}^{D,\dagger}(U; \kappa) \xrightarrow{\cong} \bigoplus_{i=0}^{t-1} \mathcal{B}$ given by $\varphi \mapsto (\varphi(\gamma_i))_{i=0, \dots, t-1}$.

Notation 6.13. We use $\mathfrak{U}_p^A(\kappa)$ and $\mathfrak{T}_l^A(\kappa)$ (for $l \notin \mathcal{S}$) to denote the infinite matrices in Proposition 4.4 for the operators \mathfrak{U}_p and \mathfrak{T}_l acting on $\bigoplus_{i=0}^{t-1} A \widehat{\otimes} \mathcal{A}$, with respect to the orthonormal basis $1_0, \dots, 1_{t-1}, z_0, \dots, z_{t-1}, z_0^2, \dots$. Here the subscripts indicate which copy of \mathcal{A} the element comes from. We use $\mathfrak{U}_p^{\mathcal{B}}(\kappa)$ and $\mathfrak{T}_l^{\mathcal{B}}(\kappa)$ (for $l \notin \mathcal{S}$) to denote the infinite matrix for the operators \mathfrak{U}_p and \mathfrak{T}_l acting on $S_{\mathcal{B}}^{D,\dagger}(U; \kappa) = \bigoplus_{i=0}^{t-1} \mathcal{B}$, with respect to the orthonormal basis $1_0, \dots, 1_{t-1}, pz_0, \dots, pz_{t-1}, p^2z_0^2, \dots$. It is clear from the definition that

$$\text{Diag}(p^{-1}; t) \mathfrak{U}_p^A(\kappa) \text{Diag}(p; t) = \mathfrak{U}_p^{\mathcal{B}}(\kappa), \quad \text{and} \quad \text{Diag}(p^{-1}; t) \mathfrak{T}_l^A(\kappa) \text{Diag}(p; t) = \mathfrak{T}_l^{\mathcal{B}}(\kappa).$$

In particular, $\text{Char}(\mathfrak{U}_p^A(\kappa), S^{D,\dagger}(U; \kappa)) = \text{Char}(\mathfrak{U}_p^{\mathcal{B}}(\kappa), S_{\mathcal{B}}^{D,\dagger}(U; \kappa))$. So to understand the U_p -slopes on $S^{D,\dagger}(U; \kappa)$, it suffices to look at the U_p -slopes on $S_{\mathcal{B}}^{D,\dagger}(U; \kappa)$.

The following lemma gives a key congruence relation between the action of a matrix in $\Sigma_0(p^m)$ on the space of overconvergent automorphic forms and on the space of *classical automorphic forms*.

Lemma 6.14. *Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in $\Sigma_0(p^m)$ with $v(a) = 0$ or 1. Then the matrix for $\|_{\kappa} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acting on \mathcal{B} (with respect to the basis $1, pz, p^2z^2, \dots$) belongs to*

$$\begin{pmatrix} \psi_m(d) \langle d \rangle^w & pA^\circ & p^2A^\circ & p^3A^\circ & \cdots \\ p^3A^\circ & \frac{a}{d} \psi_m(d) \langle d \rangle^w + p^2aA^\circ & paA^\circ & p^2aA^\circ & \cdots \\ p^4A^\circ & p^3aA^\circ & (\frac{a}{d})^2 \psi_m(d) \langle d \rangle^w + p^2a^2A^\circ & pa^2A^\circ & \cdots \\ p^5A^\circ & p^4aA^\circ & p^3a^2A^\circ & (\frac{a}{d})^3 \psi_m(d) \langle d \rangle^w + p^2a^3A^\circ & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where the (i, j) -entry of the matrix is

- $(\frac{a}{d})^i \psi_m(d) \langle d \rangle^w + p^2a^iA^\circ$ if $i = j > 0$,
- $p^{i-j+2}a^jA^\circ$ if $i > j$, and
- $p^{j-i}a^iA^\circ$ if $i < j$.

Proof. Note that $(1 + p^{m-1}z)^w \in 1 + p^3zw\mathcal{O}\langle pz, w \rangle$ since $m \geq 4$.²⁴ So Proposition 4.6 implies that (note that $p^m | c, p \nmid d$)

$$H_{\|_{\kappa} \begin{pmatrix} a & b \\ c & d \end{pmatrix}}(p^{-1}x, py) = \frac{d\psi_m(d) \langle d \rangle^w (1 + p^{-1} \frac{c}{d}x)^w}{p^{-1}cx + d - axy - pby} \in \mathcal{O}\langle w, py, axy, p^{i+2}x^i; i \in \mathbb{N} \rangle.$$

Translate this congruence into the language of matrix and note that the dominant coefficients on terms $x^i y^i$ come from the expansion of $\frac{d\psi_m(d) \langle d \rangle^w}{d - axy}$; this proves the Lemma. \square

Lemma 6.14 implies that the actions of U_p and T_l for $l \notin \mathcal{S}$ on $S_{\mathcal{B}}^{D,\dagger}(U; \kappa)$ is “very close” to the actions on the completed direct sum

$$\widehat{\bigoplus_{n \geq 0} S_2^D(U; \psi_m \omega^{-2n}; \omega^n)}.$$

More precisely, we have the following.

²⁴It is important that the zw coefficient has valuation *strictly* bigger than 2. The case $m = 3$ fails exactly at this point. See Remark 6.18.

Proposition 6.15. (1) For $l \notin \mathcal{S}$, we consider the infinite block diagonal matrix

$$\mathfrak{T}_l^{\text{cl},\infty} := \text{Diag} \left\{ \mathfrak{T}_l^{\text{cl}}(\psi_m), l \cdot \mathfrak{T}_l^{\text{cl}}(\psi_m \omega^{-2}), l^2 \cdot \mathfrak{T}_l^{\text{cl}}(\psi_m \omega^{-4}), \dots \right\}.$$

Then the difference $\mathfrak{T}_l^{\mathcal{B}}(\kappa) - \mathfrak{T}_l^{\text{cl},\infty}$ lies in the error space

$$(6.15.1) \quad \mathbf{Err} := \begin{pmatrix} pM_t(A^\circ) & pM_t(A^\circ) & p^2M_t(A^\circ) & p^3M_t(A^\circ) & \cdots \\ p^3M_t(A^\circ) & pM_t(A^\circ) & pM_t(A^\circ) & p^2M_t(A^\circ) & \cdots \\ p^4M_t(A^\circ) & p^3M_t(A^\circ) & pM_t(A^\circ) & pM_t(A^\circ) & \cdots \\ p^5M_t(A^\circ) & p^4M_t(A^\circ) & p^3M_t(A^\circ) & pM_t(A^\circ) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where the (i, j) -block entry of the matrix is

- $pM_t(A^\circ)$ if $i = j$,
- $p^{i-j+2}M_t(A^\circ)$ if $i > j$, and
- $p^{j-i}M_t(A^\circ)$ if $i < j$.

(2) Similarly, we consider the infinite block diagonal matrix

$$\mathfrak{U}_p^{\text{cl},\infty} := \text{Diag} \left(\mathfrak{U}_p^{\text{cl}}(\psi_m), p \cdot \mathfrak{U}_p^{\text{cl}}(\psi_m \omega^{-2}), p^2 \cdot \mathfrak{U}_p^{\text{cl}}(\psi_m \omega^{-4}), \dots \right).$$

Then difference $\mathfrak{U}_p^{\mathcal{B}}(\kappa) - \mathfrak{U}_p^{\text{cl},\infty}$ lies in the p -error space

$$(6.15.2) \quad \mathbf{Err}_p := \begin{pmatrix} pM_t(A^\circ) & pM_t(A^\circ) & p^2M_t(A^\circ) & p^3M_t(A^\circ) & \cdots \\ p^3M_t(A^\circ) & p^2M_t(A^\circ) & p^2M_t(A^\circ) & p^3M_t(A^\circ) & \cdots \\ p^4M_t(A^\circ) & p^4M_t(A^\circ) & p^3M_t(A^\circ) & p^3M_t(A^\circ) & \cdots \\ p^5M_t(A^\circ) & p^5M_t(A^\circ) & p^5M_t(A^\circ) & p^4M_t(A^\circ) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the (i, j) -block entry is

- $p^{i+1}M_t(A^\circ)$ if $i = j$,
- $p^{i+2}M_t(A^\circ)$ if $i > j$, and
- $p^jM_t(A^\circ)$ if $i < j$.

Moreover, the (i, i) -block entry of $\mathfrak{U}_p^{\mathcal{B}}(\kappa)$ is congruent to the matrix $p^i \cdot \mathfrak{U}_p^{\text{cl}}(\psi_{m, w-2i} \omega^{-2i})$ modulo $p^{i+2}M_t(A^\circ)$.

Proof. Note that the global elements δ appearing in the matrix of \mathfrak{U}_p or \mathfrak{T}_l for $l \notin \mathcal{S}$ in Proposition 4.4 are the same for classical or overconvergent automorphic forms for all characters. So to prove (1) and (2), it suffices to estimate the difference between the actions of each relevant δ_p on $S_{\mathcal{B}}^{D,\dagger}(U; \kappa)$ and on the completed direct sum $\widehat{\bigoplus}_{n \geq 0} S_2^D(U; \psi_m \omega^{-2n}; \omega^n)$. (Note that $l^r \cdot \mathfrak{T}_l^{\text{cl}}(\psi_m \omega^{-2r})$ is congruent modulo p to the action of T_l on the space of classical automorphic forms $S_2^D(U; \psi_m \omega^{-2r}; \omega^r)$.)

For $l \notin \mathcal{S}$, Proposition 4.4 implies that, for every $\delta_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ appearing in the expression of $\mathfrak{T}_l^{\mathcal{B}}(\kappa)$, we have $a, d \in \mathbb{Z}_p^\times$, $b, c \in \mathbb{Z}_p$, and $ad - bc = l$; so we have $ad \equiv l \pmod{p^m}$. By Lemma 6.14, $\|\kappa \delta_p$ is, modulo the expression (6.15.1) but with $t = 1$, congruent to the infinite diagonal matrix with diagonal elements

$$\psi_m(d) \langle d \rangle^w, \psi_m(d) \frac{a}{d} \langle d \rangle^w, \psi_m(d) \left(\frac{a}{d} \right)^2 \langle d \rangle^w, \dots$$

which is the same as

$$\psi_m(d), l \frac{\psi_m(d)}{d^2}, l^2 \frac{\psi_m(d)}{d^4}, \dots$$

modulo p ; it is further the same as

$$\psi_m(d), l\psi_m(d)\omega^{-2}(d), l^2\psi_m(d)\omega^{-4}(d), \dots$$

modulo p . This is the same as the contribution of δ_p to the matrix

$$\mathfrak{I}_l^{\text{cl},\infty} = \bigoplus_{r \geq 0} l^r \cdot \mathfrak{I}_l^{\text{cl}}(\psi_m \omega^{-2r}).$$

This concludes the proof of (1).

