

THE EIGENCURVE OVER THE BOUNDARY OF WEIGHT SPACE

RUOCHUAN LIU, DAQING WAN, AND LIANG XIAO

In memory of Professor Robert F. Coleman

ABSTRACT. We prove that the eigencurve associated to a definite quaternion algebra over \mathbb{Q} satisfies the following properties, as conjectured by Coleman–Mazur and Buzzard–Kilford: (a) over the boundary annuli of weight space, the eigencurve is a disjoint union of (countably) infinitely many connected components each finite and flat over the weight annuli, (b) the U_p -slopes of points on each fixed connected component are proportional to the p -adic valuations of the parameter on weight space, and (c) the sequence of the slope ratios form a union of finitely many arithmetic progressions with the same common difference. In particular, as a point moves towards the boundary on an irreducible connected component of the eigencurve, the slope converges to zero.

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1. INTRODUCTION

1.1. **Coleman–Mazur–Buzzard–Kilford Conjecture.** *Eigencurves* were introduced in the groundbreaking work of R. Coleman and B. Mazur [CM98] to study the p -adic variation of modular forms. Roughly speaking, they are rigid analytic curves that parameterize finite slope overconvergent normalized p -adic eigenforms, where the q -expansions of these overconvergent modular forms vary p -adically continuously. The study of the eigencurves has led to great success, for example in M. Kisin’s proof of the Fontaine–Mazur Conjecture for overconvergent modular forms [Kis03]. While the arithmetic properties and the local geometry of the eigencurves were extensively studied in the literature (see e.g. [Bel15⁺] for a summary), their global geometry seems to be a very intriguing and difficult topic. Only

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recently, in joint work of H. Diao with the first author [DL16], they proved the “properness”¹ of the eigencurves over weight space. In this paper, we focus on another interesting geometric property of eigencurves, namely, their behavior over the boundary annuli of weight space.

Let us be more precise. Let p be a prime number. Set $q = p$ if p odd, and $q = 4$ if $p = 2$. Let $\varphi(q)$ denote the Euler function, namely, $\varphi(q) = p - 1$ if p is odd and $\varphi(q) = 2$ if $p = 2$. We use $v(\cdot)$ and $|\cdot|$ to denote the p -adic valuation and the p -adic norm, respectively, normalized so that $v(p) = 1$ and $|p| = p^{-1}$. In particular, $v(q) = 1$ if $q = p$, and $v(q) = 2$ if $p = 2$. *Weight space* \mathcal{W} is the rigid analytic space associated to the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$, which is the union of $\varphi(q)$ open unit disks indexed by the characters of the torsion subgroup Δ of \mathbb{Z}_p^\times . Each closed point on weight space corresponds to a continuous (p -adic) character χ of \mathbb{Z}_p^\times . We take the parameter on the weight disks to be $T := T_\chi := \chi(\exp(q)) - 1$. For $r \in (0, 1)$, we use $\mathcal{W}^{>r}$ to denote the (union of) annuli where $|T| > r$; it is referred to as the “halo” of weight space by Coleman.

We fix a tame level and let \mathcal{C} denote the corresponding eigencurve, as constructed in K. Buzzard’s paper [Bu07] (which generalizes [CM98]). Each point of the eigencurve corresponds to a finite slope normalized overconvergent eigenform $f = \sum_{n \geq 0} a_n(f)q^n$.² This eigencurve admits a map wt to the weight space, known as the *weight map*, and a map a_p to $\mathbb{G}_m^{\text{rig}}$, known as the *slope map*.

$$(1.1.1) \quad \begin{array}{ccc} \mathcal{C} & \xrightarrow{a_p} & \mathbb{G}_m^{\text{rig}} \\ \downarrow \text{wt} & & \\ \mathcal{W} & & \end{array}$$

For example, if we use z_f to denote the point on \mathcal{C} corresponding to a classical normalized eigenform f of weight $k + 2 \geq 2$ and Nebentypus p -character $\chi : (\mathbb{Z}/p^m\mathbb{Z})^\times \rightarrow \mathbb{C}_p^\times$, then the image of z_f under the map a_p is the p -th Fourier coefficient $a_p(f)$ of f , and the image of z_f under the map wt is the point on \mathcal{W} corresponding to the character $x^k\chi : \mathbb{Z}_p^\times \rightarrow \mathbb{C}_p^\times$ that sends u to $u^k\chi(u)$. In particular, the value of the parameter T at this point is $T_{x^k\chi} = \exp(kq) \cdot \chi(\exp(kq)) - 1$.

For $r \in (0, 1)$, we use $\mathcal{C}^{>r}$ to denote the preimage $\text{wt}^{-1}(\mathcal{W}^{>r})$. The following is a folklore conjecture suggested by a computation of Buzzard and L. Kilford [BK05] which addresses a question asked by Coleman and Mazur [CM98].

Conjecture 1.2 (Coleman–Mazur–Buzzard–Kilford). *When $r \in (0, 1)$ is sufficiently close to 1^- , the following statements hold.*

- (1) *The space $\mathcal{C}^{>r}$ is a disjoint union of (countably infinitely many) connected components Z_1, Z_2, \dots such that the weight map $\text{wt} : Z_n \rightarrow \mathcal{W}^{>r}$ is finite and flat for each n .*
- (2) *There exist nonnegative rational numbers $\alpha_1, \alpha_2, \dots \in \mathbb{Q}$ in non-decreasing order and tending to infinity such that, for each n and each point $z \in Z_n$, we have*

$$|a_p(z)| = |T_{\text{wt}(z)}|^{\alpha_n}.$$

¹Actually the “properness” here is not in the sense of rigid analytic geometry; we refer the reader to [DL16] for its precise definition.

²This is the only time in this paper q stands for $e^{2\pi iz}$. We will not mention q -expansions again.

- (3) *The sequence $\alpha_1, \alpha_2, \dots$ is a disjoint union of finitely many arithmetic progressions, counted with multiplicity (at least when the indices are large enough).*

When the tame level is trivial and $p = 2$, this conjecture was verified using an explicit computation by Buzzard and Kilford [BK05], extending the thesis work of M. Emerton [Em98]. More explicit computations for small p and small tame levels have appeared in [Ja04, Kil08, KM12, Ro14]. A partial result that is independent of the prime p and the tame level was proved by J. Zhang and the second and third authors [WXZ14⁺].

The goal of this paper is to prove an analog of Conjecture 1.2 for overconvergent automorphic forms for definite quaternion algebras over \mathbb{Q} .

We fix some notations first. Let D be a definite quaternion algebra over \mathbb{Q} which splits at p . Fix a tame level structure; we say the tame level is *neat* if it satisfies the condition (Neat) in Subsection 2.4. Let Spc_D denote the corresponding *spectral curve* associated to the overconvergent automorphic forms for D^\times constructed by Buzzard in [Bu07], which admits maps wt and a_p similar to the eigencurve as in (1.1.1). For $r \in (0, 1)$, we denote by $\mathrm{Spc}_D^{>r}$ the preimage $\mathrm{wt}^{-1}(\mathcal{W}^{>r})$. For each character $\omega : \Delta \rightarrow \mathbb{Z}_p^\times$, we denote by \mathcal{W}_ω and $\mathcal{W}_\omega^{>r}$ the weight disk and annulus corresponding to ω , and by $\mathrm{Spc}_{D,\omega}$ and $\mathrm{Spc}_{D,\omega}^{>r}$ the preimages $\mathrm{wt}^{-1}(\mathcal{W}_\omega)$ and $\mathrm{wt}^{-1}(\mathcal{W}_\omega^{>r})$ respectively. Let $\omega_0 : \Delta \rightarrow \mathbb{Z}_p^\times$ denote the inclusion map.

The first main result is the following theorem, in which

- the constant t is equal to the dimension of the space of weight 2 automorphic forms with q -Iwahori level structure at p and tame level as above, and
- $r_{\mathrm{ord}}(\omega)$ denotes the dimension of the ordinary subspace of automorphic forms of weight 2 and character ω .

Theorem 1.3. *Let $\omega : \Delta \rightarrow \mathbb{Z}_p^\times$ be a character. Then the space $\mathrm{Spc}_D^{>1/p}$ is a disjoint union*

$$\mathrm{Spc}_D^{>1/p} = X_0 \coprod X_{(0,1)} \coprod X_1 \coprod X_{(1,2)} \coprod X_2 \coprod \cdots$$

of (possibly empty) rigid analytic spaces which are finite and flat over $\mathcal{W}^{>1/p}$ via wt , such that, for each point $x \in X_I$ with I denoting the interval $n = [n, n]$ or $(n, n + 1)$, we have

$$v(a_p(x)) \in \varphi(q)v(T_{\mathrm{wt}(x)}) \cdot I.$$

In particular, as x varies on each irreducible component of Spc_D with $\mathrm{wt}(x)$ approaching the boundary of weight space, i.e. $|T_{\mathrm{wt}(x)}| \rightarrow 1^-$, the slope $v(a_p(x)) \rightarrow 0$.

Moreover, if the tame level is neat, and we denote by $X_{n,\omega}$ and $X_{(n,n+1),\omega}$ the preimages of $\mathcal{W}_\omega^{>1/p}$ in X_n and $X_{(n,n+1)}$ respectively, then

$$\deg X_{n,\omega} = \begin{cases} r_{\mathrm{ord}}(\omega), & \text{if } n = 0, \\ r_{\mathrm{ord}}(\omega^{-1}\omega_0^{2n-2}) + r_{\mathrm{ord}}(\omega\omega_0^{-2n}), & \text{if } n \geq 1, \end{cases}$$

and

$$\deg X_{(n,n+1),\omega} = qt - r_{\mathrm{ord}}(\omega^{-1}\omega_0^{2n}) - r_{\mathrm{ord}}(\omega\omega_0^{-2n}).$$

for all $n \geq 0$. In particular, we have $\deg X_{(n,n+1),\omega} > 0$ for all $n \geq 0$.

This theorem will be proved in Subsection 3.23.

Corollary 1.4. *If the tame level is neat, then for $I = (0, 1), 1, (1, 2), 2, \dots$, we have*

$$\deg X_{I,\omega} = \deg X_{I+1,\omega\omega_0^2}.$$

In particular, the degree $\deg X_{I,\omega}$ is periodic modulo $\frac{\varphi(q)}{2}$.

In order to prove the full version of Conjecture 1.2 for Sp_D , our current technique requires to weaken the radius bound on $|T|$. The following theorem will be proved in Subsection 4.2.

Theorem 1.5. *Let $\omega : \Delta \rightarrow \mathbb{Z}_p^\times$ be a character. Then there exists $\lambda \in (0, 1)$ such that there exists a sequence of rational numbers $\alpha_0(\omega), \alpha_1(\omega), \dots$ in increasing order and tending to infinity such that $\mathrm{Sp}_D^{>\lambda}$ is a disjoint union $\coprod_{i \geq 0} Y_{i,\omega}$ of rigid analytic spaces finite and flat over $\mathcal{W}_\omega^{>\lambda}$ via wt , such that*

$$(1.5.1) \quad v(a_p(y)) = \varphi(q)v(T_{\mathrm{wt}(y)})\alpha_i(\omega)$$

for every $y \in Y_{i,\omega}$. More precisely, if the tame level is neat, then we can take $\lambda = p^{-\frac{8}{(p^2-1)t+8}}$ for $p > 2$, and $\lambda = 2^{-\frac{1}{t+1}}$ for $p = 2$.

Moreover, let M be a positive integer so that $p^{-q/p^{M-1}(p-1)} > \lambda$; we also require $M \geq 2$ if p is odd and $M \geq 4$ if $p = 2$. Then if we extend the sequence $\alpha_0(\omega), \alpha_1(\omega), \dots$ into $\tilde{\alpha}_0(\omega), \tilde{\alpha}_1(\omega), \dots$ with each $\alpha_i(\omega)$ appearing with the multiplicity $\deg Y_{i,\omega}$, then $\tilde{\alpha}_0(\omega), \tilde{\alpha}_1(\omega), \dots$ is a disjoint union of $\frac{(p-1)p^{M-1}t}{2}$ arithmetic progressions with (the same) common difference $\frac{\varphi(q)p^M}{2q^2}$. More precisely, we have

$$(1.5.2) \quad \tilde{\alpha}_{j+q-1p^M t}(\omega\omega_0^2) = \tilde{\alpha}_j(\omega) + \frac{p^M}{q^2} \quad \text{for any } j \geq 0.$$

Remark 1.6. We first remark on the content of the theorems.

- (1) It is implicit from the statements that, for each $X_{I,\omega}$ from Theorem 1.3, $X_{I,\omega} \times_{\mathcal{W}_\omega^{>1/p}} \mathcal{W}_\omega^{>\lambda}$ is the disjoint union of those $Y_{i,\omega}$ in Theorem 1.5 for which $\alpha_i(\omega) \in I$.
- (2) The bound given by Theorem 1.5 appears to depend heavily on t . It might be possible to release t to $t_{\bar{\rho}}$ by working with each residual pseudo-representation $\bar{\rho}$, where $t_{\bar{\rho}}$ denotes the dimension of the space of weight 2 automorphic forms with Iwahori level structure at p where the tame Hecke action is determined by $\bar{\rho}$. More generally, we expect our argument to continue to hold for a direct summand of the completed homology of a modular curve or a definite quaternion algebra. We will revisit this idea in a future work.
- (3) For a continuous character χ of \mathbb{Z}_p^\times , the p -adic valuation of T_χ is the same as the p -adic valuation of $\chi(c) - 1$ for any topological generator c of $(1 + q\mathbb{Z}_p)^\times$. Therefore, both theorems do not depend on our convenient choice of the generator $\exp(q)$. Moreover, the region $|T| > 1/p$ is stable under the change of variable $T \mapsto \exp(-kq)(T+1) - 1$ that recenters the weight disks around the classical weight x^k .
- (4) The radius $1/p$ of Theorem 1.3 seems to be optimal if $p > 2$. When $p = 2$, one might be able to improve the radius to $1/4$ as opposed to the $1/2$ given in Theorem 1.3. The estimate of radius λ and the positive integer M in Theorem 1.5 may not be optimal. We do not know whether it is reasonable to expect any optimal bound.
- (5) The proof of Theorem 1.3 gives rise to a certain integral model of the spectral curve near the boundary of weight space, by factoring the characteristic power series of U_p integrally. See Remark 3.25 for an elaborated discussion.

- (6) The difference between the spectral curve and the actual eigencurve is minor for the type of questions we consider in this paper, as the eigencurve is essentially a (partial) normalization of the spectral curve (and possibly changing some non-reduced structure).

Remark 1.7. We remark on the relation to the literature.