(2) can be checked similarly: for each $\delta_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ appearing in the expression of $\mathfrak{U}_p^{\mathcal{B}}(\kappa)$, we have $a \in p \cdot \mathbb{Z}_p^\times$, $d \in \mathbb{Z}_p^\times$, $b, c \in \mathbb{Z}_p$, and $ad \equiv p \pmod{p^m}$. Using Lemma 6.14 as well as the congruence $\frac{a}{d} \equiv \frac{p}{d^2} = p\omega^{-2}(d)\langle d \rangle^{-2} \pmod{p^3}$, we conclude (2) in the same way as above. \square

We now proceed to prove Theorem B.

Notation 6.16. Put $q = 1$ if $p = 2$ and $q = \frac{p-1}{2}$ if $p > 2$.

We write the characteristic series of U_p acting on $S_{\mathcal{B}}^{D,\dagger}(U; \kappa)$ as

$$\text{Char}(\mathfrak{U}_p^{\mathcal{B}}(\kappa), S_{\mathcal{B}}^{D,\dagger}(U; \kappa)) = 1 + c_1(w)X + c_2(w)X^2 + \dots \in 1 + \mathcal{O}\langle w \rangle[[X]].$$

Theorem 6.17. *Assume $m \geq 4$ as before. We have the following results regarding the Newton polygon.*

- (1) *For any $w_0 \in \mathcal{W}(x\psi_m; p^{-1})$, the Newton polygon of the power series $1 + c_1(w_0)X + \dots$ lies above the polygon starting at $(0, 0)$ with slopes given by*

$$(6.17.1) \quad \bigcup_{n=0}^{\infty} \bigcup_{r=0}^{q-1} \{ \alpha_0(\psi_m \omega^{-2r}) + qn + r, \dots, \alpha_{t-1}(\psi_m \omega^{-2r}) + qn + r \}.$$

- (2) *For each $n \in \mathbb{N}$, let λ_n denote the sum of n smallest numbers in (6.17.1). Then*

$$c_{kt}(w) \in p^{\lambda_{kt}} \cdot \mathcal{O}\langle w \rangle^\times.$$

- (3) *For any $w_0 \in \mathcal{W}(x\psi_m; p^{-1})$, the Newton polygon of the power series $1 + c_1(w_0)X + \dots$ passes through the point (kt, λ_{kt}) (which lies on the Hodge polygon in (1)). In particular, the n th slope of this Newton polygon belongs to $[\lfloor \frac{n}{t} \rfloor, \lfloor \frac{n}{t} \rfloor + 1]$.*

Proof. (1) Recall from Proposition 6.15(2), the matrix for U_p satisfies

$$\mathfrak{U}_p^{\mathcal{B}}(\kappa) - \mathfrak{U}_p^{\text{cl},\infty} \in \mathbf{Err}_p.$$

We now change the basis to

$$e_0(\psi_m), \dots, e_{t-1}(\psi_m), pe_0(\psi_m \omega^{-2})z, \dots, pe_{t-1}(\psi_m \omega^{-2})z, p^2e_0(\psi_m \omega^{-4})z^2, \dots$$

As a result, the action of U_p is given by a new matrix $\mathfrak{U}_p^{\mathcal{B},e}$ which is congruent to

$$\text{Diag}(\mathfrak{U}_p^{\text{cl},e}(\psi_m), p \cdot \mathfrak{U}_p^{\text{cl},e}(\psi_m \omega^{-2}), p^2 \cdot \mathfrak{U}_p^{\text{cl},e}(\psi_m \omega^{-4}), \dots)$$

modulo (6.15.2). In particular, for $i = 0, \dots, t-1$, the $((qn+r)t+i)$ th row of $\mathfrak{U}_p^{\mathcal{B},e}$ is entirely divisible by $p^{\alpha_i(\psi_m \omega^{-2r}) + qn+r}$. Therefore the Hodge polygon of $\mathfrak{U}_p^{\mathcal{B},e}$ lies above the Hodge polygon with slopes given by (6.17.1); this improves the result of Theorem 4.8 (when $m \geq 4$).

(2) By the proof of (1), we know that $c_{kt}(w) \in p^{\lambda_{kt}} \cdot \mathcal{O}\langle w \rangle$. It suffices to show that the reduction of $p^{-\lambda_{kt}}c_{kt}(w)$ modulo ϖ lies in $\mathbb{F}^\times \subset \mathbb{F}[w]$.

Note that, if we think of $\mathfrak{U}_p^{\mathcal{B},e}$ as an infinite block matrix with $t \times t$ -matrices as entries, its (i, j) -block for $i > j$ is entirely divisible by p^{i+2} ; so it will not contribute to the reduction of $p^{-\lambda_{kt}} c_{kt}(w)$ modulo ϖ . In other words, if M_n denotes the $t \times t$ -matrix appearing as the (n, n) -block entry of $\mathfrak{U}_p^{\mathcal{B},e}$, then

$$p^{-\lambda_{kt}} c_{kt}(w) \equiv p^{-\lambda_{kt}} \prod_{n=0}^{k-1} \det(M_n) \pmod{\varpi}.$$

Using the congruence relation discussed in (1) and Proposition 6.15(2), we see that the diagonal $t \times t$ -matrices are exactly given by

$$\mathfrak{U}_p^{\text{cl},e}(\psi_{m,w}) \text{ modulo } p^2, \quad p \cdot (\mathfrak{U}_p^{\text{cl},e}(\psi_{m,w-2}\omega^{-2}) \text{ modulo } p^2), \quad p^2 \cdot (\mathfrak{U}_p^{\text{cl},e}(\psi_{m,w-4}\omega^{-4}) \text{ modulo } p^2), \dots$$

Consequently, the reduction of $p^{-\lambda_{kt}} c_{kt}(w)$ modulo ϖ is the same as the product

$$\prod_{n=0}^{k-1} \det(\overline{\mathfrak{U}}_p^{\text{cl},e}(\psi_{m,w-2n}\omega^{-2n})) \pmod{\varpi}.$$

By Lemma 6.10, each factor lives in \mathbb{F}^\times and so is the product. (2) follows from this.

(3) Since (2) implies that the Newton polygon agrees with the Hodge polygon at points (kt, λ_{kt}) for all $k \geq 0$, the Newton polygon of the power series $1 + c_1(w_0)X + \dots$ is confined between the Hodge polygon of (1) and the polygon with vertices (kt, λ_{kt}) . (3) is immediate from this. \square

Theorem B is a corollary of Theorem 6.17 using the Jacquet-Langlands correspondence (3.7.1).

Remark 6.18. Assume $p > 2$. When $m = 3$, Theorem 6.17(1) still holds. But the argument in (2) fails in that, for example, there might be $p^2 w$ terms in $(1, 0)$ -block entry for the matrix $\mathfrak{U}_p^{\text{cl},e}$; a priori, they may have nontrivial contribution to the reduction of $p^{-\lambda_{kt}} a_{kt}(w)$ modulo ϖ . So we can only conclude that the reduction is a unit in $\mathbb{F}[[w]]$ but not necessarily a unit in $\mathbb{F}\langle w \rangle$. The slope estimate would then only work over some *open* disk of radius p^{-1} . Nonetheless, we still expect our theorem continue to hold as long as $m \geq 2$. It would be interesting to know how to extend our argument to the case $m = 2, 3$.

Corollary 6.19. *Assume $m \geq 4$ as before. Let $\text{HP}(\psi_m)$ (resp. $\text{NP}(\psi_m)$) denote the Hodge polygon (resp. Newton polygon) of the U_p -action on $S_2^D(U; \psi_m)$; we write $\text{HP}(\psi_m)(i)$ (resp. $\text{NP}(\psi_m)(i)$) for the y -coordinate of the polygon when the x -coordinate is i .*

Fix $r = 0, \dots, q-1$. Suppose that $(s_0, \text{NP}(\psi_m \omega^{-2r})(s_0))$ is a vertex of the Newton polygon $\text{NP}(\psi_m \omega^{-2r})$ and suppose that

$$(6.19.1) \quad \text{NP}(\psi_m \omega^{-2r})(s) < \text{HP}(\psi_m \omega^{-2r})(s-1) + 1^{25} \text{ for all } s = 1, \dots, s_0$$

Then for any $s = 0, \dots, s_0$, any $n \in \mathbb{Z}_{\geq 0}$, and any $w_0 \in \mathcal{W}(x\psi_m; p^{-1})$, the $(qnt + rt + s)$ th slope of the power series $1 + c_1(w_0)X + \dots$ is the s th U_p -slope on $S_2^D(U; \psi_m \omega^{-2r})$ plus $qn + r$.

Proof. As in the proof of Theorem 6.17(2), $c_{qnt+rt+s}(w_0)$ is divisible by $p^{\lambda_{qnt+rt+s}}$. The approximation in the proof of Theorem 6.17(1) also implies that, modulo $p^{\lambda_{qnt+rt+s-1}} \cdot p$, this

²⁵Note that the Newton polygon is evaluated at s and the Hodge polygon is evaluated at $s-1$.

number is equal to

$$\left(\prod_{a=0}^{nq+r-1} p^a \det \mathfrak{U}_p^{\text{cl,e}}(\psi_m \omega^{-2a}) \right) \cdot p^{(nq+r)s} \cdot (\text{coefficient of } X^s \text{ in } \text{Char}(U_p; S_2^D(\psi_m \omega^{-2r}))).$$

Under the hypothesis of the corollary, this implies that, for each s ,

- either $v(c_{qnt+rt+s}(w_0)) \geq \lambda_{qnt+rt+s-1} + 1$, in which case the Newton polygon for the U_p -action on $S_B^{D,\dagger}(U; \kappa)$ does not have a vertex at $qnt + rt + s$, or
- the valuation of $c_{qnt+rt+s}(w_0)$ is determined by the classical forms, i.e.

$$v(c_{qnt+rt+s}(w_0)) = v(c_{qnt+rt}(w_0)) + \text{NP}(\psi_m \omega^{-2r})(s) + (qn + r)s.$$

Since $(s_0, \text{NP}(\psi_m \omega^{-2r})(s_0))$ is a vertex, the $(qnt + rt + s)$ th slope, for $s = 0, \dots, s_0$, of the power series $1 + c_1(w_0)X + \dots$ agrees with the s th slope of $\text{Char}(U_p; S_2^D(\psi_m \omega^{-2r}))$ plus the normalizing factor $qn + r$. \square

Remark 6.20. We emphasize that the sequence given by s th U_p -slope on $S_2^D(U; \psi_m \omega^{-2r})$ plus $qn + r$, as n increases, is an arithmetic progression with common difference q (*but not* 1). This is due to the periodic appearance of the powers of the Teichmüller character. This agrees with the computation of Kilford and McMurdy [Kil08, KM12] in some special cases (with $m = 2$), where the common difference is 2 when $p = 5$, and is $\frac{3}{2}$ (which can be further broken up into two arithmetic progressions with common difference 3) when $p = 7$.