- (1) By G. Chenevier’s p -adic Jacquet–Langlands correspondence [Ch05], we can translate results from the case of automorphic forms for definite quaternion algebras to the case of modular forms, and hence prove a large portion of Conjecture 1.2. The only connected components of the eigencurve we cannot access by this method are the ones whose tame part are all principal series. However, see Remark 3.26(2) for a discussion of potential approaches to this case.
- (2) While the main result of [WXZ14⁺] has now become a corollary of our two main theorems (See Corollary 1.8), our proof relies on several ideas developed therein. The key improvement from [WXZ14⁺] is that we choose a better basis to estimate the Newton polygon. See Remark 2.6 for a more detailed discussion.
- (3) Some related results were proved by F. Andreatta, A. Iovita, and V. Pilloni for the usual overconvergent (Hilbert) modular forms [AIP15⁺] in the sense of Coleman–Mazur [CM98] and Andreatta–Iovita–Pilloni–Stevens [AIS14, Pi13]; they constructed a certain compactification of weight space in the category of analytic adic spaces, and showed that the sheaf of overconvergent modular forms extends. They also obtained certain results on the geometry of the eigencurve near the boundary of weight space. Although their technique appears different to ours, both works have the same goal: realizing Coleman’s idea [Cole-A] in the corresponding context. So it would be interesting to compare the two approaches.
- (4) The second half of Theorem 1.5 follows from the first half by classicality results and Atkin–Lehner theory. This argument was independently found by J. Bergdall and R. Pollack [BP15⁺].
- (5) Corollary 1.4 indicates a peculiar relation between the degrees of components of the spectral curve over one weight disk and those over another weight disk shifted by the square of the Teichmüller character. This might be related to some observation by F. Calegari, known as the “theta-cycle” phenomenon: in characteristic p , there is a θ map on the space of mod p modular forms, increasing the weight by 2; this seems to have some magical effect on the slope of modular forms (which are of characteristic zero).
- (6) In this paper, we do not touch the geometry of the eigencurves over the center of weight space, which is expected to be very complicated. We refer to [Wa98, Bu05, He05, BG15⁺, BP16⁺] for a more comprehensive discussion.

We now turn to discussing the application of our main theorem. For a character $\psi : (\mathbb{Z}/p^m\mathbb{Z})^\times \rightarrow \mathbb{C}_p^\times$ that does not factor through $(\mathbb{Z}/p^{m-1}\mathbb{Z})^\times$ (which we call p -primitive) and $k \in \mathbb{Z}_{\geq 0}$, we use $S_{k+2}^D(\psi)$ to denote the space of automorphic forms on D^\times of weight $k + 2$ with the fixed tame level, p^m -Iwahori level at p , and Nebentypus character ψ . Combining Theorems 1.3 and 1.5 with the classicality result (Proposition 2.15), one can deduce strong consequences regarding the slopes of classical automorphic forms. Roughly speaking, we prove that *knowing the slopes of weight 2 automorphic forms of $\varphi(q)$ characters with a certain conductor at p is enough to determine the slopes of all automorphic forms with*

larger conductors at p . The precise statement is as follows, whose proof will appear in Subsection 4.4.

Corollary 1.8. (1) Let ψ be a p -primitive character of $(\mathbb{Z}/p^m\mathbb{Z})^\times$ with $m \geq 2$ if $p > 2$, and $m \geq 4$ if $p = 2$. Let $\beta_0(k, \psi), \dots, \beta_{q^{-1}p^m(k+1)t-1}(k, \psi)$ denote the sequence of slopes of the U_p -action on $S_{k+2}^D(\psi)$, in non-decreasing order and counted with multiplicity. Then we have

$$(1.8.1) \quad \frac{q^2}{p^m}(\lfloor n/qt \rfloor) \leq \beta_n(k, \psi) \leq \frac{q^2}{p^m}(\lfloor n/qt \rfloor + 1),$$

for $n = 0, \dots, q^{-1}p^m(k+1)t - 1$. Note the inequalities we obtained are independent of the weight $k+2$.

(2) Let M be a positive integer so that $p^{-q/p^{M-1}(p-1)} > \lambda$; we also require $M \geq 2$ if $p > 2$ and $M \geq 4$ if $p = 2$. For each character ω of Δ , we choose a p -primitive character ψ of $(\mathbb{Z}/p^M\mathbb{Z})^\times$ as above so that $\psi|_\Delta = \omega$, and let $\beta_0(\omega), \dots, \beta_{q^{-1}p^Mt-1}(\omega)$ denote the sequence of slopes of the U_p -action on $S_2^D(\psi)$, in non-decreasing order and counted with multiplicity. (This sequence does not depend on the choice of ψ .) Then for any $k \in \mathbb{Z}_{\geq 0}$ and any p -primitive character ψ_m of $(\mathbb{Z}/p^m\mathbb{Z})^\times$ with $m \geq M$, the sequence of the slopes of the U_p -action on $S_{k+2}^D(\psi_m)$ is given by

$$\bigcup_{n=0, \dots, p^{m-M}(k+1)-1} \left\{ p^{M-m}(\beta_0(\psi_m|_\Delta \omega_0^{k-2n}) + n), \dots, p^{M-m}(\beta_{q^{-1}p^Mt-1}(\psi_m|_\Delta \omega_0^{k-2n}) + n) \right\}.$$

Another application is the following result.

Corollary 1.9. Each irreducible component of the spectral curve Sp_D contains a classical point of weight 2 (with possibly large conductor at p).

Proof. By [CM98, Corollary 1.3.13], each irreducible component extends to the boundary of weight space, where the slopes tend to zero by Theorem 1.3. So this irreducible component must contain a point above a classical weight 2 (of large conductor at p) and with slope strictly less than 1. This point must correspond to a classical automorphic form by the classicality result (see [Bu07, Proposition 4] or Proposition 2.15). \square

Remark 1.10. This corollary naturally appeared in the joint work of J. Pottharst and the third author [PX14⁺]. In that paper, we studied the parity conjecture which predicts that the vanishing orders of the L-functions of modular forms are congruent modulo 2 to the dimension of the associated Selmer groups. The basic idea is that, one can prove this in weight 2 by applying some argument involving Heegner points. The main theorem of [PX14⁺] is roughly that, on each irreducible component of the eigencurve, if the parity conjecture holds for one classical point, then it holds for all classical points. So by Corollary 1.9, any modular form, if can be translated to an automorphic form over a definite quaternion algebra (split at p), is linked to a classical automorphic form of weight 2 (of slope < 1). Hence the parity conjecture holds for that modular form.

We expect a similar argument can be applied to study the p -adic Gross–Zagier formula, to bypass the essential difficulty imposed by requiring slopes < 1 .

1.11. **Idea of the proof of Theorems 1.3 and 1.5.** We point out a few key points in the proof of the main theorems.

- (1) In Coleman’s private note [Cole-A], he advocated the idea of viewing the weight space and the eigencurve as formal schemes, as opposed to (increasing unions of) rigid analytic spaces. He pointed out that the key to realize this is to provide a certain “integral model” of the space of overconvergent modular forms *over the “halo” of weight space*. Although we shall be working with a context different from what he suggested in [Cole-A], *this viewpoint is absolutely crucial to our paper*. In the case for definite quaternion algebra we study in this paper, Coleman’s idea amounts to construct a “Banach space” over the Iwasawa algebra Λ , whose base change to each affinoid subdomain of \mathcal{W} is “close to” the Banach space of overconvergent automorphic forms in the sense of Buzzard [Bu07], at least having the same characteristic power series for U_p . In fact, this expected space is not mysterious: its dual is the coinvariant subspace of Emerton’s completed homology under the action of the unipotent radical of the Borel subgroup at p , which is a compact topological Λ -module (in the sense of [ST02]). In this paper, we present the construction using induced representations; this gives rise to a “Banach Λ -module”, which we call the space of *integral p -adic automorphic forms*. (The action is slightly twisted to match with the convention used by Buzzard [Bu04].) We refer to Remark 2.9 for the relation with Emerton’s completed homology and potential generalizations.
- (2) We choose to work with a definite quaternion algebra as opposed to the usual overconvergent modular forms, to circumvent the complication of the geometry of the modular curves, as presented in all prior works of direct computation (they all rely on the explicit equation that defines the modular curve, which is clearly inaccessible in general). In our case, the U_p -action on the space of integral p -adic automorphic forms can be written reasonably explicitly, as explained in the first part of Section 3. This was inspired by the thesis of D. Jacobs [Ja04] (a former student of Buzzard), and our generalization [WXZ14⁺].
- (3) Using the explicit description of the space of integral p -adic automorphic forms, we look at the associated infinite matrix $(P_{i,j})$ with respect to a basis originated from the Mahler basis $1, z, \binom{z}{2}, \dots$ on the space of p -adic continuous functions on \mathbb{Z}_p . A mild p -adic analysis computation (which is the core of our paper) shows that $P_{i,j} \in \mathfrak{m}_\Lambda^{\max\{0, \lfloor i/t \rfloor - \lfloor j/pt \rfloor\}}$, where \mathfrak{m}_Λ is the ideal of Λ generated by p and T . As a consequence, if we write $c_0 + c_1X + \dots \in \Lambda[[X]]$ for the characteristic power series for the U_p -operator, then c_i belongs to

$$T^{\lambda_i} \Lambda^{>1/p},$$

where λ_i is recursively defined by $\lambda_0 = 0$, and $\lambda_i - \lambda_{i-1} = \lfloor i/t \rfloor - \lfloor i/pt \rfloor$. This gives rise to a lower bound on the Newton polygon over each point of weight space with $|T| > 1/p$.

- (4) It is somewhat a lucky coincidence that, *the Newton polygon lower bound obtained in (3) partially agrees with the actual Newton polygon at classical weights*. This allows us to conclude the main theorems. This part of the argument was inspired by similar tricks used in joint work of the last two authors with C. Davis [DWX16].

1.12. **Structure of the paper.** Section 2 is devoted to constructing a certain integral model for the space of p -adic automorphic forms on a definite quaternion algebra. The action of U_p -operator on this integral model was made explicit in the first part of Section 3; and we prove Theorem 1.3 in the latter part of Section 3 using a close estimate of the Newton polygon. Section 4 is devoted to proving Theorem 1.5. In Section 5, we provide a variant of the construction given in Section 2, which can be regarded as integral models of the space of overconvergent automorphic forms.

1.13. **Acknowledgments.** We cannot emphasize enough the importance of the ideas of Robert Coleman to this paper. We thank Barry Mazur for his constant encouragement and many suggestions. We thank Vincent Pilloni for sharing his insight into Coleman's idea. We thank the anonymous referees for their impressively helpful report which greatly improved the exposition of the paper as well as simplified some arguments. We thank John Bergdall, Gaetan Chenevier, Keith Conrad, and Robert Pollack for interesting discussions. D.W. and L.X. thank the hospitality of Beijing International Center for Mathematical Research when they visited.

1.14. **Notation.** Throughout this paper, \mathbb{N} denotes the set of *positive* integers. We fix a prime number p . Set $q = p$ for $p > 2$, and $q = 4$ for $p = 2$. Let $\varphi(q)$ denote the Euler function, namely, $\varphi(q) = p - 1$ if $p > 2$ and $\varphi(q) = 2$ if $p = 2$. Write \mathbb{A} for the ring of adèles of \mathbb{Q} , and \mathbb{A}_f (resp. $\mathbb{A}_f^{(p)}$) the subring of finite adèles (resp. finite prime-to- p adèles).

For A an affinoid \mathbb{Q}_p -algebra, we use A° to denote the subring of power bounded elements. The notions $A\langle z \rangle$ and $A^\circ\langle z \rangle$ are reserved for denoting Tate algebras.

The row and column indices of matrices always start with 0. We use I_n for $n \in \mathbb{Z}_{\geq 0}$ or ∞ to denote the identity $n \times n$ -matrix.

2. AUTOMORPHIC FORMS FOR DEFINITE QUATERNION ALGEBRAS

We first discuss carefully various versions of (overconvergent) automorphic forms for definite quaternion algebras. In particular, we give a certain “integral model” of the space of p -adic automorphic forms.

Notation 2.1. We write \mathbb{Z}_p^\times as $\Delta \times (1 + q\mathbb{Z}_p)^\times$ with $\Delta \cong (\mathbb{Z}/q\mathbb{Z})^\times$. We identify $(1 + q\mathbb{Z}_p)^\times$ with \mathbb{Z}_p via $\frac{1}{q} \log(-)$. Let Λ denote the Iwasawa algebra

$$\mathbb{Z}_p[\mathbb{Z}_p^\times] \cong \mathbb{Z}_p[\Delta] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[(1 + q\mathbb{Z}_p)^\times] \cong \mathbb{Z}_p[\Delta] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[T]],$$

where T corresponds to $[\exp(q)] - 1$. Here, for $a \in \mathbb{Z}_p^\times$, we use $[a]$ to denote its image in Λ^\times ; so $[-] : \mathbb{Z}_p^\times \rightarrow \Lambda^\times$ is the *universal character* of \mathbb{Z}_p^\times . In particular, each continuous ring homomorphism $\chi : \Lambda \rightarrow \mathbb{C}_p$ defines a continuous character $\chi \circ [-] : \mathbb{Z}_p^\times \rightarrow \mathbb{C}_p^\times$ (which we still denote by χ). Conversely, all continuous \mathbb{C}_p -valued characters of \mathbb{Z}_p^\times may be obtained this way.

We use \mathfrak{m}_Λ to denote the ideal (p, T) of $\Lambda \cong \mathbb{Z}_p[\Delta] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[T]]$.

Let \mathcal{W} denote the rigid analytic space associated to Λ . For each (\mathbb{C}_p -valued) continuous character χ of \mathbb{Z}_p^\times , we write $T_\chi := \chi(\exp(q)) - 1$ for the T -coordinate of the associated point on weight space. For $r \in (0, 1) \cap p^\mathbb{Q}$, we use $\mathcal{W}^{\leq r}$ to denote the union of the disks where $|T| \leq r$; it is an affinoid subdomain of the weight space.

Following Buzzard [Bu04, Section 5], we define for $m \in \mathbb{N}$ the rigid analytic spaces

$$\begin{aligned}\mathbf{B}_{p^{-m}} &= \{z \in \mathcal{O}_{\mathbb{C}_p} \mid |z - a| \leq p^{-m} \text{ for some } a \in \mathbb{Z}_p\}; \\ \mathbf{B}_{p^{-m}}^\times &= \{z \in \mathcal{O}_{\mathbb{C}_p} \mid |z - a| \leq p^{-m} \text{ for some } a \in \mathbb{Z}_p^\times\}.\end{aligned}$$

For $m \in \mathbb{N}$ ($m \geq 2$ if $p = 2$), we say a continuous character $\chi : \mathbb{Z}_p^\times \rightarrow A^\times$ with values in an affinoid \mathbb{Q}_p -algebra A is *m-locally analytic* if for each closed point $x \in \text{Max}(A)$ and the corresponding character $\chi_x : \mathbb{Z}_p^\times \xrightarrow{\chi} A^\times \rightarrow k(x)^\times$, we have $v(T_{\chi_x}) > q/p^m(p-1)$. In this case, χ extends to a continuous homomorphism

$$\begin{aligned}\kappa : (\mathbb{Z}_p + p^m A^\circ \langle z \rangle)^\times &= \mathbb{Z}_p^\times \cdot (1 + p^m A^\circ \langle z \rangle) \longrightarrow (A^\circ \langle z \rangle)^\times \\ a \cdot x &\longmapsto \chi(a) \cdot \chi(\exp(p^m))^{(\log x)/p^m}.\end{aligned}$$

When A is a finite extension E of \mathbb{Q}_p , this means that χ extends to a homomorphism of rigid group schemes $\chi : \mathbf{B}_{p^{-m}}^\times \rightarrow \mathbb{G}_{m,E}^{\text{rig}}$. See e.g. [WXZ14⁺, (3.1.2)] for more discussion.

For $m \in \mathbb{N}$, a continuous character $\psi : \mathbb{Z}_p^\times \rightarrow E^\times$ (with E a finite extension over \mathbb{Q}_p) is called a *finite character of conductor p^m* if it factors through $(\mathbb{Z}/p^m\mathbb{Z})^\times$ but not $(\mathbb{Z}/p^{m-1}\mathbb{Z})^\times$. We say a continuous character χ of \mathbb{Z}_p^\times is *classical* if it sends x to $x^k\psi(x)$ for an integer $k \geq 0$ and a finite character ψ of conductor p^m . We write (k, ψ) for such a character; it is *m-locally analytic*, because

- (when $p > 2$) $v(T_{(k,\psi)}) \geq 1$ if $m = 1$, and $v(T_{(k,\psi)}) = 1/p^{m-2}(p-1)$ if $m \geq 2$, and
- (when $p = 2$) $v(T_{(k,\psi)}) \geq 1$ if $m = 3$, and $v(T_{(k,\psi)}) = 1/p^{m-3}$ if $m \geq 4$.

By abuse of language, we say this (k, ψ) has *conductor p^m* . In this paper, the weight of automorphic forms will be $k + 2$.

2.2. Subgroups of $\text{GL}_2(\mathbb{Z}_p)$. We consider the following subgroups of $\text{GL}_2(\mathbb{Q}_p)$ (for $m \in \mathbb{N}$):

$$\begin{aligned}\text{Iw}_{p^m} &\cong \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^m\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \supset B(\mathbb{Z}_p) \cong \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ 0 & \mathbb{Z}_p^\times \end{pmatrix} \supset N(\mathbb{Z}_p) \cong \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}, \\ T(\mathbb{Z}_p) &= \begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & \mathbb{Z}_p^\times \end{pmatrix} \quad \text{and} \quad \bar{N}(p^m\mathbb{Z}_p) \cong \begin{pmatrix} 1 & 0 \\ p^m\mathbb{Z}_p & 1 \end{pmatrix}.\end{aligned}$$

The *Iwasawa decomposition* is the isomorphism

$$(2.2.1) \quad \begin{array}{ccc} N(\mathbb{Z}_p) \times T(\mathbb{Z}_p) \times \bar{N}(p^m\mathbb{Z}_p) & \xrightarrow{\cong} & \text{Iw}_{p^m} \\ (n, t, \bar{n}) & \longmapsto & nt\bar{n}. \end{array}$$

We will often identify $\bar{N}(q\mathbb{Z}_p)$ with \mathbb{Z}_p by sending $\begin{pmatrix} 1 & 0 \\ qz & 1 \end{pmatrix}$ to z for $z \in \mathbb{Z}_p$. The Iwahori subgroup Iw_q admits an anti-involution:

$$(2.2.2) \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto g^* := \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} g^t \begin{pmatrix} 1 & 0 \\ 0 & q^{-1} \end{pmatrix} = \begin{pmatrix} a & c/q \\ qb & d \end{pmatrix}.$$

In particular, $(gh)^* = h^*g^*$ for $g, h \in \text{Iw}_q$.

Consider the natural homomorphism $\pi_d : B(\mathbb{Z}_p) \rightarrow \mathbb{Z}_p^\times$ sending $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ to d . Any continuous character of \mathbb{Z}_p^\times can be viewed as a character of $B(\mathbb{Z}_p)$ by composing with π_d . We will only consider characters of $B(\mathbb{Z}_p)$ of this form.

2.3. Induced representation. Let A be a topological ring in which p is topologically nilpotent; and let $\chi : \mathbb{Z}_p^\times \rightarrow A^\times$ be a continuous character, viewed as a character of $B(\mathbb{Z}_p)$ by composing with π_d as above. Consider the following induced representation of Iw_q :

$$\mathrm{Ind}_{B(\mathbb{Z}_p)}^{\mathrm{Iw}_q}(\chi) := \left\{ \text{continuous functions } f : \mathrm{Iw}_q \rightarrow A \mid f(bg) = \chi(b)f(g) \text{ for } b \in B(\mathbb{Z}_p), g \in \mathrm{Iw}_q \right\},$$

where instead of the usual left action, we consider the right action of $h \in \mathrm{Iw}_q$ by sending f to $f \parallel_h^\chi : g \mapsto f(gh^*)$. (The reason for our choice is to match with the convention used in Buzzard [Bu04] as shown by (2.3.2) below.) The Iwasawa decomposition (2.2.1) gives the following isomorphism, which made this induced representation explicit:

$$(2.3.1) \quad \mathrm{Ind}_{B(\mathbb{Z}_p)}^{\mathrm{Iw}_q}(\chi) \xrightarrow{\cong} \mathcal{C}(\mathbb{Z}_p; A) := \left\{ \text{continuous functions } \mathbb{Z}_p \rightarrow A \right\}$$

$$f \longmapsto h(z) := f\left(\begin{pmatrix} 1 & 0 \\ qz & 1 \end{pmatrix}\right).$$

Then the (right) action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Iw}_q$ on $\mathrm{Ind}_{B(\mathbb{Z}_p)}^{\mathrm{Iw}_q}(\chi)$ induces the following action on $\mathcal{C}(\mathbb{Z}_p; A)$:

$$(2.3.2) \quad h \parallel_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}^\chi(z) = f\left(\begin{pmatrix} 1 & 0 \\ qz & 1 \end{pmatrix} \begin{pmatrix} a & c/q \\ qb & d \end{pmatrix}\right) = f\left(\begin{pmatrix} \frac{ad-bc}{cz+d} & c/q \\ 0 & cz+d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q\frac{az+b}{cz+d} & 1 \end{pmatrix}\right) = \chi(cz+d)h\left(\frac{az+b}{cz+d}\right).$$

One checks that this action extends to an action of the monoid

$$(2.3.3) \quad \mathbf{M}_1 := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}_p) \mid q \nmid c, p \nmid d, \text{ and } ad - bc \neq 0 \right\}.$$

When A is an affinoid \mathbb{Q}_p -algebra and $\chi : \mathbb{Z}_p^\times \rightarrow A^\times$ is m_0 -locally analytic for some $m_0 \in \mathbb{N}$, for every $m \geq \max\{m_0, v(q)\}$, we can consider the m -locally analytic induced representation

$$\mathrm{Ind}_{B(\mathbb{Z}_p)}^{\mathrm{Iw}_q}(\chi)^{m, \mathrm{an}} = \left\{ f \in \mathrm{Ind}_{B(\mathbb{Z}_p)}^{\mathrm{Iw}_q}(\chi) \mid f \text{ is an analytic function on } \left(a + p^{-1}p^m\mathbb{Z}_p, 0 \right), \text{ for all } a \in q\mathbb{Z}_p \right\}.$$

Here analytic function means that the values of the function can be given by a convergent Taylor series on the specified p -adic disk. The condition that χ is m -locally analytic is used so that the action (2.3.2) is well defined.

Similar to (2.3.1), sending f to $h(z) = f\left(\begin{pmatrix} 1 & 0 \\ qz & 1 \end{pmatrix}\right)$ induces an isomorphism

$$(2.3.4) \quad \mathrm{Ind}_{B(\mathbb{Z}_p)}^{\mathrm{Iw}_q}(\chi)^{m, \mathrm{an}} \xrightarrow{\cong} \mathcal{O}_{\mathbf{B}_{qp^{-m}}} \hat{\otimes}_{\mathbb{Q}_p} A.$$

Here the latter space may be understood as the subspace of continuous functions $h \in \mathcal{C}(\mathbb{Z}_p; A)$ such that h is analytic on $a + q^{-1}p^m\mathbb{Z}_p$ for all $a \in \mathbb{Z}_p$.

We write

$$\mathrm{Ind}_{B(\mathbb{Z}_p)}^{\mathrm{Iw}_q}(\chi)^{\mathrm{lan}} = \bigcup_{m \geq m_0} \mathrm{Ind}_{B(\mathbb{Z}_p)}^{\mathrm{Iw}_q}(\chi)^{m, \mathrm{an}}$$

for the *locally analytic induced representation*; it is a subspace of $\mathrm{Ind}_{B(\mathbb{Z}_p)}^{\mathrm{Iw}_q}(\chi)$ which, under the isomorphism (2.3.1), corresponds to $\mathcal{C}^{\mathrm{an}}(\mathbb{Z}_p; A)$ consisting of locally analytic functions on \mathbb{Z}_p with values in A .

When χ is a classical character (k, ψ) of conductor $p^{m_0} \leq p^m$, $\mathrm{Ind}_{B(\mathbb{Z}_p)}^{\mathrm{Iw}_q}(\chi)^{m, \mathrm{an}}$ contains a subspace $\mathrm{Ind}_{B(\mathbb{Z}_p)}^{\mathrm{Iw}_q}(\chi)^{m, \mathrm{alg}}$ consisting of those functions whose restriction to $a + q^{-1}p^m\mathbb{Z}_p$ is a polynomial function of degree $\leq k$ for all $a \in \mathbb{Z}_p$. The isomorphism (2.3.4) induces an isomorphism between $\mathrm{Ind}_{B(\mathbb{Z}_p)}^{\mathrm{Iw}_q}(\chi)^{m, \mathrm{alg}}$ and the space $\mathrm{LP}^{m-v(q), \mathrm{deg} \leq k}(\mathbb{Z}_p; E)$ consisting of all

functions $h \in \mathcal{C}(\mathbb{Z}_p; E)$ whose restriction to $a + q^{-1}p^m\mathbb{Z}_p$ is a polynomial function of degree $\leq k$ for all $a \in \mathbb{Z}_p$.

2.4. Buzzard's overconvergent automorphic forms. We now recall Buzzard's construction of overconvergent automorphic forms following [Bu04, Section 5].

We fix a definite quaternion algebra D over \mathbb{Q} which splits at p , and we fix an isomorphism $D \otimes \mathbb{Q}_p \simeq M_2(\mathbb{Q}_p)$ and identify them, so that the groups considered in Subsection 2.2 may be viewed as subgroups of $(D \otimes \mathbb{Q}_p)^\times$. We fix the tame level structure K^p to be an open compact subgroup of $(D \otimes \mathbb{A}_f^{(p)})^\times$. We call K^p *neat* if it satisfies the following condition (see [Bu04, Section 4]):

$$(\text{Neat}) \quad \text{for any } x \in (D \otimes \mathbb{A}_f)^\times, \text{ we have } x^{-1}D^\times x \cap K^p \text{Iw}_q = \{1\}.$$

The neat condition is cofinal in the direct system of all tame level structures. For instance, for any $l \geq 3$, the full l level structure is neat.

Let χ be an m_0 -locally analytic character of \mathbb{Z}_p^\times with values in an affinoid \mathbb{Q}_p -algebra A . For $m \geq m_0$, Buzzard [Bu04] defined the space of overconvergent modular forms of weight χ and radius of convergence p^{-m} (with $m \in \mathbb{N}$ and $m \geq 2$ if $p = 2$) to be

$$S_\chi^{D, \dagger, m} := \left\{ \varphi : D^\times \backslash (D \otimes \mathbb{A}_f)^\times / K^p \rightarrow \text{Ind}_{B(\mathbb{Z}_p)}^{\text{Iw}_q}(\chi)^{m, \text{an}} \mid \varphi(xu_p) = \varphi(x) \|_{u_p}^\chi, \text{ for } u_p \in \text{Iw}_q \right\}.$$

We put $S_\chi^{D, \dagger} := \cup_{m \geq m_0} S_\chi^{D, \dagger, m}$. When $\chi = (k, \psi)$ is a classical character of conductor $p^{m_0} \leq p^m$, $S_\chi^{D, \dagger, m}$ contains the subspace of classical automorphic forms of weight $k + 2$:

$$S_{k+2}^D(K^p \text{Iw}_{p^m}; \psi) := \left\{ \varphi : D^\times \backslash (D \otimes \mathbb{A}_f)^\times / K^p \rightarrow \text{Ind}_{B(\mathbb{Z}_p)}^{\text{Iw}_q}(\chi)^{m, \text{alg}} \mid \varphi(xu_p) = \varphi(x) \|_{u_p}^\chi, \text{ for } u_p \in \text{Iw}_q \right\}.$$

2.5. The U_p -operator. We choose a decomposition of the double coset:

$$\text{Iw}_q \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \text{Iw}_q = \prod_{j=0}^{p-1} \text{Iw}_q v_j,$$

for example with $v_j = \begin{pmatrix} p & 0 \\ jq & 1 \end{pmatrix}$, and define

$$(2.5.1) \quad U_p(\varphi) := \sum_{j=0}^{p-1} \varphi|_{v_j}^\chi, \quad \text{with } (\varphi|_{v_j}^\chi)(g) := \varphi(gv_j^{-1}) \|_{v_j}^\chi.$$

This definition does not depend on the choice of the coset representatives v_j . The operator U_p naturally acts on $S_\chi^{D, \dagger, m}$, and preserves the subspace $S_{k+2}^D(K^p \text{Iw}_{p^m}; \psi)$ if χ is a classical character (k, ψ) of conductor $p^{m_0} \leq p^m$.

Remark 2.6. Buzzard [Bu04] allows variants of the construction by taking equivariant functions under a (smaller) Iwahori subgroup Iw_{p^α} and with values in $\text{Ind}_{B(\mathbb{Z}_p)}^{\text{Iw}_{p^\alpha}}(\chi)^{\beta, \text{an}}$, namely, those functions that are analytic on $a + p^{\beta-\alpha}\mathbb{Z}_p$ for all $a \in \mathbb{Z}_p$ (under the identification $h(z) = f\left(\begin{pmatrix} 1 & 0 \\ p^\alpha z & 1 \end{pmatrix}\right)$), as long as $\alpha \geq 1$ and $\beta \geq \max\{\alpha, m_0, v(q)\}$. But [Bu04, Lemma 4] says that the characteristic power series for the U_p -action on the space of overconvergent automorphic forms does not depend on this general flexibility. In [WXZ14⁺], we exploit the construction when $\alpha = \beta = m_0$, i.e., taking equivariant functions under the Iwahori subgroup $\text{Iw}_{p^{m_0}}$ and having the target of φ to be $A\langle z \rangle$. In contrast, we focus on the other extreme $\alpha = v(q)$ and $\beta = m_0$ in this paper: letting the disks become smaller and smaller as m_0 goes

to infinity, while keeping the setup to take equivariant functions under the Iw_q -action; this is adapted to the goal of finding an integral model.

2.7. Integral model of the space of p -adic automorphic forms. Consider the universal character $[-] : \mathbb{Z}_p^\times \rightarrow \Lambda^\times$, viewed as a character of $B(\mathbb{Z}_p)$ by composing with π_d . (So the coefficient ring will be the Iwasawa algebra $A = \Lambda$.) Define the space of *integral p -adic automorphic forms for D* to be

$$S_{\text{int}}^D := \left\{ \varphi : D^\times \backslash (D \otimes \mathbb{A}_f)^\times / K^p \rightarrow \text{Ind}_{B(\mathbb{Z}_p)}^{\text{Iw}_q}([-]) \mid \varphi(xu_p) = \varphi(x) \|_{u_p}^{[-]}, \text{ for } u_p \in \text{Iw}_q \right\}.$$

Since the coefficient ring Λ is not a Banach algebra, S_{int}^D is not a Banach space in the literal sense. But as shown later in (2.11.1) and Proposition 2.17, at least when the tame level is neat, S_{int}^D is the completion of an (countably) infinite direct sum of Λ 's. Moreover, the topological space S_{int}^D carries a continuous action of U_p , defined using the same formula (2.5.1). (Here and later, we do not discuss the tame Hecke operators as we will not use them. Clearly, this space carries a natural action of the tame Hecke algebra; see for example [WXZ14⁺, §3.4].)