Example 6.21. We provide an example to better understand the strength of (6.19.1). Consider the explicit example in Section 5 with $D = \mathbb{Q}\langle \mathbf{i}, \mathbf{j} \rangle$ and $p = 3$. We first consider the $m = 3$ case where we take U to be

$$(6.21.1) \quad U = D^\times(\mathbb{Z}_2) \times \prod_{l \neq 2,3} \text{GL}_2(\mathbb{Z}_l) \times \begin{pmatrix} \mathbb{Z}_3^\times & \mathbb{Z}_3 \\ 27\mathbb{Z}_3 & 1 + 3\mathbb{Z}_3 \end{pmatrix}$$

and ψ_3 to be a character of \mathbb{Z}_3^\times of conductor 27. Then $S_2^D(U; \psi_3)$ is 3-dimensional, and the action of U_3 on the a basis is given by

$$(6.21.2) \quad \begin{pmatrix} \zeta_9 & \zeta_9^2 & \zeta_9^8 \\ \zeta_9^4 & \zeta_9^2 & \zeta_9^5 \\ \zeta_9^7 & \zeta_9^2 & \zeta_9^2 \end{pmatrix}.$$

Its Newton polygon has slopes $\frac{1}{6}, \frac{1}{2}$, and $\frac{5}{6}$ and the Hodge polygon has slopes 0, $\frac{1}{2}$, and 1.

For the case $m = 4$, we take U to be as in (6.21.1) except the number 27 is replaced by 81. We take the character ψ_4 to have conductor 81. Then $S_2^D(U; \psi_4)$ is 9-dimensional, and the action of U_3 on a basis is given by

$$(6.21.3) \quad \begin{pmatrix} \zeta^{19} & 0 & 0 & 0 & \zeta^2 & \zeta^{17} & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta^{13} & 0 & 0 & 0 & \zeta^{20} & \zeta^{23} \\ 0 & \zeta^{11} & \zeta^2 & 0 & 0 & 0 & \zeta^7 & 0 & 0 \\ \zeta & 0 & 0 & 0 & \zeta^2 & \zeta^8 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta^{22} & 0 & 0 & 0 & \zeta^{20} & \zeta^{14} \\ 0 & \zeta^{11} & \zeta^{20} & 0 & 0 & 0 & \zeta^{16} & 0 & 0 \\ \zeta^{10} & 0 & 0 & 0 & \zeta^2 & \zeta^{26} & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta^4 & 0 & 0 & 0 & \zeta^{20} & \zeta^5 \\ 0 & \zeta^{11} & \zeta^{11} & 0 & 0 & 0 & \zeta^{25} & 0 & 0 \end{pmatrix},$$

where ζ is a primitive 27th root of unity. The Newton polygon of this matrix has slopes $\frac{1}{18}, \frac{1}{6}, \frac{5}{18}, \dots, \frac{17}{18}$, and the Hodge polygon has slopes $0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, 1$. In this case, the number s_0 in Corollary 6.19 can be taken to be 6; so we can determine about “two thirds” of all slopes using Corollary 6.19.

One can verify Proposition 6.5 by checking $\overline{M}^T M = 3I$ for the matrix M in (6.21.2) or (6.21.3), here the overline means to take the complex conjugation.

We now return to the general case.

Theorem 6.22. *Assume $m \geq 4$ as before. Let $\text{ord}(\psi_m \omega^{-2n})$ denote the dimension of the ordinary part of $S_2^D(U; \psi_m \omega^{-2n})$, or equivalently, the multiplicity of slope 0 in $\text{NP}(\psi_m \omega^{-2n})$.*

Then the spectral variety $\text{Spc}_D \times_{\mathcal{W}} \mathcal{W}(x\psi_m; p^{-1})$ is a disjoint union of subvarieties

$$X_0, X_{(0,1]}, X_{(1,2]}, X_{(2,3]}, \dots$$

such that each subvariety is finite and flat over $\mathcal{W}(x\psi_m; p^{-1})$, and for any closed point $x \in X_{(n,n+1]}$ (resp. $x \in X_0$), we have $v(a_p(x)) \in (n, n+1]$ (resp. $v(a_p(x)) = 0$). Moreover, the degree of $X_{(n,n+1]}$ over $\mathcal{W}(x\psi_m; p^{-1})$ is exactly

$$t + \text{ord}(\psi_m \omega^{-2n-2}) - \text{ord}(\psi_m \omega^{-2n}).$$

In particular, this number depends only on $n \bmod q$.

Proof. It suffices to show that, for a fixed $n \in \mathbb{Z}_{\geq 0}$ and any $w_0 \in \mathcal{W}(x\psi_m; p^{-1})$, the number of slopes of $1 + c_1(w_0)X + \dots$ less than or equal to n , is *independent of w_0* and is equal to $nt + \text{ord}(\psi_m \omega^{-2n})$. If so, the subspace

$$X_{[0,n]} = \{(x, w_0) \in \text{Spc}_D \times_{\mathcal{W}} \mathcal{W}(x\psi_m; p^{-1}) \mid v(a_p(x)) \leq n\}$$

is finite and flat of degree $nt + \text{ord}(\psi_m \omega^{-2n})$ over $\mathcal{W}(x\psi_m; p^{-1})$; and it follows that $X_{[0,n]}$ is both open (by definition) and closed (by finiteness) in $\text{Spc}_D \times_{\mathcal{W}} \mathcal{W}(x\psi_m; p^{-1})$, and hence a union of connected components. The theorem then follows.

To estimate the number of slopes less than or equal to n , we use the Hodge polygon lower bound in Theorem 6.17. It then suffices to prove that

$$(6.22.1) \quad v(c_{nt+\text{ord}(\psi_m \omega^{-2n})}(w_0)) = \lambda_{nt+n} \cdot \text{ord}(\psi_m \omega^{-2n}), \text{ and } v(c_{nt+s}(w_0)) > \lambda_{nt+ns} \text{ for } s > \text{ord}(\psi_m \omega^{-2n}).$$

We again go back to the slope estimate in the proof of Theorem 6.17 (like in the proof of Corollary 6.19); it is easy to deduce that $c_{nt+s}(w_0)$ for $s \geq \text{ord}(\psi_m \omega^{-2n})$ is congruent to

$$\left(\prod_{i=0}^{n-1} \det \mathfrak{L}_p^{\text{cl,e}}(\psi_m \omega^{-2i}) \right) \cdot p^{ns} \cdot \left(\text{coefficient of } X^s \text{ in } \text{Char}(U_p; S_2^D(\psi_m \omega^{-2n})) \right)$$

modulo $p^{\lambda_{nt+ns+1}}$. The valuation inequalities (6.22.1) follow from this congruence relation. \square

Remark 6.23. We certainly expect that $X_{(i,i+1]}$ is the disjoint union of $X_{(i,i+1]} \coprod X_{i+1}$ (with the obvious meaning); but we do not know how to prove this because, apriori, the error terms from w might present an obstruction.

Remark 6.24. Using Corollary 6.19 and the argument above, we can show that, when there is a vertex $(s_0, \text{NP}(\psi_m \omega^{-2r})(s_0))$ of the Newton polygon $\text{NP}(\psi_m \omega^{-2r})$ as in Corollary 6.19,

we can get a further decomposition of $X_{(qn+r, qn+r+1]}$ separating those points whose a_p -slopes are first s_0 U_p -slopes on $S_2^D(U; \psi_m \omega^{-2r})$ plus $qn + r$.

7. TECHNIQUES FOR SEPARATION BY RESIDUAL PSEUDO-REPRESENTATIONS

We motivate this section by pointing out that the power of Corollary 6.19 is largely determined by how close the Newton polygon is to the Hodge polygon. The application of this result is largely limited as the level subgroup U gets smaller. A natural idea to loosen the condition (6.19.1) is to separate the space of automorphic forms using the *tame* Hecke algebras.

In fact, we will show that one can obtain a natural direct sum decomposition of the space of overconvergent automorphic forms according to the residual Galois pseudo-representations attached. Furthermore, we can reproduce main theorems of the previous section for each direct summand. We also emphasize that this decomposition should have its own interest.

We keep the notation as in the previous section. In particular, we assume Hypothesis 6.1: $m \geq 4$.

7.1. Pseudo-representations. Let $G_{\mathbb{Q}, \mathcal{S}}$ denote the Galois group of the maximal extension of \mathbb{Q} unramified outside \mathcal{S} (see Subsection 3.1 for \mathcal{S}). Let R be a (topological) ring. A (2-dimensional) *pseudo-representation* is a (continuous) map $\rho : G_{\mathbb{Q}, \mathcal{S}} \rightarrow R$ such that, for $g_i \in G_{\mathbb{Q}, \mathcal{S}}$, we have $\rho(1) = 2$, $\rho(g_1 g_2) = \rho(g_2 g_1)$, and

$$\rho(g_1)\rho(g_2)\rho(g_3) + \rho(g_1 g_2 g_3) + \rho(g_1 g_3 g_2) = \rho(g_1)\rho(g_2 g_3) + \rho(g_2)\rho(g_1 g_3) + \rho(g_3)\rho(g_1 g_2).$$

Let $\rho : G_{\mathbb{Q}, \mathcal{S}} \rightarrow \mathcal{O}$ be a pseudo-representation.

- If $\chi : G_{\mathbb{Q}, \mathcal{S}} \rightarrow \mathcal{O}^\times$ is a continuous character, then $(\rho \otimes \chi)(g) := \rho(g)\chi(g)$ is a pseudo-representation.
- We use $\bar{\rho} : G_{\mathbb{Q}, \mathcal{S}} \rightarrow \mathbb{F}$ to denote the reduction $\bar{\rho}(g) := \rho(g) \bmod \varpi$; it is called the *residual pseudo-representation* associated to ρ .
- The (residual) pseudo-representation is uniquely determined by its evaluation on the geometric Frobenius: $\rho(\text{Frob}_l)$ for $l \notin \mathcal{S}$.

It is known that to each automorphic representation π appearing in $S_2^D(U; \psi_m)$, there exists a pseudo-representation $\rho_\pi : G_{\mathbb{Q}, \mathcal{S}} \rightarrow \mathcal{O}$ such that $\rho(\text{Frob}_l) = a_l(\pi)$ for all $l \notin \mathcal{S}$. We say that a residual pseudo-representation $\bar{\rho} : G_{\mathbb{Q}, \mathcal{S}} \rightarrow \mathbb{F}$ *appears* in a space of automorphic forms $S_2^D(U; \psi_m)$ if there is an automorphic representation π appearing in $S_2^D(U; \psi_m)$ such that the reduction of the associated pseudo-representation is $\bar{\rho}$.