Note that Λ (or $\Lambda[\frac{1}{p}]$) is not an affinoid algebra, one needs to pass over an affinoid subdomain of \mathcal{W} to compare S_{int}^D with the usual space of overconvergent automorphic forms. More precisely, for $m_0 \in \mathbb{N}_{\geq 4}$, we write $\mathcal{W}_{m_0} := \mathcal{W}^{\leq p^{-1/p}m_0^{-4}}$ and use $[-]_{m_0} : \mathbb{Z}_p^\times \rightarrow \mathcal{O}_{\mathcal{W}_{m_0}}^\times$ to denote the induced universal character (the radius here is not optimal), which is m_0 -locally analytic. Then $S_{\text{int}}^D \widehat{\otimes}_\Lambda \mathcal{O}_{\mathcal{W}_{m_0}}$ contains $S_{[-]_{m_0}}^{D,\dagger} = \cup_{m \geq m_0} S_{[-]_{m_0}}^{D,\dagger,m}$ and hence $S_{[-]_{m_0}}^{D,\dagger,m_0}$ as subspaces.

Remark 2.8. In his thesis [Go88], F. Gouvêa proved that U_p is not a compact operator on the space of p -adic modular forms. Since S_{int}^D is an integral model of the space of p -adic automorphic forms for D , it is very likely that the U_p -action on S_{int}^D is not compact in the sense of Definition 5.1. However, a key observation of this work is that one can still define the characteristic power series for the U_p -action with respect to a carefully chosen orthonormal basis of S_{int}^D . See Theorem 3.16 and the discussion in Section 5.

Remark 2.9. The Λ -dual space of S_{int}^D may be identified with $P(\mathbb{Z}_p) = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ -coinvariants of Emerton's completed homology of the Shimura variety associated to D^\times :

$$\tilde{S}^D := \left(\varprojlim_{m \rightarrow \infty} \mathbb{Z}_p [D^\times \backslash (D \otimes \mathbb{A}_f)^\times / K^p K_{p,m}] \right)_{P(\mathbb{Z}_p)},$$

where $K_{p,m} := (1 + p^m M_2(\mathbb{Z}_p))^\times$. (Rigorously speaking, the U_p -action on both sides might be normalized slightly differently; this is because we chose to follow the convention of Buzzard [Bu04].) More naturally, we should have taken the invariants under $N(\mathbb{Z}_p)$, which would lead to a theory of eigensurface over $\mathcal{W} \times \mathcal{W}$ "homogeneous" along one factor of \mathcal{W} . But it is custom to simplify the picture by taking a "slice" of the eigensurface to study the eigencurve.

For general algebraic groups G over \mathbb{Q} which is quasi-split at p and compact modulo center at the archimedean place, one can construct the corresponding integral model by taking the coinvariants of the completed homology of the Shimura variety under the unipotent subgroup of the chosen Borel subgroup at p , or equivalently consider a similar induced representation. These two viewpoints are essentially the same, as explained in [Lo11, Section 3.10]. More generally, it might be possible to extend this construction to general G by looking at the completed homology groups of the associated locally symmetric space.

We also point out that our construction is closely tied to the étale (or Betti) realization of the eigencurve or the corresponding automorphic forms. One can also realize the space of automorphic forms using their de Rham realization, as in the original construction of Coleman–Mazur [CM98] and Andreatta–Iovita–Pilloni–Stevens [AIS14, Pi13]. In the recent work of Andreatta, Iovita, and Pilloni [AIP15⁺], they constructed some integral model of the space of overconvergent Hilbert modular forms, developing an idea of Coleman [Cole-A] (see also Remark 5.6).

Hypothesis 2.10. For simplicity, from now on we assume that K^p is neat. But we shall insert discussions throughout on reducing the argument from the general case to the neat case.

2.11. Explicit presentation of (overconvergent) automorphic forms. One can give an explicit description of the space of (overconvergent) automorphic forms. For this, we decompose $(D \otimes \mathbb{A}_f)^\times$ into (a disjoint union of) double cosets $\coprod_{i=0}^{t-1} D^\times \gamma_i K^p \text{Iw}_q$, for some elements $\gamma_0, \gamma_1, \dots, \gamma_{t-1} \in (D \otimes \mathbb{A}_f)^\times$. The condition (Neat) implies that the natural map $D^\times \times K^p \text{Iw}_q \rightarrow D^\times \gamma_i K^p \text{Iw}_q$ for each i sending (δ, u) to $\delta \gamma_i u$ is bijective. Since D^\times is dense in $(D \otimes \mathbb{Q}_p)^\times$, we may and will take each γ_i so that its p -component $\gamma_{i,p}$ is just 1.³

Evaluating each function φ at these chosen γ_i 's, we have an explicit description of various spaces of automorphic forms:

$$(2.11.1) \quad \begin{aligned} S_{\text{int}}^D &\xrightarrow{\cong} \bigoplus_{i=0}^{t-1} \mathcal{C}(\mathbb{Z}_p; \Lambda) \\ S_\chi^{D, \dagger, m} &\xrightarrow{\cong} \bigoplus_{i=0}^{t-1} \mathcal{O}_{\mathbb{B}_{q^p-m}} \widehat{\otimes}_{\mathbb{Q}_p} A \\ \varphi &\longmapsto (\varphi(\gamma_i))_{i=0, \dots, t-1}, \end{aligned}$$

where $\chi : \mathbb{Z}_p^\times \rightarrow A^\times$ is an m -analytic character of \mathbb{Z}_p^\times with values in an affinoid \mathbb{Q}_p -algebra A .

2.12. Characteristic power series. From the explicit presentation (2.11.1), we see that $S_\chi^{D, \dagger, m}$ is an orthonormalizable Banach A -module, i.e. there exists an orthonormal basis $(e_i)_{i \in \mathbb{Z}_{\geq 0}}$ such that $S_\chi^{D, \dagger, m} \cong \widehat{\bigoplus}_{i \geq 0} A e_i$. With respect to this basis, the action of U_p is given by an infinite matrix, say P . Moreover, the action of U_p is compact (see e.g. [Bu07, Lemma 12.2]), namely, it is a uniform limit of A -linear operators whose images are finite A -modules. We define the *characteristic power series* of the U_p -action on $S_\chi^{D, \dagger, m}$ to be

$$\text{Char}(U_p; S_\chi^{D, \dagger, m}) := \det(I_\infty - XP) \in A[[X]].$$

This power series converges and does not depend on the choice of the orthonormal basis $(e_i)_{i \in \mathbb{Z}_{\geq 0}}$.

Definition 2.13. For $r \in (0, 1) \cap p^\mathbb{Q}$, let $[-]_{\leq r} : \mathbb{Z}_p^\times \rightarrow \mathcal{O}_{\mathbb{W}^{\leq r}}^\times$ denote the universal character. Choose $m \in \mathbb{N}$ such that $r < p^{-q/p^m(p-1)}$ so that the universal character is m -locally analytic. The *spectral curve* $\text{Spc}_D^{\leq r}$ over $\mathcal{W}^{\leq r}$ is defined to be the (scheme theoretic) zero locus of the characteristic power series $\text{Char}(U_p; S_{[-]_{\leq r}}^{D, \dagger, m})$ inside $\mathcal{W}^{\leq r} \times \mathbb{G}_m^{\text{rig}}$. By [Bu04, Lemma 4], $\text{Char}(U_p; S_{[-]_{\leq r}}^{D, \dagger, m})$ and hence $\text{Spc}_D^{\leq r}$ does not depend on the choice of m , and it is compatible

³It was pointed out to us by Emerton and Ren (independently) that the assumption $\gamma_{i,p} = 1$ is not essentially needed to prove Proposition 3.1(3) later.

as r varies. We put $\mathrm{Spc}_D := \cup_{r \in (0,1)} \mathrm{Spc}_D^{\leq r}$ to be the *spectral curve* with tame level K^p . The natural projection wt to weight space \mathcal{W} is called the *weight map*. The composition of $x \mapsto x^{-1}$ on $\mathbb{G}_m^{\mathrm{rig}}$ and the natural projection $\mathrm{Spc}_D \rightarrow \mathbb{G}_m^{\mathrm{rig}}$ is called the *slope map* and denoted by $a_p : \mathrm{Spc}_D \rightarrow \mathbb{G}_m^{\mathrm{rig}}$.

Remark 2.14. One may reduce the non-neat case to the neat case as follows. Choose a split prime $l \geq 3$ of D different from p , and intersect K^p with the full level l structure. The spectral curve for this new tame level structure (which is neat) is endowed with a $\mathrm{GL}_2(\mathbb{Z}/l\mathbb{Z})$ -action which commutes with the weight map. Since $|\mathrm{GL}_2(\mathbb{Z}/l\mathbb{Z})|$ is invertible on weight space, the spectral curve for K^p is a union of connected components of the spectral curve for this new tame level.

We record the following classicality result for later use. It establishes a basic link between the classical theory of automorphic forms and the theory of the overconvergent ones.

Proposition 2.15 ([Bu04, Proposition 4]). *Let $\chi = (k, \psi) : \mathbb{Z}_p^\times \rightarrow E^\times$ be a classical character of conductor p^m . Let $0 \neq \varphi \in S_{\chi}^{D, \dagger}$ be an eigenvector for U_p with non-zero eigenvalue λ .*

- If $v(\lambda) < k + 1$, then φ is classical, i.e. $\varphi \in S_{k+2}^D(K^p \mathrm{Iw}_{p^m}; \psi)$.
- If $v(\lambda) > k + 1$, then $\varphi \notin S_{k+2}^D(K^p \mathrm{Iw}_{p^m}; \psi)$.

2.16. One-variable p -adic analysis. Before proceeding, we need some one-variable p -adic analysis, as developed by P. Colmez [Colm10].

Recall that $\mathcal{C}(\mathbb{Z}_p; \mathbb{Z}_p)$ carries a supremum norm:

$$\text{for } f \in \mathcal{C}(\mathbb{Z}_p; \mathbb{Z}_p), \quad |f|_{\mathbb{Z}_p} := \max_{z \in \mathbb{Z}_p} |f(z)|.$$

The functions $1, z, \dots, \binom{z}{n}, \dots$ form an orthonormal basis of $\mathcal{C}(\mathbb{Z}_p; \mathbb{Z}_p)$, called the *Mahler basis*. In other words, any $f \in \mathcal{C}(\mathbb{Z}_p; \mathbb{Z}_p)$ admits a *Mahler expansion*:

$$(2.16.1) \quad f(z) = \sum_{n \geq 0} a_n \binom{z}{n}, \quad \text{where all } a_n \in \mathbb{Z}_p; \quad \text{and } \lim_{n \rightarrow \infty} |a_n| = 0.$$

For $m \geq v(q)$, recall that $\mathcal{O}_{\mathbf{B}_{qp^{-m}}}$ is the subspace of $\mathcal{C}(\mathbb{Z}_p; \mathbb{Q}_p)$ consisting of those continuous functions that admits a local Taylor expansion over $a + q^{-1}p^m\mathbb{Z}_p$ for each $a \in \mathbb{Z}_p$. It carries a natural norm

$$\text{for } f \in \mathcal{O}_{\mathbf{B}_{qp^{-m}}}, \quad |f|_{qp^{-m}, \mathrm{an}} := \max_{z \in \mathbf{B}_{qp^{-m}}(\mathbb{C}_p)} |f(z)|.$$

By [Colm10, Théorème 1.29], the functions

$$\left\lfloor \frac{n}{q^{-1}p^m} \right\rfloor! \cdot \binom{z}{n}; \quad n \in \mathbb{Z}_{\geq 0}$$

form an orthonormal basis of $\mathcal{O}_{\mathbf{B}_{qp^{-m}}}$ for the norm $|\cdot|_{qp^{-m}, \mathrm{an}}$. Note that this orthonormal basis is not the one we commonly use in the context of studying $S_{[-]_m}^{D, \dagger, m}$ (e.g. in [Bu04] and in [WXZ14⁺]).

In the rest of the paper, we always equip $\Lambda[[X]]$ with the \mathfrak{m}_Λ -adic topology.

Proposition 2.17. *Using the isomorphism (2.11.1), the space S_{int}^D admits an orthonormal basis (over Λ) given by*

$$1_0, \dots, 1_{t-1}, z_0, \dots, z_{t-1}, \binom{z_0}{2}, \dots, \binom{z_i}{2}, \dots,$$

where the subscript i indicates that the term comes from the i th direct summand. Let $P = (P_{m,n})_{m,n \in \mathbb{Z}_{\geq 0}}$ denote the corresponding infinite matrix for the U_p -action, with coefficients in Λ . Suppose that the (uniform) limit of power series

$$\text{Char}(P) := \det(I_\infty - XP) = \lim_{n \rightarrow \infty} \det \left(I_n - X(P_{i,j})_{i,j=0,\dots,n-1} \right) \in \Lambda[[X]]$$

exists (which we shall prove in Theorem 3.16). Then it agrees with the characteristic power series of the U_p -action on each $S_{[-]_m}^{D,\dagger,m}$.

Proof. By [Colm10, Théorème 1.29] we cited above, the functions

$$\left\lfloor \frac{n}{q^{-1}p^m} \right\rfloor! \cdot \binom{z_i}{n} \text{ for } i = 0, \dots, t-1 \text{ and } n \in \mathbb{Z}_{\geq 0}$$

form an orthonormal basis of $S_{[-]_m}^{D,\dagger,m}$. If P' denotes the infinite matrix of U_p -action on this basis, then P and P' are conjugated by an infinite diagonal matrix with diagonal entries

$$\underbrace{1, \dots, 1}_t, \underbrace{1, \dots, 1}_t, \dots, \underbrace{\left\lfloor \frac{n}{q^{-1}p^m} \right\rfloor!, \dots, \left\lfloor \frac{n}{q^{-1}p^m} \right\rfloor!}_t, \dots$$

So taking the limit of the characteristic polynomial of the first $r \times r$ -minors, as r goes to infinity gives

$$\text{Char}(U_p; S_{[-]_m}^{D,\dagger,m}) = \det(I_\infty - XP') = \det(I_\infty - XP),$$

provided that the latter is well defined. □

Remark 2.18. Once again, we point out that the definition of $\text{Char}(P)$ depends on the choice of the orthonormal basis; in particular, it a priori depends on the choice of the coset representatives (see Remark 5.3), if we had not shown that it agrees with the characteristic power series $\text{Char}(U_p; S_{[-]_m}^{D,\dagger,m})$. See Section 5 for more discussion.

3. ESTIMATION OF THE NEWTON POLYGON

The advantage of working with a definite quaternion algebra is that the action of the U_p -operators may be written in a relatively explicit form. This was first observed by Buzzard and carried out by his student Jacobs [Ja04] (in one example), and later carefully optimized by the second and third authors and Zhang in [WXZ14⁺].

In this section, we will first revisit this explicit presentation of the U_p -operator. Then we give an estimate of the explicit formula for the U_p -operator and provide a lower bound of the Newton polygon for the U_p -action that is valid over weight space $\mathcal{W}^{>1/p}$. Luckily, this lower bound agrees with the actual Newton polygon at infinitely many points. Theorem 1.3 follows from a careful analysis of the Newton polygon at these points, as proved at the end of this section.

Proposition 3.1. *In terms of the isomorphism (2.11.1), the U_p -operator on S_{int}^D can be described by the following commutative diagram.*

$$\begin{array}{ccc} S_{\text{int}}^D & \xrightarrow{\varphi \mapsto (\varphi(\gamma_i))} & \bigoplus_{i=0}^{t-1} \mathcal{C}(\mathbb{Z}_p; \Lambda) \\ \varphi \mapsto U_p \varphi \downarrow & & \downarrow \mathfrak{U}_p \\ S_{\text{int}}^D & \xrightarrow{\varphi \mapsto (\varphi(\gamma_i))} & \bigoplus_{i=0}^{t-1} \mathcal{C}(\mathbb{Z}_p; \Lambda). \end{array}$$

Here the right vertical arrow \mathfrak{U}_p is given by a $t \times t$ matrix with the following description.

- (1) Each entry of \mathfrak{U}_p is a sum of operators of the form $\|_{\delta_p}^{[-]}$, where δ_p is the p -component of a global element $\delta \in D^\times$.
- (2) There are exactly p such operators appearing in each row and each column of \mathfrak{U}_p .
- (3) Each δ_p appearing above belongs to $\begin{pmatrix} p\mathbb{Z}_p & \mathbb{Z}_p \\ q\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$.

Proof. We reproduce the proof from [WXZ14⁺, Proposition 4.4] to make this paper more self-contained. For each γ_i , we have

$$(U_p \varphi)(\gamma_i) = \sum_{j=0}^{p-1} \varphi(\gamma_i v_j^{-1}) \|_{v_j}^{[-]}.$$

Write each $\gamma_i v_j^{-1}$ uniquely as $\delta_{i,j}^{-1} \gamma_{\lambda_{i,j}} u_{i,j}$ with $\delta_{i,j} \in D^\times$, $\lambda_{i,j} \in \{0, \dots, t-1\}$, and $u_{i,j} \in K^p \text{Iw}_q$. Then we have

$$(U_p \varphi)(\gamma_i) = \sum_{j=0}^{p-1} \varphi(\delta_{i,j}^{-1} \gamma_{\lambda_{i,j}} u_{i,j}) \|_{v_j}^{[-]} = \sum_{j=0}^{p-1} \varphi(\gamma_{\lambda_{i,j}}) \|_{u_{i,j,p} v_j}^{[-]},$$

where $u_{i,j,p}$ is the p -component of $u_{i,j}$. Substitute back in $u_{i,j} v_j = \gamma_{\lambda_{i,j}}^{-1} \delta_{i,j} \gamma_i$ and note the fact that both γ_i and $\gamma_{\lambda_{i,j}}$ have trivial p -component by our choice in Subsection 2.11. We have

$$(U_p \varphi)(\gamma_i) = \sum_{j=0}^{p-1} \varphi(\gamma_{\lambda_{i,j}}) \|_{\delta_{i,j,p}}^{[-]},$$

where $\delta_{i,j,p}$ is the p -component of the global element $\delta_{i,j} \in D^\times$. We now check the description of each $\delta_{i,j,p}$:

$$\delta_{i,j,p} = u_{i,j,p} v_j \in \text{Iw}_q \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \text{Iw}_q \subseteq \begin{pmatrix} p\mathbb{Z}_p & \mathbb{Z}_p \\ q\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}.$$

This concludes the proof of the proposition. □

Remark 3.2. By choosing the representatives γ_i 's more carefully, one can ensure that each global element δ appearing above has norm exactly p . This was used in a somewhat crucial way in [WXZ14⁺].

Now, to understand the action of the U_p -operator, it suffices to understand the action of $\|_{\delta_p}^{[-]}$ on $\mathcal{C}(\mathbb{Z}_p; \Lambda)$ for each $\delta_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} p\mathbb{Z}_p & \mathbb{Z}_p \\ q\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$. For later use, we will generalize our discussion to all δ_p in the monoid \mathbf{M}_1 (as defined in (2.3.3)).

3.3. More Mahler expansions. Recall that every function $f \in \mathcal{C}(\mathbb{Z}_p; \mathbb{Z}_p)$ admits a Mahler expansion $f(z) = \sum_{n \geq 0} a_n(f) \binom{z}{n}$ with $a_n(f) \in \mathbb{Z}_p$ and $\lim_{n \rightarrow \infty} a_n(f) = 0$. These *Mahler coefficients* $a_n(f)$ can be determined by the following process: for $f(z) \in \mathcal{C}(\mathbb{Z}_p; \mathbb{Z}_p)$, we write

$$\tilde{\Delta}(f)(z) = \tilde{\Delta}^{(1)}(f)(z) = f(z+1) - f(z) \text{ and } \tilde{\Delta}^{(m+1)}(f)(z) = \tilde{\Delta}(\tilde{\Delta}^{(m)}(f))(z) \text{ for } m \in \mathbb{N}.$$

Set $\tilde{\Delta}^{(0)}(f) = f$. Then for $m \in \mathbb{Z}_{\geq 0}$, we have

$$(3.3.1) \quad a_m(f) = \tilde{\Delta}^{(m)}(f)(0) = \sum_{i=0}^m (-1)^{m-i} \binom{m}{i} f(i).$$

Proposition 3.4. *Consider the action of $\delta_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{M}_1$ on $\mathcal{C}(\mathbb{Z}_p; \Lambda)$. Let $P(\delta_p) = (P_{m,n}(\delta_p))_{m,n \in \mathbb{Z}_{\geq 0}}$ denote the infinite matrix for this action with respect to the orthonormal Mahler basis $1, z, \binom{z}{2}, \dots$. Then*

$$(3.4.1) \quad P_{m,n}(\delta_p) = \tilde{\Delta}^{(m)} \left(\binom{(az+b)/(cz+d)}{n} \cdot [(cz+d)] \right) \Big|_{z=0}.$$

Proof. For a function $f \in \mathcal{C}(\mathbb{Z}_p; \Lambda)$, we write its Mahler coefficients $a_n(f) \in \Lambda$ as an infinite column vector. So the entry $P_{m,n}(\delta_p)$ corresponds to the m th Mahler coefficient of the function

$$\binom{z}{n} \Big\| \begin{bmatrix} - \\ a & b \\ c & d \end{bmatrix} = \binom{(az+b)/(cz+d)}{n} \cdot [(cz+d)].$$

Note that the universal character $[-]$ cannot be applied to the formal expression $cz+d$; but $[cz+d]$ still makes sense as a continuous function on \mathbb{Z}_p with values in Λ . Comparing with (3.3.1), we have

$$P_{m,n}(\delta_p) = a_m \left(\binom{(az+b)/(cz+d)}{n} \cdot [(cz+d)] \right) = \tilde{\Delta}^{(m)} \left(\binom{(az+b)/(cz+d)}{n} \cdot [(cz+d)] \right) \Big|_{z=0}. \quad \square$$

3.5. More p -adic analysis. We start by listing the following three useful equalities, which can be checked easily.

$$(3.5.1) \quad \tilde{\Delta}^{(m)}(fg)(z) = \sum_{i=0}^m \binom{m}{i} \tilde{\Delta}^{(m-i)}(f)(z+i) \tilde{\Delta}^{(i)}(g)(z).$$

$$(3.5.2) \quad \tilde{\Delta}^{(m)} \binom{z}{n} = \begin{cases} \binom{z}{n-m} & \text{if } n \geq m \\ 0 & \text{otherwise.} \end{cases}$$

$$(3.5.3) \quad \binom{x_1 + \dots + x_r}{n} = \sum_{i_1 + \dots + i_r = n} \binom{x_1}{i_1} \dots \binom{x_r}{i_r}.$$

As a corollary of (3.5.2), if $f = \sum_{n \geq 0} a_n(f) \binom{z}{n}$ is the Mahler expansion of a function $f \in \mathcal{C}(\mathbb{Z}_p; \mathbb{Z}_p)$, then

$$(3.5.4) \quad \tilde{\Delta}^{(m)}(f)(z) = \sum_{n \geq m} a_n(f) \binom{z}{n-m}.$$

Definition 3.6. We say a continuous function $f \in \mathcal{C}(\mathbb{Z}_p; \mathbb{Z}_p)$ is a *polynomial function of degree $\leq n$* (for $n \in \mathbb{Z}_{\geq 0}$) if the Mahler coefficients $a_i(f) = 0$ for $i > n$.

- Lemma 3.7.** (1) A continuous function $f \in \mathcal{C}(\mathbb{Z}_p; \mathbb{Z}_p)$ is a polynomial function of degree $\leq n$ if and only if $\tilde{\Delta}^{(n+1)}(f) = 0$.
(2) If f and g are polynomial functions on \mathbb{Z}_p of degree $\leq n$ and $\leq m$, respectively, then fg is a polynomial function of degree $\leq m + n$.
(3) If $f \in \mathcal{C}(\mathbb{Z}_p; \mathbb{Z}_p)$ is a polynomial function of degree $\leq r$, then $\binom{f(z)}{n}$ is a polynomial function of degree $\leq rn$.

Proof. (1) If $f(z) = \sum_{i \geq 0} a_i(f) \binom{z}{i}$ is the Mahler expansion of f , then

$$\tilde{\Delta}^{(n+1)}(f) = \sum_{i \geq n+1} a_i(f) \binom{z}{i-n-1}.$$

Then $\tilde{\Delta}^{(n+1)}(f) = 0$ if and only if $a_i = 0$ for all $i \geq n + 1$. (1) is proved.

The rest of the lemma follows immediately from the fact that the polynomial functions of degree $\leq n$ are precisely the polynomials in $\mathbb{Q}_p[z]$ of degree at most n which map \mathbb{Z}_p to \mathbb{Z}_p . \square

In addition to the degree, the following convenient definition is tailored for our computation.

Definition-Proposition 3.8. We say a continuous function $f \in \mathcal{C}(\mathbb{Z}_p; \mathbb{Z}_p)$ has *tilted degree $\leq n$* (for $n \in \mathbb{Z}_{\geq 0}$) if the following equivalent conditions are satisfied:

- (1) for any $m \in \mathbb{N}$, $\tilde{\Delta}^{(m)}(f)$ is a (continuous) function on \mathbb{Z}_p that takes value in $p^{m-n}\mathbb{Z}_p$;
(2) writing down the Mahler expansion of $f(z) = \sum_{j \geq 0} a_j(f) \binom{z}{j}$, then $v(a_j(f)) \geq j - n$.

Note that the assumption $f \in \mathcal{C}(\mathbb{Z}_p; \mathbb{Z}_p)$ implies that condition (1) for $m \leq n$ and condition (2) for $j \leq n$ hold automatically.

Proof. Let $f(z) = \sum_{j \geq 0} a_j(f) \binom{z}{j}$ be the Mahler expansion of f . By (3.5.4), $\tilde{\Delta}^{(m)}(f) = \sum_{n \geq m} a_n(f) \binom{z}{n-m}$. Since the Mahler basis forms an orthonormal basis of $\mathcal{C}(\mathbb{Z}_p; \mathbb{Z}_p)$, $\tilde{\Delta}^{(m)}(f)$ takes value in $p^{m-n}\mathbb{Z}_p$ if and only if $a_j(f) \in p^{m-n}\mathbb{Z}_p$ for all $j \geq m \geq n$, which is equivalent to $v(a_j(f)) \geq j - n$ for all $j \geq n$. \square

Remark 3.9. If $f \in \mathcal{C}(\mathbb{Z}_p; \mathbb{Z}_p)$ is a polynomial function of degree $\leq n$, then it has tilted degree $\leq n$.

- Lemma 3.10.** (1) If $f \in \mathcal{C}(\mathbb{Z}_p; \mathbb{Z}_p)$ and $n \in \mathbb{N}$, then f has tilted degree $\leq n$ if and only if $\tilde{\Delta}(f)$ has tilted degree $\leq n - 1$.
(2) If f and g are \mathbb{Z}_p -valued continuous functions on \mathbb{Z}_p of tilted degree $\leq m$ and $\leq n$, respectively. Then fg has tilted degree $\leq m + n$.

Proof. (1) is clear from Definition-Proposition 3.8(2) because $a_j(f) = a_{j-1}(\tilde{\Delta}(f))$.

(2) We check it using Definition-Proposition 3.8(1). By (3.5.1), we have

$$\tilde{\Delta}^{(r)}(fg)(z) = \sum_{i=0}^r \binom{r}{i} \tilde{\Delta}^{(r-i)}(f)(z+i) \tilde{\Delta}^{(i)}(g)(z).$$

Each term on the right hand side has valuation at least $(r - i - m) + (i - n) = r - m - n$. So fg has tilted degree $\leq m + n$. \square

To understand the expression (3.4.1), we need the following estimates.

Lemma 3.11. *Let $g(z) = b_0 + b_1z + \cdots + b_rz^r \in \mathbb{Z}_p[z]$. If there exists some $s \in \mathbb{R}_{\geq 0}$ such that $v(b_i) \geq is$ for all i , then we can rewrite $g(z)$ as*

$$\sum_{i=0}^r c_i \cdot i! \binom{z}{i}$$

with $v(c_i) \geq is$ for all i .

Proof. : If we compare the coefficients of each degree, we see that

$$\begin{aligned} b_r &= c_r, \\ b_{r-1} &= c_{r-1} + c_r \alpha_{r,r-1}, \\ b_{r-2} &= c_{r-2} + c_{r-1} \alpha_{r-1,r-2} + c_r \alpha_{r,r-2}, \\ &\dots\dots, \end{aligned}$$

where $\alpha_{i,j}$ are the coefficients of z^j in the product $z(z-1)\cdots(z-i+1)$, which is of course an integer. By reverse induction, we see that $v(b_r) \geq rs$ implies that of $v(c_r) \geq rs$, and $v(b_{r-1}) \geq (r-1)s$ implies that of c_{r-1}, \dots . This concludes the lemma. \square

Lemma 3.12. *For a function $f(z) = a_0 + pa_1z + p^2a_2z^2 + \cdots \in \mathbb{Z}_p[[pz]]$ and an integer $n \geq 0$, the expression $\binom{f(z)}{n}$ has tilted degree $\leq \lfloor \frac{n}{p} \rfloor$.*

Proof. By approximation, we may assume that $f(z)$ is a polynomial. Put

$$g(z) = f(z)(f(z) - 1)\cdots(f(z) - n + 1) \in \mathbb{Z}_p[pz]$$

so that $\binom{f(z)}{n} = \frac{1}{n!}g(z)$. By Lemma 3.11, we may rewrite $g(z)$ as

$$\sum_{k=0}^r c_k \cdot k! p^k \binom{z}{k}$$

with $v(c_k) \in \mathbb{Z}_p$ for all k .