The goal of this section is to decompose the space $S_{\mathcal{B}}^{D, \dagger}(U; \kappa)$ according to the residual pseudo-representations appearing in the space of *weight two* classical automorphic forms.²⁶ The key is to use the tame Hecke action to break up the space $S_{\mathcal{B}}^{D, \dagger}(U; \kappa)$. We start with the decomposition over the space of classical automorphic forms.

Notation 7.2. We use $\mathcal{B}(U; \psi_m)$ to denote all residual pseudo-representations $\bar{\rho}$ that appear in $S^{\text{cl}} := \bigoplus_{r=0}^{q-1} S_2^D(U; \psi_m \omega^{-2r}; \omega^r)$. For each pair of distinct residual pseudo-representations $\bar{\rho}, \bar{\rho}' \in \mathcal{B}(U; \psi)$, we pick a prime $l_{\bar{\rho}, \bar{\rho}'} \notin \mathcal{S}$ such that $\bar{\rho}(\text{Frob}_{l_{\bar{\rho}, \bar{\rho}'}}) \neq \bar{\rho}'(\text{Frob}_{l_{\bar{\rho}, \bar{\rho}'}})$. We fix a lift $\tilde{a}_{l_{\bar{\rho}, \bar{\rho}'}}(\bar{\rho}) \in \mathcal{O}$ of $\bar{\rho}(\text{Frob}_{l_{\bar{\rho}, \bar{\rho}'}})$ and a lift $\tilde{a}_{l_{\bar{\rho}, \bar{\rho}'}}(\bar{\rho}') \in \mathcal{O}$ of $\bar{\rho}'(\text{Frob}_{l_{\bar{\rho}, \bar{\rho}'}})$.

²⁶It should not be too surprise to see that we only need weight two modular forms, as it was already observed by Serre [Se96] that all modular residual pseudo-representations appears in weight two.

For $\bar{\rho} \in \mathcal{B}(U; \psi_m)$, consider the following tame Hecke operator

$$P_{\bar{\rho}} := \prod_{\bar{\rho}' \neq \bar{\rho}} (T_{l_{\bar{\rho}, \bar{\rho}'}} - \tilde{a}_{l_{\bar{\rho}, \bar{\rho}'}}(\bar{\rho}')) / (\tilde{a}_{l_{\bar{\rho}, \bar{\rho}'}}(\bar{\rho}) - \tilde{a}_{l_{\bar{\rho}, \bar{\rho}'}}(\bar{\rho}')).$$

Note that $P_{\bar{\rho}}$ defines an endomorphism of the integral model $S_2^D(U; \psi_m \omega^{-2r}; \omega^r; \mathcal{O})$ for each r . The operator $P_{\bar{\rho}}$ depends on the choice of the lifts $\tilde{a}_{l_{\bar{\rho}, \bar{\rho}'}}(\bar{\rho})$ and $\tilde{a}_{l_{\bar{\rho}, \bar{\rho}'}}(\bar{\rho}')$'s.

Lemma 7.3. *Fix $r \in \{0, \dots, q-1\}$. Let $P_{\bar{\rho}, r}$ denote the action of $P_{\bar{\rho}}$ on the space of classical automorphic forms $S_2^D(U; \psi_m \omega^{-2r}; \omega^r; \mathcal{O})$. Then $P_{\bar{\rho}, r}^2 \equiv P_{\bar{\rho}, r} \pmod{\varpi}$. The limit*

$$\tilde{P}_{\bar{\rho}, r}^{\text{cl}} := \lim_{n \rightarrow \infty} (P_{\bar{\rho}, r})^{p^n}$$

exists and it is the projection to the direct sum $V(\bar{\rho})_r$ of subspaces $V(\pi)$ over all automorphic representations π appearing in $S_2^D(U; \psi_m \omega^{-2r}; \omega^r)$ for which the associated pseudo-representation reduces to $\bar{\rho}$. In particular, we have

$$(\tilde{P}_{\bar{\rho}, r})^2 = \tilde{P}_{\bar{\rho}, r}, \quad \tilde{P}_{\bar{\rho}, r} \tilde{P}_{\bar{\rho}', r} = 0 \text{ if } \bar{\rho} \neq \bar{\rho}', \quad \text{and} \quad \sum_{\bar{\rho} \in \mathcal{B}(U; \psi_m)} \tilde{P}_{\bar{\rho}, r} = \text{id}.$$

Moreover, the definition of $\tilde{P}_{\bar{\rho}, r}$ is independent of the choice of the lifts $\tilde{a}_{l_{\bar{\rho}, \bar{\rho}'}}(\bar{\rho})$ and $\tilde{a}_{l_{\bar{\rho}, \bar{\rho}'}}(\bar{\rho}')$'s; and it defines a direct sum decomposition of the integral model

$$S_2^D(U; \psi_m \omega^{-2r}; \omega^r; \mathcal{O}) \cong \bigoplus_{\bar{\rho} \in \mathcal{B}(U; \psi_m)} V(\bar{\rho}; \mathcal{O})_r.$$

Proof. Note that, $P_{\bar{\rho}, r}$ acts on each $V(\pi)$ by some element in (ϖ) if $\bar{\rho}_\pi \neq \bar{\rho}$, and by some 1-unit if $\bar{\rho}_\pi = \bar{\rho}$. The Lemma follows from this immediately. \square

The upshot is that one can extend the decomposition above to the case of overconvergent automorphic forms.

7.4. Some infinite matrices. For each r , we identify $S_2^D(U; \psi_m \omega^{-2r}; \omega^r)$ with $\bigoplus_{i=0}^{t-1} E$ by evaluating the automorphic forms at $\gamma_0, \gamma_1, \dots, \gamma_{t-1}$. This way, the operators $P_{\bar{\rho}, r}$ and $\tilde{P}_{\bar{\rho}, r}$ are represented by two $t \times t$ -matrices $\mathfrak{P}_{\bar{\rho}, r}^{\text{cl}}, \tilde{\mathfrak{P}}_{\bar{\rho}, r}^{\text{cl}} \in M_t(\mathcal{O})$.

We use $\mathfrak{P}_{\bar{\rho}}^{\text{cl}, \infty}$ (resp. $\tilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl}, \infty}$) to denote the infinite block diagonal matrix whose diagonal block-entries are $\mathfrak{P}_{\bar{\rho}, 0}^{\text{cl}}, \mathfrak{P}_{\bar{\rho}, 1}^{\text{cl}}, \dots$ (resp. $\tilde{\mathfrak{P}}_{\bar{\rho}, 0}^{\text{cl}}, \tilde{\mathfrak{P}}_{\bar{\rho}, 1}^{\text{cl}}, \dots$).

Note that $P_{\bar{\rho}}$ only involves Hecke operators; so it also acts on the space of overconvergent automorphic forms $S_B^{D, \dagger}(U; \kappa)$. Let $\mathfrak{P}_{\bar{\rho}}^{\mathcal{B}}(\kappa)$ denote the matrix for $P_{\bar{\rho}}$ under the basis given by $1_0, \dots, 1_{t-1}, pz_0, \dots, pz_{t-1}, p^2 z_0^2, \dots$ as in Notation 6.13.

By Proposition 6.15(1), we have that

$$(7.4.1) \quad \mathfrak{P}_{\bar{\rho}}^{\mathcal{B}}(\kappa) \equiv \mathfrak{P}_{\bar{\rho}}^{\text{cl}, \infty} \text{ modulo the error space } \mathbf{Err} \text{ in (6.15.1).}$$

The next Proposition says that we can improve the infinite matrix $\mathfrak{P}_{\bar{\rho}}^{\mathcal{B}}(\kappa)$ into a projection, as we did above; so that we factor out the subspace of overconvergent automorphic forms corresponding to the Galois pseudo-representation $\bar{\rho}$.

Proposition 7.5. *Keep the notation as above.*

- (1) *We have $(\tilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl}, \infty})^2 = \tilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl}, \infty}$, $\tilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl}, \infty} \tilde{\mathfrak{P}}_{\bar{\rho}'}^{\text{cl}, \infty} = 0$ if $\bar{\rho} \neq \bar{\rho}'$, and $\sum_{\bar{\rho} \in \mathcal{B}(U; \psi_m)} \tilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl}, \infty} = I_\infty$, where $I_\infty := \text{Diag}(1)$ denotes the infinite identity matrix.*

(2) *The limit*

$$\tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) := \lim_{n \rightarrow \infty} (\mathfrak{P}_{\bar{\rho}}^{\mathcal{B}}(\kappa))^{p^n}$$

exists. Moreover, we have

$$(\tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa))^2 = \tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa), \quad \tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa)\tilde{\mathfrak{P}}_{\bar{\rho}'}^{\mathcal{B}}(\kappa) = 0 \text{ for } \bar{\rho} \neq \bar{\rho}', \text{ and } \sum_{\bar{\rho} \in \mathcal{B}(U; \psi_m)} \tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) = I_{\infty}.$$

(3) *We have a decomposition of Banach A -modules respecting the U_p -action:*

$$S_{\mathcal{B}}^{D, \dagger}(U; \kappa) = \bigoplus_{\bar{\rho} \in \mathcal{B}(U; \psi_m)} \tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) S_{\mathcal{B}}^{D, \dagger}(U; \kappa).$$

Consequently, we have a product formula for the characteristic series

$$\text{Char}(U_p; S_{\mathcal{B}}^{D, \dagger}(U; \kappa)) = \prod_{\bar{\rho} \in \mathcal{B}(U; \psi_m)} \text{Char}(U_p; \tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) S_{\mathcal{B}}^{D, \dagger}(U; \kappa)).$$

(4) *We have the following congruence relation: for every $\bar{\rho} \in \mathcal{B}(U; \psi_m)$, the difference of the infinite matrices $\tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) - \tilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl}, \infty}$ belongs to the space \mathbf{Err} in (6.15.1).*

Proof. (1) follows from the corresponding properties of $\tilde{\mathfrak{P}}_{\bar{\rho}, r}^{\text{cl}}$ in Lemma 7.3.