Since $\binom{f(z)}{n}$ is a continuous \mathbb{Z}_p -valued function, it suffices to prove that when $k > \lfloor \frac{n}{p} \rfloor$, $v(k!p^k/n!) \geq k - \lfloor \frac{n}{p} \rfloor$. To this end, note that in this case $\lfloor \frac{k}{p^\ell} \rfloor \geq \lfloor \frac{n}{p^{\ell+1}} \rfloor$ for any $\ell \in \mathbb{N}$. Thus

$$v(k!p^k) = k + \lfloor \frac{k}{p} \rfloor + \lfloor \frac{k}{p^2} \rfloor + \cdots \geq k + \lfloor \frac{n}{p^2} \rfloor + \lfloor \frac{n}{p^3} \rfloor + \cdots = k - \lfloor \frac{n}{p} \rfloor + v(n!).$$

We are done. \square

Lemma 3.13. *For a function $f(z) = a_0 + a_1z + pa_2z^2/2 + p^2a_3z^3/3 + \cdots + p^{k-1}a_kz^k/k + \cdots$ with $a_n \in \mathbb{Z}_p$ and an integer $m \geq 0$, the expression $\binom{f(z)}{m}$ has tilted degree $\leq m$.*

Proof. By approximation, we may suppose that $f(z)$ is a polynomial function. Moreover, by the binomial identity (3.5.3) together with the additive property of the tilted degree (Lemma 3.10), we may assume that $f(z) = p^{n-1}a_nz^n/n$ is a monomial with $n \in \mathbb{N}$ (the case $f(z) = a_0$ is trivial). When $n = 1$, this follows from the easy bound: $\binom{a_1z}{m}$ is a polynomial function of degree m , so we are done by Remark 3.9. So we assume $n > 1$ for the rest of the proof.

Fix n and put $g(z) = f(z)(f(z) - 1) \cdots (f(z) + m - 1) \in \mathbb{Z}_p[\frac{p^{n-1}}{n}z^n]$; its degree is nm . By Lemma 3.11, we can rewrite $\binom{f(z)}{m} = g(z)/m!$ as

$$\sum_{k=0}^{nm} c_k k! p^{\frac{n-1-v(n)}{n}k} / m! \cdot \binom{z}{k},$$

where c_k belongs to $\mathbb{Z}_p[p^{1/n}]$. Now we hope to show that for any $k \leq mn$,

$$(3.13.1) \quad v(c_k k! p^{\frac{n-1-v(n)}{n}k} / m!) \geq \max\{k - m, 0\}.$$

When $k \leq m$, this follows from the fact that $\binom{f(z)}{m}$ is a continuous \mathbb{Z}_p -valued function on \mathbb{Z}_p . So we may assume that $k > m$ (and $k \leq mn$ as the degree of $g(z)$ is just mn). Simplifying the terms of (3.13.1), we see that it suffices to show (by ignoring the contribution of $v(c_k)$)

$$(3.13.2) \quad v(k!) + m \geq v(m!) + \frac{1 + v(n)}{n}k.$$

We separate several cases:

- (a) If $p \nmid n$, then we need to show that $v(k!) + m \geq v(m!) + k/n$. But this is just a combination of $v(k!) \geq v(m!)$ (as $k \geq m$) and $m \geq k/n$ (as $k \leq nm$).
- (b) If $n \geq 2p$ and $(p, n) \neq (2, 4)$, then $\frac{1+v(n)}{n} \leq 1/p$. But $v(k!/m!) \geq \lfloor \frac{k-m}{p} \rfloor \geq \frac{k-m}{p} - \frac{p-1}{p}$. So (3.13.2) follows from

$$v(k!/m!) + m \geq \frac{k}{p} + \frac{p-1}{p}m - \frac{p-1}{p} \geq \frac{k}{p} \geq \frac{1+v(n)}{n}k.$$

- (c) If $n = p$, the inequality (3.13.2) might fail. So we have to go back to the beginning to show directly that $\binom{ap^{p-2}z^p}{m}$ has tilted degree $\leq m$, when $a \in \mathbb{Z}_p$. We prove this by induction on m ; the case of $m = 0$ is void. Now suppose this is proved for all numbers strictly less than m and we prove it for m . We need to show that

$$\tilde{\Delta} \left(\binom{ap^{p-2}z^p}{m} \right) \text{ has tilted degree } \leq m - 1.$$

We argue as in Lemma 3.7(3). Note that

$$\begin{aligned} \tilde{\Delta} \left(\binom{ap^{p-2}z^p}{m} \right) &= \binom{ap^{p-2}(z+1)^p}{m} - \binom{ap^{p-2}z^p}{m} \\ &\stackrel{(3.5.3)}{=} \sum_{j=1}^m \binom{ap^{p-2}((z+1)^p - z^p)}{j} \binom{ap^{p-2}z^p}{m-j}. \end{aligned}$$

Note that $ap^{p-2}((z+1)^p - z^p) \in \mathbb{Z}_p[pz]$; so Lemma 3.12 shows the first factor has tilted degree $\leq \lfloor \frac{j}{p} \rfloor \leq j-1$. The second factor has tilted degree $\leq m-j$ by induction. By Lemma 3.10, the sum has tilted degree $\leq m-1$. We are done.

- (d) If $(p, n) = (2, 4)$, we may proceed as in (c) to show directly that $\binom{2az^4}{m}$ has tilted degree $\leq m$. The only non-trivial part is the inductive step: we need to show that $\binom{2a(z+1)^4 - 2az^4}{j}$ has tilted degree $\leq j-1$ for $j \in \mathbb{N}$. To this end, note that $2a(z+1)^4 - 2az^4 = a(8z^3 + 12z^2 + 8z + 2) \in \mathbb{Z}_2[2z]$, and then apply Lemma 3.12.

□

Proposition 3.14. (1) When $\delta_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} p\mathbb{Z}_p & \mathbb{Z}_p \\ q\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$, the coefficient $P_{m,n}(\delta_p)$ belongs to

$$\mathfrak{m}_\Lambda^{\max\{m-\lfloor n/p \rfloor, 0\}}.$$

(2) When $\delta_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{M}_1$, the coefficient $P_{m,n}(\delta_p)$ belongs to

$$\mathfrak{m}_\Lambda^{\max\{m-n, 0\}}.$$

Proof. The coefficient $P_{m,n}(\delta_p)$ for sure belongs to Λ because the operator $||_{\delta_p}^{[-]}$ preserves $\mathcal{C}(\mathbb{Z}_p; \Lambda)$. This explains the max on the exponents.

Put $f(z) = (az + b)/(cz + d) \in \mathbb{Z}_p[[z]]$. Write $d = d_0 \cdot \langle d \rangle$ with $d_0 \in \Delta = (\mathbb{Z}/q\mathbb{Z})^\times$ and $\langle d \rangle \in (1 + q\mathbb{Z}_p)^\times$, so that $[cz + d]$ can be written as

$$[d_0] \cdot [(cz + d)/d_0] = [d_0] \cdot (1 + T)^{\log((cz+d)/d_0)/q}.$$

Put $g(z) = \log((cz + d)/d_0)/q$; it is of the form considered in Lemma 3.13. Now we have

$$\tilde{\Delta}^{(m)} \left(\left(\binom{(az+b)/(cz+d)}{n} [cz+d] \right) \Big|_{z=0} \right) = [d_0] \cdot \sum_{r \geq 0} T^r \cdot \tilde{\Delta}^{(m)} \left(\left(\binom{f(z)}{n} \binom{g(z)}{r} \right) \Big|_{z=0} \right).$$

- By Lemma 3.13, $\binom{g(z)}{r}$ has tilted degree $\leq r$.
- In case (1), we have $f(z) \in \mathbb{Z}_p[[pz]]$. So by Lemma 3.12, $\binom{f(z)}{n}$ has tilted degree $\leq \lfloor \frac{n}{p} \rfloor$.
- In case (2), note that $f(z)$ is of the form considered in Lemma 3.13. Thus $\binom{f(z)}{n}$ has tilted degree $\leq n$.

Using Lemma 3.10 again, we see that the tilted degree of $\binom{f(z)}{n} \binom{g(z)}{r}$ is $\leq r + \lfloor \frac{n}{p} \rfloor$ in case (1), and is $\leq r + n$ in case (2). So the T^r -coefficients of (3.4.1) has valuation at least

$$m - \left\lfloor \frac{n}{p} \right\rfloor - r \text{ in case (1), and } m - n - r \text{ in case (2).}$$

This is exactly what we need to prove. □

Lemma 3.15. *The rigid space associated to the ring*

$$\Lambda^{>1/p} := \Lambda[[pT^{-1}]] = \mathbb{Z}_p[[T, pT^{-1}]] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Delta]$$

is $\mathcal{W}^{>1/p}$. The ideal $\mathfrak{m}_\Lambda \Lambda^{>1/p}$ is the same as the principal ideal (T) .

Proof. This is clear, noting that $p = pT^{-1} \cdot T$. □

In the following, we will work over $\Lambda^{>1/p}$ instead.

Theorem 3.16. *Recall that $t = \#D^\times \setminus (D \otimes \Delta_f)^\times / K^p \text{Iw}_q$. Let $P = (P_{m,n})_{m,n \in \mathbb{Z}_{\geq 0}}$ denote the infinite matrix of the U_p -action with respect to the basis*

$$1_0, \dots, 1_{t-1}, z_0, \dots, z_{t-1}, \binom{z_0}{2}, \dots, \binom{z_i}{2}, \dots,$$

as in Proposition 2.17. The characteristic power series

$$\text{Char}(P) := \lim_{n \rightarrow \infty} \det(1 - X(P_{i,j})_{i,j=0,\dots,n-1}) = \sum_{n \geq 0} c_n X^n \in \Lambda[[X]]$$

is well defined. Moreover, we have

$$(3.16.1) \quad c_n \in T^{\lambda(n)} \cdot \Lambda^{>1/p} \text{ for } n \in \mathbb{Z}_{\geq 0},^4$$

where $\lambda(0) = 0, \lambda(1), \dots$ is a sequence of integers determined by

$$\lambda(i+1) - \lambda(i) = \left\lfloor \frac{i}{t} \right\rfloor - \left\lfloor \frac{i}{pt} \right\rfloor.$$

Proof. Combining the estimate in Proposition 3.14 and the explicit description of the U_p -operator in Proposition 3.1, we see that the (m, n) -entry of the infinite matrix P satisfies

$$(3.16.2) \quad P_{m,n} \in \mathfrak{m}_\Lambda^{\max\{\lfloor m/t \rfloor - \lfloor n/pt \rfloor, 0\}}.$$

In particular, modulo \mathfrak{m}_Λ^r for each $r \in \mathbb{N}$, the infinite matrix is strict upper triangular except the first $\lfloor ptr/(p-1) \rfloor \times \lfloor ptr/(p-1) \rfloor$ -minor. Then modulo \mathfrak{m}_Λ^r , the characteristic polynomial of the first $s \times s$ -minor is the same as the characteristic polynomial of the first $\lfloor ptr/(p-1) \rfloor \times \lfloor ptr/(p-1) \rfloor$ -minor for $s \geq \lfloor ptr/(p-1) \rfloor$. This implies that $\text{Char}(P) \in \Lambda[[X]]$ is well defined.

To compute $\text{Char}(P)$, we work with a bigger coefficient ring $\Lambda^{>1/p}$. So we have

$$(3.16.3) \quad \mathfrak{m}_\Lambda^a \cdot T^b \Lambda^{>1/p} = T^{a+b} \Lambda^{>1/p} \text{ for any integers } a, b \text{ such that } a, a+b \in \mathbb{Z}_{\geq 0}.$$

We now conjugate the matrix P by the infinite diagonal matrix whose diagonal entries are

$$\underbrace{1, \dots, 1}_t, \underbrace{T, \dots, T}_t, \underbrace{T^2, \dots, T^2}_t, \dots;$$

let $P' = (P'_{m,n})_{m,n \in \mathbb{Z}_{\geq 0}}$ denote the matrix we get this way. Then we have

$$P'_{m,n} \in \mathfrak{m}_\Lambda^{\max\{\lfloor m/t \rfloor - \lfloor n/pt \rfloor, 0\}} \cdot T^{\lfloor n/t \rfloor - \lfloor m/t \rfloor} \Lambda^{>1/p} \stackrel{(3.16.3)}{\subseteq} T^{\max\{\lfloor n/t \rfloor - \lfloor n/pt \rfloor, \lfloor n/t \rfloor - \lfloor m/t \rfloor\}} \Lambda^{>1/p}.$$

In particular, the entries of P' in the n -th column all lie in $T^{\lfloor n/t \rfloor - \lfloor n/pt \rfloor} \Lambda^{>1/p}$. So $\text{Char}(P) = \text{Char}(P')$ has the property given in (3.16.1). \square

Remark 3.17. Comparing with [WXZ14⁺], the major advantage of our estimate is that the basis we choose in this paper allows us to extend the estimate to the entire annuli, as opposed to just small disks near the boundary of weight space in [WXZ14⁺]. Moreover, we shall see later that the estimate in Theorem 3.16 is already *sharp* for infinitely many n . This magical fact allows us to deduce the strong Theorem 1.3.

By Proposition 2.17 (where the missing condition is checked by Theorem 3.16), the characteristic power series of U_p on the space of overconvergent automorphic forms $S_{[-]_m}^{D, \dagger, m}$ (for any m) is $\sum_{n \geq 0} c_n X^n \in \Lambda[[X]]$ with c_n bounded as in (3.16.1).

Now, we fix a character ω of Δ . By abuse of notation, we still use c_n to denote its image under the quotient map $\Lambda \rightarrow \mathbb{Z}_p[[T]]$ by evaluating Δ using ω . Write $c_n(T) = \sum_{m \geq 0} b_{n,m} T^m$ for $b_{n,m} \in \mathbb{Z}_p$.

⁴In the very recent preprint [JN16⁺] of Johansson and Newton, they gave a more conceptual proof of the estimate in this Theorem.

Corollary 3.18. *For $T \in \mathbb{C}_p$ with $0 < v(T) < 1$, we have $v(c_n(T)) \geq \lambda(n)v(T)$ for every $n \geq 0$, with equality holding if and only if $b_{n,\lambda(n)} \in \mathbb{Z}_p^\times$. Moreover, if $b_{n,\lambda(n)} \notin \mathbb{Z}_p^\times$, then*

$$v(c_n(T)) \geq \lambda(n)v(T) + \min\{v(T), 1 - v(T)\}.$$

As a consequence, the Newton polygon of $\sum_{n \geq 0} c_n(T)X^n$ always lies above the polygon with vertices $(n, \lambda(n)v(T))$ for all $n \geq 0$.

Proof. First note that if $\sum_{m \in \mathbb{Z}} d_m T^m \in \Lambda^{>1/p}$, then $v(d_m) \geq \max\{0, -m\}$. Combining this fact with (3.16.1), we get

$$v(b_{n,m}) \geq \max\{\lambda(n) - m, 0\}.$$

For $T \in \mathbb{C}_p$ with $0 < v(T) < 1$, we deduce

$$(3.18.1) \quad v(b_{n,m}T^m) \geq \max\{\lambda(n) - m, 0\} + mv(T) \geq \lambda(n)v(T),$$

with the second equality holding if and only if $m = \lambda(n)$. It follows that we always have $v(b_{n,m}T^m) \geq \lambda(n)v(T)$, with equality holding if and only if $m = \lambda(n)$ and $b_{n,\lambda(n)}$ is a p -adic unit in \mathbb{Z}_p . The rest of the corollary is clear. \square

We call the polygon given in Corollary 3.18 the *lower bound polygon* of $\sum_{n \geq 0} c_n(T)X^n$.

In the next we consider the ordinary locus of the (entire) spectral curve. In fact, it is very well known among the experts that the ordinary locus of the eigencurve is nothing but the Hida family of ordinary modular forms. Consequently, it should be finite and flat over weight space. Because of the lack of proper reference, we give a proof about this fact in our case as follows.

Theorem 3.19. *Let X^{ord} denote the locus of the spectral curve Sp_D where $a_p(x)$ has valuation zero. Then the following holds:*

- (1) X^{ord} is finite and flat over \mathcal{W} ;
- (2) X^{ord} is a union of connected components of Sp_D .