For (2), we observe that $\mathfrak{P}_{\bar{\rho}}^{\mathcal{B}}(\kappa) \equiv \mathfrak{P}_{\bar{\rho}}^{\text{cl}, \infty} \pmod{\varpi}$ by Proposition 6.15(1). So by Lemma 7.3,

$$(7.5.1) \quad (\mathfrak{P}_{\bar{\rho}}^{\mathcal{B}}(\kappa))^2 \equiv \mathfrak{P}_{\bar{\rho}}^{\mathcal{B}}(\kappa) \pmod{\varpi}.$$

Easy induction proves that $(\mathfrak{P}_{\bar{\rho}}^{\mathcal{B}}(\kappa))^{p^{i+1}} \equiv (\mathfrak{P}_{\bar{\rho}}^{\mathcal{B}}(\kappa))^{p^i} \pmod{\varpi^i}$; so the limit $\tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) := \lim_{i \rightarrow \infty} (\mathfrak{P}_{\bar{\rho}}^{\mathcal{B}}(\kappa))^{p^i}$ exists. The property $(\tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa))^2 = \tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa)$ also follows from (7.5.1).

Now for two pseudo-representations $\bar{\rho} \neq \bar{\rho}'$ in $\mathcal{B}(U; \psi_m)$, we have

$$\mathfrak{P}_{\bar{\rho}}^{\mathcal{B}}(\kappa)\mathfrak{P}_{\bar{\rho}'}^{\mathcal{B}}(\kappa) \equiv \mathfrak{P}_{\bar{\rho}}^{\text{cl}, \infty}\mathfrak{P}_{\bar{\rho}'}^{\text{cl}, \infty} \equiv 0 \pmod{\varpi}.$$

It then follows that $\tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa)\tilde{\mathfrak{P}}_{\bar{\rho}'}^{\mathcal{B}}(\kappa) = 0$ (note that it is important to know that $\mathfrak{P}_{\bar{\rho}}^{\mathcal{B}}(\kappa)$ commutes with $\mathfrak{P}_{\bar{\rho}'}^{\mathcal{B}}(\kappa)$ because both operators can be expressed in terms of Hecke operators.)

Similarly, we start with

$$\sum_{\bar{\rho} \in \mathcal{B}(U; \psi_m)} \mathfrak{P}_{\bar{\rho}}^{\mathcal{B}}(\kappa) \equiv \sum_{\bar{\rho} \in \mathcal{B}(U; \psi_m)} \mathfrak{P}_{\bar{\rho}}^{\text{cl}, \infty} \equiv I_{\infty} \pmod{\varpi}$$

Raising it to p^i th power implies that

$$I_{\infty} \equiv \sum_{\bar{\rho} \in \mathcal{B}(U; \psi_m)} (\mathfrak{P}_{\bar{\rho}}^{\mathcal{B}}(\kappa))^{p^i} \pmod{\varpi^i}.$$

Here we used the fact that $\mathfrak{P}_{\bar{\rho}}^{\mathcal{B}}(\kappa)\mathfrak{P}_{\bar{\rho}'}^{\mathcal{B}}(\kappa) \equiv 0 \pmod{\varpi}$ for $\bar{\rho} \neq \bar{\rho}'$ and once again the crucial commutativity of $\mathfrak{P}_{\bar{\rho}}^{\mathcal{B}}(\kappa)$'s. Taking limit shows that $\sum_{\bar{\rho} \in \mathcal{B}(U; \psi_m)} \tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) = I_{\infty}$.

(3) follows from (2) and the fact that U_p commutes with each $\tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa)$, as this operator is a limit of polynomials in tame Hecke operators.

We now check (4). First recall some basic properties of the error space \mathbf{Err} defined in (6.15.1). For $M_1, M_2 \in \mathbf{Err}$, it is easy to see that $M_1 M_2 \in \mathbf{Err}$ and $\mathfrak{P}_{\bar{\rho}}^{\text{cl}, \infty} M_1, M_1 \mathfrak{P}_{\bar{\rho}}^{\text{cl}, \infty} \in \mathbf{Err}$. Thus

$$(7.5.2) \quad (\mathfrak{P}_{\bar{\rho}}^{\mathcal{B}}(\kappa))^{p^n} - (\mathfrak{P}_{\bar{\rho}}^{\text{cl}, \infty})^{p^n} = (\mathfrak{P}_{\bar{\rho}}^{\text{cl}, \infty} + (\mathfrak{P}_{\bar{\rho}}^{\mathcal{B}}(\kappa) - \mathfrak{P}_{\bar{\rho}}^{\text{cl}, \infty}))^{p^n} - (\mathfrak{P}_{\bar{\rho}}^{\text{cl}, \infty})^{p^n} \in \mathbf{Err}$$

because $\mathfrak{P}_\rho^{\mathcal{B}}(\kappa) - \mathfrak{P}_\rho^{\text{cl},\infty} \in \mathbf{Err}$ by (7.4.1). Taking limit proves (4). \square

Caution 7.6. It is important to point out that, in (7.5.2), since $\mathfrak{P}_\rho^{\mathcal{B}}(\kappa)$ and $\mathfrak{P}_\rho^{\text{cl},\infty}$ do not commute with each other, we *cannot* use binomial expansion formula to improve the congruence (7.5.2); hence the limit $\widetilde{\mathfrak{P}}_\rho^{\mathcal{B}}(\kappa)$ is *not* a block diagonal matrix. So Proposition 7.5(4) is the best congruence we could hope for.

Remark 7.7. We should point out that decomposing a Banach Hecke module according to pseudo-Galois representations $\bar{\rho}$ is a quite formal process and can be done in a much greater generality. However, it is often difficult to control the factor corresponding to each $\bar{\rho}$. The advantage of our situation is that we can give a good “model” of the factor corresponding to each $\bar{\rho}$.

7.8. $\bar{\rho}$ -part of classical automorphic forms. Recall from Lemma 7.3 that the space of classical automorphic forms $S_2^D(U; \psi_m \omega^{-2r}; \omega^r; \mathcal{O})$ for each r is written as the direct sum $\bigoplus_{\bar{\rho} \in \mathcal{B}(U; \psi_m)} V(\bar{\rho}, \mathcal{O})_r$. We put $V(\bar{\rho})_r = V(\bar{\rho}; \mathcal{O})[\frac{1}{p}]$ and $d_{\bar{\rho},r} := \dim V(\bar{\rho})_r$; the number depends only on $r \bmod q$.

Note that the operator U_p acts on each $V(\bar{\rho}, \mathcal{O})_r$. By Corollary 6.8 (and Subsection 4.7(6)), the Hodge slopes $\alpha_0(\bar{\rho})_r \leq \dots \leq \alpha_{d_{\bar{\rho},r}-1}(\bar{\rho})_r$ of the U_p -action on each $V(\bar{\rho}, \mathcal{O})_r$ belong to $[0, 1]$; so are the Newton slopes. We pick a basis $e_0(\bar{\rho})_r, \dots, e_{d_{\bar{\rho},r}-1}(\bar{\rho})_r$ of $V(\bar{\rho}, \mathcal{O})_r$ such that, the corresponding matrix $\mathfrak{U}_p^{\text{cl},\bar{\rho},r}$ of the U_p -action has i th row divisible by $p^{\alpha_i(\bar{\rho})_r}$.

Providing $S_2^D(U; \psi_m \omega^{-2r}; \omega^r)$ with the natural basis of evaluation at $\gamma_0, \dots, \gamma_{t-1}$, and each $V(\bar{\rho})_r$ with the basis above, we write $\mathfrak{C}_{\bar{\rho},r}$ and $\mathfrak{D}_{\bar{\rho},r}$ for the matrices for the natural inclusion and the natural projection $\widetilde{\mathfrak{P}}_{\bar{\rho},r}^{\text{cl}}$:

$$V(\bar{\rho})_r \xrightarrow{\mathfrak{C}_{\bar{\rho},r}} S_2^D(U; \psi_m \omega^{-2r}; \omega^r) \xrightarrow{\mathfrak{D}_{\bar{\rho},r}} V(\bar{\rho})_r.$$

So $\mathfrak{C}_{\bar{\rho},r}$ is a $t \times d_{\bar{\rho},r}$ -matrix and $\mathfrak{D}_{\bar{\rho},r}$ is a $d_{\bar{\rho},r} \times t$ -matrix such that $\mathfrak{C}_{\bar{\rho},r} \mathfrak{D}_{\bar{\rho},r} = \widetilde{\mathfrak{P}}_{\bar{\rho},r}^{\text{cl}}$ and $\mathfrak{D}_{\bar{\rho},r} \mathfrak{C}_{\bar{\rho},r} = I_{d_{\bar{\rho},r}}$.

7.9. A model for the $\bar{\rho}$ -part of overconvergent automorphic forms. Proposition 7.5 allows us to reduce the study of the U_p -action on $S_{\mathcal{B}}^{D,\dagger}(U; \kappa)$ to the U_p -action on each subspace $\widetilde{\mathfrak{P}}_\rho^{\mathcal{B}}(\kappa) S_{\mathcal{B}}^{D,\dagger}(U; \kappa)$, which we call the $\bar{\rho}$ -part of $S_{\mathcal{B}}^{D,\dagger}(U; \kappa)$. This space is slightly too abstract as pointed out in Remark 7.7; we need to give it a “model”: $V(\bar{\rho})_A^\infty$.

We set

$$S_A^{\text{cl},\infty,\circ} := \widehat{\bigoplus_{r \geq 0}} S_2^D(U; \psi_m \omega^{-2r}; \omega^r; \mathcal{O}) \otimes_{\mathcal{O}} A^\circ, \text{ and } S_A^{\text{cl},\infty} := S_A^{\text{cl},\infty,\circ} \otimes_{\mathcal{O}} E.$$

We define the U_p -action on this space to be $\bigoplus_{r \geq 0} p^r \cdot U_p$. Let $\mathfrak{U}_p^{\text{cl},\infty}$ denote the matrix for this action with respect the standard basis given by evaluation at $\gamma_0, \dots, \gamma_{t-1}$ of each of the summand; this matrix is the infinite block diagonal matrix whose diagonal components are $p^r \cdot \mathfrak{U}_p^{\text{cl}}(\psi_m \omega^{-2r})$.

We put

$$V(\bar{\rho})_A^{\infty,\circ} := \widehat{\bigoplus_{r \geq 0}} V(\bar{\rho}; \mathcal{O})_r \otimes_{\mathcal{O}} A^\circ, \text{ and } V(\bar{\rho})_A^\infty = V(\bar{\rho})_A^{\infty,\circ} \otimes_{\mathcal{O}} E.$$

We define the U_p -action on this space given by $\bigoplus p^r \cdot U_p$; the corresponding matrix with respect to the chosen basis on each $V(\bar{\rho}; \mathcal{O})_r$ is an infinite block diagonal matrix $\mathfrak{U}_p^{\bar{\rho}, \infty}$ whose diagonal components are $p^r \cdot \mathfrak{U}_p^{\text{cl}, \bar{\rho}, r}$.

We write

$$\mathfrak{C}_{\bar{\rho}}^{\infty} := \widehat{\bigoplus}_{r \geq 0} \mathfrak{C}_{\bar{\rho}, r} : V(\bar{\rho})_A^{\infty} \rightarrow S_A^{\text{cl}, \infty} \quad \text{and} \quad \mathfrak{D}_{\bar{\rho}}^{\infty} := \widehat{\bigoplus}_{r \geq 0} \mathfrak{D}_{\bar{\rho}, r} : S_A^{\text{cl}, \infty} \rightarrow V(\bar{\rho})_A^{\infty}$$

for the natural inclusion and projection, respectively. So we have $\mathfrak{D}_{\bar{\rho}}^{\infty} \mathfrak{C}_{\bar{\rho}}^{\infty} = I_{\infty}$, and $\widetilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl}, \infty} = \mathfrak{C}_{\bar{\rho}}^{\infty} \mathfrak{D}_{\bar{\rho}}^{\infty}$ is the infinite block diagonal matrix composed of $\widetilde{\mathfrak{P}}_{\bar{\rho}, r}^{\text{cl}}$.

On the infinite level, we use the letter Φ to denote the following identification

$$(7.9.1) \quad S_B^{D, \dagger}(U; \kappa) = \bigoplus_{i=0}^{t-1} E\langle w, pz \rangle = \bigoplus_{i=0}^{t-1} \widehat{\bigoplus}_{n \geq 0} E\langle w \rangle (pz)^n \cong S_A^{\text{cl}, \infty},$$

where the first and the last equality are given by evaluation at the elements $\gamma_0, \gamma_1, \dots, \gamma_{t-1}$. This isomorphism does not respect the actions of the Hecke operators literally but we will show later that it approximately does.

Proposition 7.10. *The following two natural morphisms are isomorphisms*

$$\begin{aligned} \varphi_{\bar{\rho}} : \widetilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) S_B^{D, \dagger}(U; \kappa) &\subseteq S_B^{D, \dagger}(U; \kappa) \xrightarrow[\cong]{(7.9.1)} S_A^{\text{cl}, \infty} \xrightarrow{\mathfrak{D}_{\bar{\rho}}^{\infty}} V(\bar{\rho})_A^{\infty}; \\ \psi_{\bar{\rho}} : V(\bar{\rho})_A^{\infty} &\xrightarrow{\mathfrak{C}_{\bar{\rho}}^{\infty}} S_A^{\text{cl}, \infty} \xrightarrow[\cong]{(7.9.1)^{-1}} S_B^{D, \dagger}(U; \kappa) \xrightarrow{\widetilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa)} \widetilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) S_B^{D, \dagger}(U; \kappa). \end{aligned}$$

Moreover, $\psi_{\bar{\rho}}^{-1} = (1 + \epsilon) \circ \varphi_{\bar{\rho}}$ for some endomorphism $\epsilon : V(\bar{\rho})_A^{\infty} \rightarrow V(\bar{\rho})_A^{\infty}$ which, under the basis $\{e_j(\bar{\rho})_r \mid j = 0, \dots, d_{\bar{\rho}, r} - 1 \text{ and } r \geq 0\}$, is an infinite matrix in

$$\mathbf{Err}_{\bar{\rho}} := \begin{pmatrix} pM_{d_{\bar{\rho}, 0}}(A^{\circ}) & pM_{d_{\bar{\rho}, 0} \times d_{\bar{\rho}, 1}}(A^{\circ}) & p^2M_{d_{\bar{\rho}, 0} \times d_{\bar{\rho}, 2}}(A^{\circ}) & p^3M_{d_{\bar{\rho}, 0} \times d_{\bar{\rho}, 3}}(A^{\circ}) & \cdots \\ p^3M_{d_{\bar{\rho}, 1} \times d_{\bar{\rho}, 0}}(A^{\circ}) & pM_{d_{\bar{\rho}, 1}}(A^{\circ}) & pM_{d_{\bar{\rho}, 1} \times d_{\bar{\rho}, 2}}(A^{\circ}) & p^2M_{d_{\bar{\rho}, 1} \times d_{\bar{\rho}, 3}}(A^{\circ}) & \cdots \\ p^4M_{d_{\bar{\rho}, 2} \times d_{\bar{\rho}, 0}}(A^{\circ}) & p^3M_{d_{\bar{\rho}, 2} \times d_{\bar{\rho}, 1}}(A^{\circ}) & pM_{d_{\bar{\rho}, 2} \times d_{\bar{\rho}, 2}}(A^{\circ}) & pM_{d_{\bar{\rho}, 2} \times d_{\bar{\rho}, 3}}(A^{\circ}) & \cdots \\ p^5M_{d_{\bar{\rho}, 3} \times d_{\bar{\rho}, 0}}(A^{\circ}) & p^4M_{d_{\bar{\rho}, 3} \times d_{\bar{\rho}, 1}}(A^{\circ}) & p^3M_{d_{\bar{\rho}, 3} \times d_{\bar{\rho}, 2}}(A^{\circ}) & pM_{d_{\bar{\rho}, 3}}(A^{\circ}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the (i, j) -block entry is

- $pM_{d_{\bar{\rho}, i}}(A^{\circ})$ if $i = j$,
- $p^{i-j+2}M_{d_{\bar{\rho}, i} \times d_{\bar{\rho}, j}}(A^{\circ})$ if $i > j$, and
- $p^{j-i}M_{d_{\bar{\rho}, i} \times d_{\bar{\rho}, j}}(A^{\circ})$ if $i < j$.

Proof. We first take the composition

$$(7.10.1) \quad \begin{aligned} \varphi_{\bar{\rho}} \circ \psi_{\bar{\rho}} - I_{\infty} &= \mathfrak{D}_{\bar{\rho}}^{\infty} \widetilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) \mathfrak{C}_{\bar{\rho}}^{\infty} - I_{\infty} \\ &= \mathfrak{D}_{\bar{\rho}}^{\infty} \widetilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl}, \infty} \mathfrak{C}_{\bar{\rho}}^{\infty} - I_{\infty} + \mathfrak{D}_{\bar{\rho}}^{\infty} (\widetilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) - \widetilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl}, \infty}) \mathfrak{C}_{\bar{\rho}}^{\infty}. \end{aligned}$$

Note that $\mathfrak{D}_{\bar{\rho}}^{\infty} \widetilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl}, \infty} \mathfrak{C}_{\bar{\rho}}^{\infty} - I_{\infty} = \mathfrak{D}_{\bar{\rho}}^{\infty} \mathfrak{C}_{\bar{\rho}}^{\infty} \mathfrak{D}_{\bar{\rho}}^{\infty} \mathfrak{C}_{\bar{\rho}}^{\infty} - I_{\infty} = 0$ and

$$\mathfrak{D}_{\bar{\rho}}^{\infty} (\widetilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) - \widetilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl}, \infty}) \mathfrak{C}_{\bar{\rho}}^{\infty} \in \mathfrak{D}_{\bar{\rho}}^{\infty} \cdot \mathbf{Err} \cdot \mathfrak{C}_{\bar{\rho}}^{\infty} \subseteq \mathbf{Err}_{\bar{\rho}},$$

where the last inclusion uses the fact that $\mathfrak{C}_{\bar{\rho}}^{\infty}$ and $\mathfrak{D}_{\bar{\rho}}^{\infty}$ are block diagonal matrices (but not with square blocks though). Since all matrices in $I_{\infty} + \mathbf{Err}_{\bar{\rho}}$ are invertible, $\varphi_{\bar{\rho}} \circ \psi_{\bar{\rho}}$ is an isomorphism. Thus it suffices to prove that $\psi_{\bar{\rho}}$ is surjective.

For this, we need only to show the surjectivity of $\psi_{\bar{\rho}} \circ \mathfrak{D}_{\bar{\rho}}^{\infty}$. Note that

$$\psi_{\bar{\rho}} \circ \mathfrak{D}_{\bar{\rho}}^{\infty} = \tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) \mathfrak{C}_{\bar{\rho}}^{\infty} \mathfrak{D}_{\bar{\rho}}^{\infty} = \tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) \tilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl},\infty} = \tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) (I_{\infty} + (\tilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl},\infty} - \tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa))).$$

By Proposition 7.5(4), the operator $I_{\infty} + (\tilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl},\infty} - \tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa)) \in I_{\infty} + \mathbf{Err}$ is an isomorphism. Then the surjectivity of $\psi_{\bar{\rho}} \circ \mathfrak{D}_{\bar{\rho}}^{\infty}$ follows from the surjectivity of $\tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa)$ onto $\tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) S_{\mathcal{B}}^{D,\dagger}(U; \kappa)$. This then concludes the proof of both $\varphi_{\bar{\rho}}$ and $\psi_{\bar{\rho}}$ being isomorphisms.

Finally, we observe that (7.10.1) implies that

$$\psi_{\bar{\rho}}^{-1} = \left(I_{\infty} + \mathfrak{D}_{\bar{\rho}}^{\infty} (\tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) - \tilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl},\infty}) \mathfrak{C}_{\bar{\rho}}^{\infty} \right)^{-1} \circ \varphi_{\bar{\rho}} = (I_{\infty} + \epsilon) \circ \varphi_{\bar{\rho}}$$

for the infinite matrix $\epsilon = \mathfrak{D}_{\bar{\rho}}^{\infty} (\tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) - \tilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl},\infty}) \mathfrak{C}_{\bar{\rho}}^{\infty} \in \mathbf{Err}_{\bar{\rho}}$. \square

Notation 7.11. Fix $\bar{\rho} \in \mathcal{B}(U; \psi_m)$ a residual pseudo-representation. Let $\text{HP}_{\bar{\rho},r}$ (resp. $\text{NP}_{\bar{\rho},r}$) denote the Hodge polygon (resp. Newton polygon) of the matrix $\mathfrak{U}_p^{\text{cl},\bar{\rho},r}$; let $\text{HP}_{\bar{\rho},r}(i)$ (resp. $\text{NP}_{\bar{\rho},r}(i)$) denote the y -coordinate of the polygon when the x -coordinate is i . Let $\alpha_0(\bar{\rho})_r \leq \dots \leq \alpha_{d_{\bar{\rho},r}-1}(\bar{\rho})_r$ denote the slopes of $\text{HP}_{\bar{\rho},r}$ in non-decreasing order. Let $\text{ord}_{\bar{\rho},r}$ denote the multiplicity of the slope 0 in $\text{NP}_{\bar{\rho},r}$.