Proof. Applying Remark 2.14, it reduces to the case that tame level is neat. Then it suffices to prove this over each connected component of \mathcal{W} ; so we fix a character ω of Δ throughout and denote by $[-]_\omega$ the universal character. Let X_ω^{ord} denote the corresponding components. Recall that the characteristic power series of U_p on $S_{[-]_\omega}^{D,\dagger}$ is equal to $\sum_{n=0}^{\infty} c_n(T)X^n$. Let d be the maximal index such that $c_d(T)$ is a unit in $\mathbb{Z}_p[[T]]$, or equivalently, the constant term of $c_d(T)$ is a p -adic unit in \mathbb{Z}_p ; such a d must exist by Corollary 3.18. We claim that X_ω^{ord} is finite and flat of degree d over \mathcal{W}_ω . Indeed, for each weight character $\chi : \mathbb{Z}_p^\times \rightarrow \mathbb{C}_p^\times$ whose restriction to Δ is ω , the Newton polygon of $\sum_{n=0}^{\infty} c_n(T_\chi)X^n$ has slope zero in the first d segment and the vertices with x -coordinate strictly bigger than d must have y -coordinate at least $\min\{1, v(T_\chi)\}$, and the corresponding slopes eventually tend to infinity. This implies that X_ω^{ord} is finite and flat over \mathcal{W}_ω of degree d , and is an affinoid subdomain when restricted to the fiber of each affinoid subdomain of \mathcal{W}_ω . Moreover, the first non-zero slope of the Newton polygon at χ can be bounded away from 0 uniformly over any affinoid subdomain; so X^{ord} is disconnected from its complement. \square

Remark 3.20. For a classical weight (k, ψ) , its fiber in X^{ord} exactly corresponds to the ordinary part of the space of automorphic forms $S_{(k,\psi)}^{D,\dagger}$. By the Zariski density of classical weights in weight space, one can show that X^{ord} is the same as the spectral Hida family (e.g. by the standard control theorem for Hida families ([Hi02, Theorem 7.1(5)])).

Recall that $r_{\text{ord}}(\omega)$ denotes the dimension of the ordinary subspace of automorphic forms of weight 2 and character ω .

Corollary 3.21 (Hida). *The degree of X_{ω}^{ord} over \mathcal{W}_{ω} is $r_{\text{ord}}(\omega)$. As a consequence, for any integer $k \in \mathbb{Z}$ and any finite character ψ of conductor p^m , the slope zero subspace of $S_{(k,\psi)}^{D,\dagger}$ has dimension $r_{\text{ord}}(\psi|_{\Delta} \cdot \omega_0^k)$.*

Proof. In the course of the proof of Theorem 3.19, we already saw that the degree of X_{ω}^{ord} over \mathcal{W}_{ω} is equal to the dimension of the ordinary subspace of $S_{\chi}^{D,\dagger}(K^p)$ for every weight character $\chi \in \mathcal{W}_{\omega}$. In particular, we can choose $\chi = \omega$. The rest part follows from the fact that (k, ψ) belongs to the weight disk corresponding to $\psi|_{\Delta} \cdot \omega_0^k$. \square

Let $m \in \mathbb{N}_{\geq 2}$, and let ψ be a finite character of conductor p^m . For $k \in \mathbb{Z}_{\geq 0}$, using an isomorphism analogous to (2.11.1), we see that $S_{k+2}^D(K^p \text{Iw}_{p^m}; \psi)$ is isomorphic to the direct sum of t copies of $\text{LP}^{m-v(q), \text{deg} \leq k}(\mathbb{Z}_p; E)$. So in total,

$$(3.21.1) \quad \dim S_{k+2}^D(K^p \text{Iw}_{p^m}; \psi) = (k+1)q^{-1}p^m t.$$

To prove Theorem 1.3, we also need the following result on Atkin–Lehner theory.

Proposition 3.22 (Atkin–Lehner). *We use $\alpha_0(\psi), \dots, \alpha_{(k+1)q^{-1}p^m t - 1}(\psi)$ to denote the slopes of U_p acting on $S_{k+2}^D(K^p \text{Iw}_{p^m}, \psi)$ in non-decreasing order. Then we have*

$$\alpha_i(\psi) = k + 1 - \alpha_{(k+1)q^{-1}p^m t - 1 - i}(\psi^{-1}) \quad \text{for } i = 0, \dots, (k+1)q^{-1}p^m t - 1.$$

In particular, the total sum of the U_p -slopes of $S_{k+2}^D(K^p \text{Iw}_{p^m}; \psi) \oplus S_{k+2}^D(K^p \text{Iw}_{p^m}; \psi^{-1})$ is $(k+1)^2 q^{-1} p^m t$.

Proof. This is well known to the experts. We here give a proof as follows. Firstly note that the base change of $S_{k+2}^D(K^p \text{Iw}_{p^m}; \psi)$ to \mathbb{C} is isomorphic to the corresponding classical space of automorphic forms for D . Since ψ has conductor p^m while the level structure at p is Iw_{p^m} , by applying Jacquet–Langlands and [LW12, Proposition 2.8], we see that for every automorphic representation π appearing in $S_{k+2}^D(K^p \text{Iw}_{p^m}; \psi)$, its p -component π_p is a principal series of $\text{GL}_2(\mathbb{Q}_p)$ whose corresponding two characters of \mathbb{Q}_p^{\times} are $\text{unr}(\alpha)$ and $\text{unr}(\alpha^{-1}) \otimes \omega_p$, where $\text{unr}(?)$ is an unramified character of \mathbb{Q}_p^{\times} sending p to $?$ and ω_p is the p -component of the Hecke character associated to ψ . Moreover, the U_p -eigenvalue on the Iw_{p^m} fixed vector of π_p is $\alpha p^{(k+1)/2}$. But we can twist the representation π by a central Hecke character associated to ψ^{-1} ; then the resulting automorphic representation would appear in $S_{k+2}^D(K^p \text{Iw}_{p^m}; \psi^{-1})$ (or rather its base change to \mathbb{C}) instead. Moreover, the U_p -eigenvalue on the Iw_{p^m} fixed vector of the p -component of this automorphic representation is $\alpha^{-1} p^{(k+1)/2}$. In conclusion, one can pair the U_p -eigenvalues of $S_{k+2}^D(K^p \text{Iw}_{p^m}; \psi)$ and the U_p -eigenvalues of $S_{k+2}^D(K^p \text{Iw}_{p^m}; \psi^{-1})$ so that they multiply to p^{k+1} . Our assertion on slopes follows from this. \square

3.23. Proof of Theorem 1.3. By virtue of Remark 2.14, we only need to treat the case that the tame level is neat. To this end, it suffices to get the desired decomposition for each $\text{Sp}_{D,\omega}^{>1/p}$. We consider the classical weights $\chi_k = (k, \psi)$ of conductor q^2 with $k \in \mathbb{Z}_{\geq 0}$, such that $\chi_k|_{\Delta} = \omega$. The corresponding T -coordinates T_{χ_k} have valuation $\frac{q}{\varphi(q^2)} = \frac{p}{q(p-1)} < 1$. The space of overconvergent automorphic forms $S_{\chi_k}^{D,\dagger}$ contains the subspace $S_{k+2}^D(K^p \text{Iw}_{q^2}; \psi)$ of classical automorphic forms. Put $n_k = kqt$ for $k \in \mathbb{Z}_{\geq 0}$. By (3.21.1), we have

$$\dim S_{k+2}^D(K^p \text{Iw}_{q^2}; \psi) = (k+1)qt = n_{k+1}.$$

Step I: The first important observation is that the Newton polygon of $\sum_{n \geq 0} c_n(T_{\chi_k})X^n$ touches the lower bound polygon at the points

$$\mathcal{P}_k := (n_{k+1}, \lambda(n_{k+1})v(T_{\chi_k})).$$

On the one hand, Proposition 3.22 says that the sum of all U_p -slopes on $S_{k+2}^D(K^p \text{Iw}_{q^2}; \psi) \oplus S_{k+2}^D(K^p \text{Iw}_{q^2}; \psi^{-1})$ is $(k+1)^2 qt$. By Proposition 2.15, one deduces that the set of all U_p -slopes on $S_{k+2}^D(K^p \text{Iw}_{q^2}; \psi) \oplus S_{k+2}^D(K^p \text{Iw}_{q^2}; \psi^{-1})$ is exactly the set of the first n_{k+1} U_p -slopes in each of $S_{(k,\psi)}^{D,\dagger}$ and $S_{(k,\psi^{-1})}^{D,\dagger}$. It follows that the sum of the first n_{k+1} U_p -slopes in each of $S_{(k,\psi)}^{D,\dagger}$ and $S_{(k,\psi^{-1})}^{D,\dagger}$ is also $(k+1)^2 qt$.

On the other hand, for each of $T_{(k,\chi)}$ and $T_{(k,\chi^{-1})}$, when the x -coordinate is $(k+1)qt$, the y -coordinate of the lower bound polygon is

$$\begin{aligned} \frac{p}{q(p-1)} \lambda((k+1)qt) &= \frac{p}{q(p-1)} \sum_{n=0}^{(k+1)qt-1} \left(\left\lfloor \frac{n}{t} \right\rfloor - \left\lfloor \frac{n}{pt} \right\rfloor \right) \\ (3.23.1) \quad &= \frac{p}{q(p-1)} \left(t \sum_{n=0}^{(k+1)q-1} n - pt \sum_{n=0}^{\frac{(k+1)q}{p}-1} n \right) \\ &= \frac{(k+1)^2 qt}{2}. \end{aligned}$$

This exactly agrees (!) with half of the sum of the first n_{k+1} U_p -slopes on each of $S_{(k,\psi)}^{D,\dagger}$ and $S_{(k,\psi^{-1})}^{D,\dagger}$. In particular, we see that the sum of first n_{k+1} U_p -slopes on $S_{(k,\psi)}^{D,\dagger}$ (resp. $S_{(k,\psi^{-1})}^{D,\dagger}$) is $(k+1)^2 qt/2$. That is, the Newton polygon of $\sum_{n \geq 0} c_n(T_{\chi_k})X^n$ passes through the point

$$((k+1)qt, (k+1)^2 qt/2) = (n_{k+1}, \lambda(n_{k+1})v(T_{\chi_k})).$$

We admit that the matching of two sums of U_p -slopes is quite lucky in our case, which we do not know how to reproduce in too much more generality. From now on, we may work only with $S_{(k,\psi)}^{D,\dagger}$. (For each k , we fix one ψ of conductor q^2 that satisfy $\omega_0^k \psi|_{\Delta} = \omega$.)

Step II: We deduce the decomposition of $\text{Sp}_D^{>1/p}$ from the touching of polygons.

To proceed, first note that $\lfloor \frac{n_k}{t} \rfloor - \lfloor \frac{n_k}{pt} \rfloor = kq - kqp^{-1} = k\varphi(q)$. So if $i \in \mathbb{Z}$ and $n_{k+1} - i \geq 0$, then

$$\lambda(n_{k+1} - i) \geq \lambda(n_{k+1}) - (k+1)\varphi(q)i$$

with equality if and only if $i \in [-t, t]$. Since $|T_{\chi_k}| = p^{-p/q(p-1)} \in (1/p, 1)$, by Corollary 3.18, we have

$$(3.23.2) \quad v(c_n(T_{\chi_k})) = \lambda(n)v(T_{\chi_k}) \text{ if and only if } b_{n,\lambda(n)} \text{ is a } p\text{-adic unit in } \mathbb{Z}_p.$$

We have previously shown that the Newton polygon of $\sum_{n \geq 0} c_n(T_{\chi_k})X^n$ exactly goes through the point $\mathcal{P}_k = (n_{k+1}, \lambda(n_{k+1})v(T_{\chi_k}))$. Note that this does not force $b_{n_{k+1},\lambda(n_{k+1})}$ to be a p -adic unit, as the point \mathcal{P}_k may not be a vertex for the Newton polygon. We thus suppose that the line segment of the Newton polygon of $\sum_{n \geq 0} c_n(T_{\chi_k})X^n$ passing through \mathcal{P}_k lies over $[n_{k+1}^-, n_{k+1}^+]$ in its x -coordinate. It is clear that if $n_{k+1}^- \neq n_{k+1}^+$, then this line segment has slope $(k+1)\varphi(q)$. It follows that

$$n_{k+1}^- \in [n_{k+1} - t, n_{k+1}] \quad \text{and} \quad n_{k+1}^+ \in [n_{k+1}, n_{k+1} + t].$$

Moreover, the equivalence (3.23.2) implies that n_{k+1}^- (resp. n_{k+1}^+) is the minimal index in $[n_{k+1} - t, n_{k+1}]$ (resp. maximal index in $[n_{k+1}, n_{k+1} + t]$) such that

$$b_{n_{k+1}^-, \lambda(n_{k+1}^-)} \text{ (resp. } b_{n_{k+1}^+, \lambda(n_{k+1}^+)}) \text{ is a } p\text{-adic unit in } \mathbb{Z}_p.$$

For a uniform treatment later, we set $n_0^- = 0$ and n_0^+ the maximal index in $[0, t]$ such that $b_{n_0^+, 0}$ is a p -adic unit.

Now, if we specialize to any point $T \in \mathcal{W}_\omega^{>1/p}$, we must have (for all $i \in \mathbb{Z}_{\geq 0}$)

$$v(c_{n_k - i}(T)) \geq v(T)\lambda(n_k - i) \geq v(T) \cdot (\lambda(n_k) - k\varphi(q)i),$$

where the first inequality is a strict inequality if $n_k - t \leq n_k - i < n_k^-$ (by the minimality of n_k^-) and the second inequality is a strict inequality if $n_k - i < n_k - t$. In summary, for $i \in \mathbb{Z}_{\geq 0}$, we have the inequality

$$v(c_{n_k - i}(T)) \geq v(T) \cdot (\lambda(n_k) - k\varphi(q)i),$$

which becomes a strict inequality if $n_k - i < n_k^-$ and becomes an equality if $n_k - i = n_k^-$. Similarly, we have the inequality

$$v(c_{n_k + i}(T)) \geq v(T) \cdot (\lambda(n_k) + k\varphi(q)i),$$

which becomes a strict inequality if $n_k + i > n_k^+$ and becomes an equality if $n_k + i = n_k^+$. Moreover, by Corollary 3.18, we see that the differences in all strict inequalities are at least $\min\{v(T), 1 - v(T)\}$.

In summary, we conclude that for every $T \in \mathbb{C}_p$ with $0 < v(T) < 1$, if $n_k^- \neq n_k^+$, then the points

$$(n_k^-, \lambda(n_k^-)v(T)) \text{ and } (n_k^+, \lambda(n_k^+)v(T))$$

are two consecutive vertices of the Newton polygon of $\sum_{n \geq 0} c_n(T)X^n$. Furthermore, the line segment connecting these two vertices has slope $k\varphi(q)v(T)$, and passes through the point $(n_k, \lambda(n_k)v(T))$. Otherwise, $n_k^- = n_k = n_k^+$ is a vertex of the Newton polygon of $\sum_{n \geq 0} c_n(T)X^n$.