Write the characteristic power series of U_p on $\tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) S_{\mathcal{B}}^{D,\dagger}(U; \kappa)$ as

$$\text{Char}(U_p, \tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) S_{\mathcal{B}}^{D,\dagger}(U; \kappa)) = 1 + c_{\bar{\rho},1}(w)X + c_{\bar{\rho},2}(w)X^2 + \dots \in 1 + \mathcal{O}\langle w \rangle \llbracket X \rrbracket.$$

Its zero in $\mathcal{W}(x\psi_m, p^{-1}) \times \mathbb{G}_{m,\text{rig}}$ is the spectral curve $\text{Spc}_{\bar{\rho}}$. We have

$$\text{Spc} \times_{\mathcal{W}} \mathcal{W}(x\psi_m; p^{-1}) = \bigcup_{\bar{\rho} \in \mathcal{B}(U; \psi_m)} \text{Spc}_{\bar{\rho}}.$$

Theorem 7.12. Assume $m \geq 4$ as before. Theorem 6.17, Corollary 6.19, and Theorem 6.22 hold for each $\bar{\rho} \in \mathcal{B}(U; \psi_m)$, in the following sense.

- (1) For any $w_0 \in \mathcal{W}(x\psi_m, p^{-1})$, the Newton polygon of the power series $1 + c_{\bar{\rho},1}(w_0)X + \dots$ lies above the polygon starting at $(0, 0)$ with slopes given by

$$(7.12.1) \quad \bigcup_{r=0}^{\infty} \{ \alpha_0(\bar{\rho})_r + r, \alpha_1(\bar{\rho})_r + r, \dots, \alpha_{d_{\bar{\rho},r}-1}(\bar{\rho})_r + r \}$$

- (2) For each $n \in \mathbb{N}$, let $\lambda_{\bar{\rho},n}$ denote the sum of n smallest numbers in (7.12.1). Then

$$c_{\bar{\rho},n}(w) \in p^{\lambda_{\bar{\rho},n}} \cdot \mathcal{O}\langle w \rangle^{\times}, \quad \text{for all } n \text{ of the form } n = n_{\bar{\rho},k} = \sum_{r=0}^k d_{\bar{\rho},r}$$

In particular, for any $w_0 \in \mathcal{W}(x\psi_m; p^{-1})$, the Newton polygon of the power series $1 + c_{\bar{\rho},1}(w_0)X + \dots$ passes through the point $(n, \lambda_{\bar{\rho},n})$ for $n = n_{\bar{\rho},k}$.

- (3) Fix $r = 0, \dots, q-1$. Suppose that $(s_0, \text{NP}_{\bar{\rho},r}(s_0))$ is a vertex of the Newton polygon $\text{NP}_{\bar{\rho},r}$ and suppose that

$$(7.12.2) \quad \text{NP}_{\bar{\rho},r}(s) < \text{HP}_{\bar{\rho},r}(s-1) + 1 \text{ for all } s = 1, \dots, s_0$$

Then for any $s = 0, \dots, s_0$, any $n \in \mathbb{Z}_{\geq 0}$, and any $w_0 \in \mathcal{W}(x\psi_m; p^{-1})$, the $(n_{\bar{\rho},qn+r} + s)$ th slope of the power series $1 + c_1(w_0)X + \dots$ is the s th U_p -slope on $V(\bar{\rho})_r$ plus $qn + r$.

(4) The spectral variety $\text{Spc}_{\bar{\rho}}$ is a disjoint union of subvarieties

$$X_{\bar{\rho},0}, X_{\bar{\rho},(0,1]}, X_{\bar{\rho},(1,2]}, X_{\bar{\rho},(2,3]}, \dots$$

such that each subvariety is finite and flat over $\mathcal{W}(x\psi_m; p^{-1})$, and for any closed point $x \in X_{\bar{\rho},?}$, we have $v(a_p(x)) \in ?$. Moreover, the degree of $X_{\bar{\rho},(r,r+1]}$ over $\mathcal{W}(x\psi_m; p^{-1})$ is exactly

$$d_{\bar{\rho},r+1} + \text{ord}_{\bar{\rho},r+1} - \text{ord}_{\bar{\rho},r}.$$

(5) Keep the notation and hypothesis as in (3) and (4). For all $n \in \mathbb{Z}_{\geq 0}$ and for a number $\beta > 0$ appearing in the first s_0 U_p -slopes on $V(\bar{\rho})_r$, the closed points $x \in X_{\bar{\rho},(qn+r,qn+r+1]}$ for which $v(a_p(x)) = \beta + qn + r$ form a connected component of $X_{\bar{\rho},(qn+r,qn+r+1]}$. It is finite and flat over $\mathcal{W}(x\psi_m; p^{-1})$ of degree equal to the multiplicity of β in the set of U_p -slopes of $V(\bar{\rho})_r$.

Proof. By Proposition 7.10, both $\varphi_{\bar{\rho}}$ and $\psi_{\bar{\rho}}$ are isomorphisms of Banach spaces. So we have

$$\text{Char}(U_p; \tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) S_{\mathcal{B}}^{D,\dagger}(U; \kappa)) = \text{Char}((\psi_{\bar{\rho}})^{-1} \circ \mathfrak{U}_p^{\mathcal{B}} \circ \psi_{\bar{\rho}}; V(\bar{\rho})_A^{\infty}).$$

Recall from Proposition 6.15(2) that the infinite block diagonal matrix $\mathfrak{U}_p^{\text{cl},\infty} = \text{Diag}\{\mathfrak{U}_p^{\text{cl}}(\psi_m), p \cdot \mathfrak{U}_p^{\text{cl}}(\psi_m \omega^{-2}), p^2 \cdot \mathfrak{U}_p^{\text{cl}}(\psi_m \omega^{-4}), \dots\}$ satisfies

$$\mathfrak{U}_p^{\mathcal{B}}(\kappa) - \mathfrak{U}_p^{\text{cl},\infty} \in \text{the error space } \mathbf{Err}_p \text{ in (6.15.2).}$$

We introduce the following error space

$$\mathbf{Err}_{\bar{\rho},p} := \begin{pmatrix} pM_{d_{\bar{\rho},0}}(A^\circ) & pM_{d_{\bar{\rho},0} \times d_{\bar{\rho},1}}(A^\circ) & p^2M_{d_{\bar{\rho},0} \times d_{\bar{\rho},2}}(A^\circ) & p^3M_{d_{\bar{\rho},0} \times d_{\bar{\rho},3}}(A^\circ) & \cdots \\ p^3M_{d_{\bar{\rho},1} \times d_{\bar{\rho},0}}(A^\circ) & p^2M_{d_{\bar{\rho},1}}(A^\circ) & p^2M_{d_{\bar{\rho},1} \times d_{\bar{\rho},2}}(A^\circ) & p^3M_{d_{\bar{\rho},1} \times d_{\bar{\rho},3}}(A^\circ) & \cdots \\ p^4M_{d_{\bar{\rho},2} \times d_{\bar{\rho},0}}(A^\circ) & p^4M_{d_{\bar{\rho},2} \times d_{\bar{\rho},1}}(A^\circ) & p^3M_{d_{\bar{\rho},2} \times d_{\bar{\rho},2}}(A^\circ) & p^3M_{d_{\bar{\rho},2} \times d_{\bar{\rho},3}}(A^\circ) & \cdots \\ p^5M_{d_{\bar{\rho},3} \times d_{\bar{\rho},0}}(A^\circ) & p^5M_{d_{\bar{\rho},3} \times d_{\bar{\rho},1}}(A^\circ) & p^5M_{d_{\bar{\rho},3} \times d_{\bar{\rho},2}}(A^\circ) & p^4M_{d_{\bar{\rho},3}}(A^\circ) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the (i, j) -block entry is

- $p^{i+1}M_{d_{\bar{\rho},i}}(A^\circ)$ if $i = j$,
- $p^{i+2}M_{d_{\bar{\rho},i} \times d_{\bar{\rho},j}}(A^\circ)$ if $i > j$, and
- $p^jM_{d_{\bar{\rho},i} \times d_{\bar{\rho},j}}(A^\circ)$ if $i < j$.

Rewrite the composite $(\psi_\pi)^{-1} \circ \mathfrak{U}_p^{\mathcal{B}} \circ \psi_\pi$ as

$$\begin{aligned} (\psi_{\bar{\rho}})^{-1} \circ \mathfrak{U}_p^{\mathcal{B}} \circ \psi_{\bar{\rho}} &= (\mathbf{id} + \epsilon) \mathfrak{D}_{\bar{\rho}}^{\infty} \tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) \mathfrak{U}_p^{\mathcal{B}} \mathfrak{C}_{\bar{\rho}}^{\infty} \\ &= \mathfrak{D}_{\bar{\rho}}^{\infty} \tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) \mathfrak{U}_p^{\mathcal{B}} \mathfrak{C}_{\bar{\rho}}^{\infty} + \epsilon \mathfrak{D}_{\bar{\rho}}^{\infty} \tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) \mathfrak{U}_p^{\mathcal{B}} \mathfrak{C}_{\bar{\rho}}^{\infty} \\ (7.12.3) \quad &= \mathfrak{D}_{\bar{\rho}}^{\infty} \tilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl},\infty} \mathfrak{U}_p^{\text{cl},\infty} \mathfrak{C}_{\bar{\rho}}^{\infty} + \mathfrak{D}_{\bar{\rho}}^{\infty} (\tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) \mathfrak{U}_p^{\mathcal{B}} - \tilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl},\infty} \mathfrak{U}_p^{\text{cl},\infty}) \mathfrak{C}_{\bar{\rho}}^{\infty} + \epsilon \mathfrak{D}_{\bar{\rho}}^{\infty} \tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) \mathfrak{U}_p^{\mathcal{B}} \mathfrak{C}_{\bar{\rho}}^{\infty}. \end{aligned}$$

Here the second equality in the first line follows from the commutativity of $\tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa)$ and $\mathfrak{U}_p^{\mathcal{B}}$ as they are (limits of) Hecke operators. It suffices to understand each of the terms.

- The first term $\mathfrak{D}_{\bar{\rho}}^{\infty} \tilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl},\infty} \mathfrak{U}_p^{\text{cl},\infty} \mathfrak{C}_{\bar{\rho}}^{\infty}$ of (7.12.3) exactly gives the action of U_p on the space of classical automorphic forms.
- By Proposition 6.15, we easily deduce that

$$\begin{aligned} \tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) \mathfrak{U}_p^{\mathcal{B}} - \tilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl},\infty} \mathfrak{U}_p^{\text{cl},\infty} &= \tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) (\mathfrak{U}_p^{\mathcal{B}} - \mathfrak{U}_p^{\text{cl},\infty}) + (\tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa) - \tilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl},\infty}) \mathfrak{U}_p^{\text{cl},\infty} \\ &\in (\tilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl},\infty} + \mathbf{Err}) \cdot \mathbf{Err}_p + \mathbf{Err} \cdot \mathfrak{U}_p^{\text{cl},\infty} \subseteq \mathbf{Err}_p; \end{aligned}$$

so the middle term of (7.12.3)

$$\mathfrak{D}_{\bar{\rho}}^{\infty}(\tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa)\mathfrak{U}_p^{\mathcal{B}} - \tilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl},\infty}\mathfrak{U}_p^{\text{cl},\infty})\mathfrak{C}_{\bar{\rho}}^{\infty} \in \mathfrak{D}_{\bar{\rho}}^{\infty} \cdot \mathbf{Err}_p \cdot \mathfrak{C}_{\bar{\rho}}^{\infty} \subseteq \mathbf{Err}_{\bar{\rho},p}.$$

(iii) We write

$$\epsilon \mathfrak{D}_{\bar{\rho}}^{\infty} \tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa)\mathfrak{U}_p^{\mathcal{B}}\mathfrak{C}_{\bar{\rho}}^{\infty} = \epsilon \mathfrak{D}_{\bar{\rho}}^{\infty}(\tilde{\mathfrak{P}}_{\bar{\rho}}^{\mathcal{B}}(\kappa)\mathfrak{U}_p^{\mathcal{B}} - \tilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl},\infty}\mathfrak{U}_p^{\text{cl},\infty})\mathfrak{C}_{\bar{\rho}}^{\infty} + \epsilon \mathfrak{D}_{\bar{\rho}}^{\infty} \tilde{\mathfrak{P}}_{\bar{\rho}}^{\text{cl},\infty}\mathfrak{U}_p^{\text{cl},\infty}\mathfrak{C}_{\bar{\rho}}^{\infty}$$

The second term belongs to $\mathbf{Err}_{\bar{\rho},p}$ because $\epsilon \in \mathbf{Err}_{\bar{\rho}}$ by Proposition 7.10. For the first term, we use the argument in (ii) to see that it belongs to

$$\epsilon \cdot \mathfrak{D}_{\bar{\rho}}^{\infty} \cdot \mathbf{Err}_p \cdot \mathfrak{C}_{\bar{\rho}}^{\infty} \subseteq \mathbf{Err}_{\bar{\rho},p}.$$

Combining the computation above, we see that $(\psi_{\bar{\rho}})^{-1} \circ \mathfrak{U}_p^{\mathcal{B}} \circ \psi_{\bar{\rho}}$ belongs to

$$(7.12.4) \quad \begin{pmatrix} \mathfrak{U}_p^{\text{cl},\bar{\rho},0} & & & & \\ & p \cdot \mathfrak{U}_p^{\text{cl},\bar{\rho},1} & & & \\ & & p^2 \cdot \mathfrak{U}_p^{\text{cl},\bar{\rho},2} & & \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix} + \mathbf{Err}_{\bar{\rho},p}.$$

At this point, (1)–(4) of the Theorem can be proved in the same way as they were proved in Theorem 6.17, Corollary 6.19, and Theorem 6.22, with the modifications indicated below.

(1) already follows from the estimate (7.12.4) because each $\mathfrak{U}_p^{\text{cl},\bar{\rho},r}$ is already written in the form adapted to its Hodge polygon.

For (2), we need to consider the action of $P_{\bar{\rho}}$ on the space $S_2^D(U; \psi_{m,w}\omega^{-2r})$ (see (6.9.2) for the definition). Let $\tilde{P}_{\bar{\rho}}$ denote the limit $\lim_{n \rightarrow \infty} (P_{\bar{\rho}})^{p^n}$. By the same argument as in Proposition 7.5, we have $\tilde{P}_{\bar{\rho}}^2 = \tilde{P}_{\bar{\rho}}$, $\tilde{P}_{\bar{\rho}}\tilde{P}_{\bar{\rho}'}$ for $\bar{\rho} \neq \bar{\rho}'$, and $\sum_{\bar{\rho} \in \mathcal{B}(U; \psi_m)} \tilde{P}_{\bar{\rho}} = \text{id}$. We use $V(\bar{\rho}, w)_r$ to denote the image $\tilde{P}_{\bar{\rho}} S_2^D(U; \psi_{m,w}\omega^{-2r})$, which is isomorphic to $V(\bar{\rho})_r \otimes_{\mathcal{O}} \mathcal{O}/p^2\mathcal{O}[w]$, as an \mathcal{O} -module. Let $\mathfrak{U}_p^{\text{cl},\bar{\rho},w,r}$ denote the matrix for the U_p -action on $V(\bar{\rho}, w)_r$ with respect to the basis $e_0(\bar{\rho})_r, \dots, e_{d_{\bar{\rho},r}-1}(\bar{\rho})_r$; its i th row is divisible by $p^{\alpha_i(\bar{\rho})_r}$, and all coefficients on w is divisible by p . We use $\bar{\mathfrak{U}}_p^{\text{cl},\bar{\rho},w,r}$ to denote the matrix given by dividing the i th row of $\mathfrak{U}_p^{\text{cl},\bar{\rho},w,r}$ by $p^{\alpha_i(\bar{\rho})_r}$. As argued in the proof of Theorem 6.17(2), it suffices to prove that $\det \bar{\mathfrak{U}}_p^{\text{cl},\bar{\rho},w,r}$ belongs to $\mathbb{F}^{\times} \subseteq \mathbb{F}[w]$ for each r . However, this follows from the fact that the product

$$\prod_{\bar{\rho} \in \mathcal{B}(U; \psi_m)} \det \bar{\mathfrak{U}}_p^{\text{cl},\bar{\rho},w,r} = \det \bar{\mathfrak{U}}_p^{\text{cl},e}(\psi_{m,w}\omega^{-2r}) \in \mathbb{F}^{\times}.$$

(3) and (4) follow from the arguments in Corollary 6.19 and Theorem 6.22 with no essential changes. (5) follows from (3) immediately. \square

REFERENCES

- [Bu04] K. Buzzard, On p -adic families of automorphic forms, in *Modular curves and abelian varieties*, 23–44, *Progr. Math.*, **224**, Birkhäuser, Basel, 2004.
- [Bu05] K. Buzzard, Questions about slopes of modular forms, *Automorphic forms. I, Astérisque* No. **298** (2005), 1–15.
- [Bu07] K. Buzzard, Eigenvarieties, in *L-functions and Galois representations*, 59–120, *London Math. Soc. Lecture Note Ser.*, **320**, Cambridge Univ. Press, Cambridge, 2007.
- [BC04] K. Buzzard and F. Calegari, A counterexample to the Gouvêa-Mazur conjecture. *C. R. Math. Acad. Sci. Paris* **338** (2004), no. **10**, 751–753.

- [BK05] K. Buzzard and L. Kilford, The 2-adic eigencurve at the boundary of weight space, *Compos. Math.* **141** (2005), no. 3, 605–619.
- [Ch05] G. Chenevier, Une correspondance de Jacquet-Langlands p -adique, *Duke Math. J.* **126** (2005), no. 1, 161–194.
- [Cl05] L. Clay, *Some Conjectures About the Slopes of Modular Forms*, PhD thesis, Northwestern University, June 2005.
- [Co96] R. Coleman, Classical and overconvergent modular forms. *Invent. Math.* **124** (1996), no. 1-3, 215–241.
- [Co97] R. Coleman, Classical and overconvergent modular forms of higher level, *J. Théor. Nombres Bordeaux* **9** (1997), no. 2, 395–403.
- [Co13] R. Coleman, The Eisenstein family, *Proc. Amer. Math. Soc.* **141** (2013), no. 9, 2945–2950.
- [CM98] R. Coleman and B. Mazur, The eigencurve, in *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, 1–113, *London Math. Soc. Lecture Note Ser.*, **254**, Cambridge Univ. Press, Cambridge, 1998.
- [DWX13] C. Davis, D. Wan, and L. Xiao, Newton slopes for Artin-Scheier-Witt towers, [arXiv:1310.5311](https://arxiv.org/abs/1310.5311).
- [DS05] F. Diamond, J. Shurman, *First course in modular forms*. Graduate Texts in Mathematics, **228**. Springer-Verlag, New York, 2005. xvi+436 pp.
- [DL] H. Diao and R. Liu, On properness of the eigencurve, [arXiv:1309.0577](https://arxiv.org/abs/1309.0577).
- [Em98] M. Emerton, 2-adic modular forms of minimal slope, *Thesis at Harvard University*, 1998.
- [Go01] F. Gouvêa, Where the slopes are. *J. Ramanujan Math. Soc.* **16** (2001), no. 1, 75–99.
- [GM92] F. Gouvêa and B. Mazur, Families of modular eigenforms, *Math. Comp.*, **58**, 793–806 (1992).
- [GM93] F. Gouvêa and B. Mazur, On the characteristic power series of the U operator, *Ann. Inst. Fourier, Grenoble*, **43**, 301–302 2(1993).
- [Ja04] D. Jacobs, Slopes of Compact Hecke Operators, *thesis*, University of London, Imperial College, 2004.
- [Kil08] L. J. P. Kilford, On the slopes of the U_5 operator acting on overconvergent modular forms. *J. Thor. Nombres Bordeaux* **20** (2008), no. 1, 165–182.
- [KM12] L. J. P. Kilford and K. McMurdy, Slopes of the U_7 operator acting on a space of overconvergent modular forms, *LMS J. Comput. Math.* **15** (2012), 113–139.
- [Kis09] M. Kisin, The Fontaine-Mazur conjecture for GL_2 . *J. Amer. Math. Soc.* **22** (2009), no. 3, 641–690.
- [Lo07] D. Loeffler, Spectral expansions of overconvergent modular functions, *Int. Math. Res. Not.* 2007, no. **16**.
- [Pi13] V. Pilloni, Overconvergent modular forms, *Ann. Inst. Fourier* **63** (2013), no. 1, 219–239.
- [PX] J. Pottharst and L. Xiao, On the parity conjecture in finite slope, *in preparation*.
- [Ro13] D. Roe, The 3-adic Eigencurve at the boundary of weight space, [arXiv:1310.7354](https://arxiv.org/abs/1310.7354).
- [Se96] J. P. Serre, Two letters on quaternions and modular forms (mod p), *Israel J. Math.* **95** (1996), 281–299.
- [Wa98] D. Wan, Dimension variation of classical and p -adic modular forms, *Invent. Math.* **133** (1998), no. 2, 449–463.

DAQING WAN, 340 ROWLAND HALL, DEPARTMENT OF MATHEMATICS, IRVINE, CA 92697.
E-mail address: dwan@math.uci.edu

LIANG XIAO, 340 ROWLAND HALL, DEPARTMENT OF MATHEMATICS, IRVINE, CA 92697.
E-mail address: lxiao@math.uci.edu

JUN ZHANG, CHERN INSTITUTE OF MATHEMATICS, NANKAI UNIVERSITY, TIANJIN 300071, P.R.CHINA
E-mail address: zhangjun04@mail.nankai.edu.cn