The decomposition of the spectral curve follows from this. More precisely, for $I = k = [k, k]$ or $(k, k + 1)$ with $k \in \mathbb{Z}_{\geq 0}$, we define $X_{I, \omega}$ to be the open subspace of $\mathrm{Sp}_D^{>1/p}$ such that for each point $z \in X_{I, \omega}$, we have

$$v(a_p(z)) \in \varphi(q)v(T_{\mathrm{wt}(z)}) \cdot I.$$

By our previous estimates and applying [Bu07, Corollary 4.3], these are in fact finite flat over $\mathcal{W}_\omega^{>1/p}$, and are affinoid subdomains when restricted to fibers of any affinoid subdomain of $\mathcal{W}_\omega^{>1/p}$. It follows that these $X_{I, \omega}$'s are unions of connected components of $\mathrm{Sp}_D^{>1/p}$. Regarding the degrees, we must have

$$(3.23.3) \quad \sum_{j=0}^{k-1} (\deg X_{j, \omega} + \deg X_{(j, j+1), \omega}) = n_k^- \in [n_k - t, n_k] \text{ and}$$

$$(3.23.4) \quad \sum_{j=0}^{k-1} (\deg X_{j, \omega} + \deg X_{(j, j+1), \omega}) + \deg X_{k, \omega} = n_k^+ \in [n_k, n_k + t].$$

Step III: It remains to compute the degrees of $X_{I,\omega}$'s. It is clear that $X_{0,\omega}$ coincides with the restriction of X_ω^{ord} , which is introduced in the proof of Theorem 3.19, to $\mathcal{W}_\omega^{>1/p}$. Then by Corollary 3.21, $\deg X_{0,\omega}$ is equal to the dimension of slope zero subspace in $S_\omega^{D,\dagger}$. That is,

$$\deg X_{0,\omega} = n_0^+ = r_{\text{ord}}(\omega).$$

(One subtlety of the argument here is that: we can not directly apply the previous part of Theorem 1.3 because the weight $\omega \notin \mathcal{W}^{>1/p}$. We have to employ Corollary 3.21 instead.) For $k \geq 0$, first note that $n_{k+1} - n_{k+1}^-$ is equal to the dimension of slope $k+1$ subspace in $S_{k+2}^D(K^p\text{Iw}_{q^2}; \psi)$. By Atkin–Lehner theory (Proposition 3.22) and Proposition 2.15, the multiplicity is the same as the dimension of slope zero subspace in $S_{(k,\psi^{-1})}^{D,\dagger}$. Using Corollary 3.21 again, we deduce that

$$n_{k+1} - n_{k+1}^- = r_{\text{ord}}(\psi^{-1}|_\Delta \cdot \omega_0^k) = r_{\text{ord}}(\omega^{-1}\omega_0^{2k})$$

because $\psi^{-1}|_\Delta \cdot \omega_0^k = \chi_k^{-1}|_\Delta \cdot \omega_0^{2k} = \omega^{-1}\omega_0^{2k}$.

To compute $n_{k+1}^+ - n_{k+1}$, we recall the following exact sequence (cf. [Jo11])

$$0 \rightarrow S_{k+2}^D(K^p\text{Iw}_{q^2}, \psi) \rightarrow S_{(k,\psi)}^{D,\dagger} \xrightarrow{\left(\frac{d}{dz}\right)^{k+1}} S_{(-k-2,\psi)}^{D,\dagger} \rightarrow 0.$$

This exact sequence is equivariant for the U_p -action on the first two spaces, and the $p^{k+1}U_p$ -action on the third space. It is clear that $n_{k+1}^+ - n_{k+1}$ is equal to the codimension of $S_{k+2}^D(K^p\text{Iw}_{q^2}, \psi)$ in the slope $\leq k+1$ subspace in $S_{(k,\psi)}^{D,\dagger}$. The latter in turn is equal to the dimension of slope zero subspace of $S_{(-k-2,\psi)}^{D,\dagger}$ by the exact sequence. Using Corollary 3.21, we thus obtain

$$n_{k+1}^+ - n_{k+1} = r_{\text{ord}}(\psi|_\Delta \cdot \omega_0^{-k-2}) = r_{\text{ord}}(\omega\omega_0^{-2k-2}).$$

The final degree is computed by

$$\deg X_{k,\omega} = n_k^+ - n_k^- = (n_k^+ - n_k) + (n_k - n_k^-) = \begin{cases} r_{\text{ord}}(\omega), & \text{if } k = 0, \\ r_{\text{ord}}(\omega^{-1}\omega_0^{2k-2}) + r_{\text{ord}}(\omega\omega_0^{-2k}), & \text{if } k \geq 1, \end{cases}$$

and

$$\begin{aligned} \deg X_{(k,k+1),\omega} &= n_{k+1}^- - n_k^+ = n_{k+1} - n_k - (n_{k+1} - n_{k+1}^-) - (n_k^+ - n_k) \\ &= qt - r_{\text{ord}}(\omega^{-1}\omega_0^{2k}) - r_{\text{ord}}(\omega\omega_0^{-2k}). \end{aligned}$$

This concludes the proof of Theorem 1.3. \square

The following interesting consequence of Theorem 1.3 is pointed out to us by Chenevier. We are grateful to him for allowing us to include it in this paper.

Proposition 3.24. *Let C be an irreducible component of Spc_D . If C is finite over weight space, then C is inside the ordinary locus.*

Proof. Under the assumption, the weight map $C \rightarrow \mathcal{W}$ is finite and flat (the flatness is ensured by [CM98, Theorem C]). Thus the analytic function a_p on C , which is nowhere vanishing and bounded by 1, has a norm g down to some weight disk \mathcal{W}_ω . It is clear that

the analytic function g is also a nowhere vanishing and bounded by 1, so it has the form $p^n h$ where h is a unit in $\mathbb{Z}_p[[T]]$. In particular, this shows that for all $w \in \mathcal{W}_\omega$,

$$\sum_{x \in C, \text{wt}(x)=w} v(a_p(x)) = n.$$

But Theorem 1.3 says that $v(a_p(x))$ goes to 0 above the boundary of \mathcal{W}_ω , so $n = 0$. Thus $v(a_p(x)) = 0$ for all $x \in C$, concluding the proposition. \square

Remark 3.25. We note that the existence of n_k^\pm in the proof of Theorem 1.3 in fact implies that, for $n = n_k^\pm$, $c_n(T)$ is equal to T^{λ_n} times a unit in $\Lambda^{>1/p}$. Then, a standard factorization argument shows (see e.g. [Ke09, Proposition 3.2.2] for the argument) that we can factor $\text{Char}(P)$ into the following product

$$P_0(X) \cdot P_{(0,1)}(X) \cdot P_1(X) \cdot P_{(1,2)}(X) \cdots$$

such that each $P_I(X) \in \Lambda^{>1/p}[[X]]$ is the characteristic polynomial corresponding to the component X_I . This gives an integral model \mathfrak{X}_I of each X_I (in the case that the tame level is neat).

An intriguing and pressing future question is: what is the arithmetic property at the “special fibers” of \mathfrak{X}_n and $\mathfrak{X}_{(n,n+1)}$? In particular, the pseudo-representation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ can be extended to the integral model; what can we say about the representation over the special fiber of these formal schemes? We hope to come back to this question in a future work. At the same time, we encourage the readers to explore more applications of Coleman’s idea on studying the integral model of the eigencurve.

An alternative way to understand the integral model of the eigencurve is to “compactify” the weight space in the category of adic spaces, by viewing the boundary part $\mathcal{W}^{>1/p}$ in the T -adic world and adding a point at the “boundary” on each disk (whose residue field is $\mathbb{F}_p((T))$). Then one can extend the spectral curve over these “boundary points.” We refer to [AIP15⁺] and the recent preprint [JN16⁺] for more discussion on this viewpoint.

Remark 3.26. We discuss some potential generalization of our main theorem.

- (1) As pointed out in Remark 1.7(1), our result cannot access the eigencurve with trivial tame level structure. There are two possible strategies to remedy this. One is to work with modular symbols; it might be possible to replicate the argument in this paper under that setup. We encourage interested readers to explore this possibility. Another approach is to base change (in a p -adic family) to a real quadratic field F in which p splits. Then the eigensurface for the unramified definite quaternion algebra D over F should be the same as the eigensurface for overconvergent Hilbert modular forms over F . Then the analogous results for the unramified eigencurve should follow from that of the eigensurface for D (if the latter case may be proved).
- (2) Now, take a general algebraic group G over \mathbb{Q} which is quasi-split at p and whose \mathbb{R} -points are compact modulo center. Then one can construct the associated eigenvariety \mathcal{C}_G (as carried out in [Lo11]). Our ultimate optimistic expectation is that an analogue of Proposition 3.14 still holds true.⁵ In particular, we can still see the characteristic power series of the U_p -operators on the space of integral automorphic forms (with respect to an appropriate basis). Nonetheless, the classicality argument

⁵This was recently verified by Johansson and Newton [JN16⁺].

and the touching of Newton polygon are no longer available, at least not in a naive way. We strongly encourage interested readers to investigate in this issue.

4. DISTRIBUTION OF U_p -SLOPES

This section is devoted to proving Theorem 1.5. Our argument for the first half of the theorem is modeled on the proof of [DWX16, Theorem 3.8], whose upshot is to give a suitable upper bound for the Newton polygon. The rest of the theorem, i.e. the arithmetic progression statement about the ratios, then follows easily from Atkin–Lehner theory.

To start with, we first point out an upper bound of the Newton polygon of $\sum_{n=0}^{\infty} c_n(T)X^n$ when $0 < v(T) < 1$. Indeed, in the course of the proof of Theorem 1.3, we already show that the Newton polygon of $\sum_{n \geq 0} c_n(T)X^n$ passes through the points $(n_k, \lambda(n_k)v(T))$ for all $k \geq 0$. Therefore, we deduce that the Newton polygon of $\sum_{n \geq 0} c_n(T)X^n$ always lies below the polygon with vertices $(n_k, \lambda(n_k)v(T))$ for all $k \geq 0$. We call this polygon the *upper bound polygon* of $\sum_{n=0}^{\infty} c_n(T)X^n$.

Lemma 4.1. *The maximal vertical difference between the lower bound polygon and the upper bound polygon is $(p^2 - 1)tv(T)/8$ for $p > 2$, and $tv(T)$ for $p = 2$.*

Proof. We only treat the case $p > 2$, the case $p = 2$ being similar. Note that the lower bound polygon and the upper bound polygon touch at the vertices $(n_k, \lambda(n_k)v(T))$ for $k \geq 0$. It is sufficient to bound their vertical difference over $x \in [n_k, n_{k+1}]$. By (3.23.1), we first get $\lambda(n_{k+1}) = (k + 1)^2 p(p - 1)t/2$. A short computation then shows that the restriction of the upper bound polygon on $[n_k, n_{k+1}]$ is a linear function with slope $(k + \frac{1}{2})(p - 1)v(T)$. On the other hand, for every integer $a \in [0, p - 1]$, by Theorem 3.16, we know that the restriction of the lower bound polygon on $[n_k + at, n_k + (a + 1)t]$ is a linear function with slope $(k(p - 1) + a)v(T)$. We therefore deduce that the maximal vertical difference over $[n_k, n_{k+1}]$ is achieved when $a = \frac{p-1}{2}$.

In that case, put $n = n_k + (p - 1)t/2$. It is straightforward to see the vertical difference at $x = n$ is, by looking at the incremental differences of slopes built from the vertex $(n_k, \lambda(n_k)v(T))$,

$$\begin{aligned} & \sum_{i=n_k}^{n-1} \left(\left(k + \frac{1}{2} \right) (p - 1) - \left(\left\lfloor \frac{i}{t} \right\rfloor - \left\lfloor \frac{i}{pt} \right\rfloor \right) \right) v(T) \\ &= tv(T) \sum_{j=kp}^{kp + \frac{p-1}{2} - 1} \left(\left(k + \frac{1}{2} \right) (p - 1) - \left(j - \left\lfloor \frac{j}{p} \right\rfloor \right) \right) \\ &= tv(T) \sum_{j=0}^{\frac{p-1}{2} - 1} \left(\frac{p-1}{2} - j \right) = \frac{1}{8} (p^2 - 1)tv(T). \quad \square \end{aligned}$$

4.2. Proof of Theorem 1.5. We first show the existence of λ , the sequence $\alpha_0(\omega), \alpha_1(\omega), \dots$ and the desired decomposition for $\mathrm{Spc}_{D,\omega}^{>\lambda}$. For this purpose, by virtue of Remark 2.14, it is sufficient to treat the case that the tame level is neat. Also, we assume $p > 2$, the case $p = 2$ being similar. We will proceed as in the proof of Theorem 1.3. That is, it suffices to show that for $T \in \mathbb{C}_p$ with $0 < v(T) < \frac{8}{(p^2-1)t+8}$, the ratios to $v(T)$ of the slopes (counted

with multiplicity) of the Newton polygon of $\sum_{n \geq 0} c_n(T)X^n$ are independent of the choice of T .

Recall that the Newton polygon of $\sum_{n \geq 0} c_n(T)X^n$ is the convex hull of points $(n, v(c_n(T)))$ for all $n \geq 0$. We consider those points which lie below the upper bound polygon.

Claim: If $(l, v(c_l(T_0)))$ lies strictly below the upper bound polygon for some $l \in \mathbb{N}$ and $T_0 \in \mathbb{C}_p$ with $0 < v(T_0) < \frac{8}{(p^2-1)t+8}$, then there exists a *unique* integer $m(l) \geq \lambda(l)$ such that for every $T \in \mathbb{C}_p$ with $0 < v(T) < \frac{8}{(p^2-1)t+8}$, $(l, v(c_l(T)))$ lies strictly below the upper bound polygon and $v(c_l(T)) = m(l)v(T)$.

Granting the claim, we conclude that there exists a (finite or infinite) set of positive integers $\{l_i\}_{i \in I}$ such that if $0 < v(T) < \frac{8}{(p^2-1)t+8}$, then the Newton polygon of $\sum_{n \geq 0} c_n(T)X^n$ is the convex hull of points

$$\{(n_k, \lambda(n_k)v(T))\}_{k \geq 0} \coprod \{(l_i, m(l_i)v(T))\}_{i \in I}.$$

It is then clear that the ratios to $v(T)$ of the slopes of this polygon are independent of T . This yields the existence of the sequence $\alpha_0(\omega), \alpha_1(\omega), \dots$ and the desired decomposition for $\text{Spc}_{D,\omega}^{>\lambda}$.

We now proceed to show the claim. First note that if $0 < v(T) < \frac{8}{(p^2-1)t+8}$ and $m < \lambda(l)$, then by (3.18.1), we get

$$(4.2.1) \quad v(b_{l,m}T^m) \geq \lambda(l) - m + mv(T) \geq \lambda(l)v(T) - v(T) + 1 > \lambda(l)v(T) + \frac{(p^2-1)tv(T)}{8}.$$

On the other hand, since $(l, v(c_l(T_0)))$ lies strictly below the upper bound polygon, by Lemma 4.1, we get

$$v(c_l(T_0)) - \lambda(l)v(T_0) < \frac{(p^2-1)tv(T_0)}{8}.$$

Hence for $m < \lambda(l)$, we obtain

$$(4.2.2) \quad v(c_l(T_0)) < \lambda(l)v(T_0) + \frac{(p^2-1)tv(T_0)}{8} < v(b_{l,m}T_0^m)$$

by specializing (4.2.1) to $T = T_0$. Therefore, there must be some $m \geq \lambda(l)$ such that $v(b_{l,m}T_0^m) \leq v(c_l(T_0))$. Let $m(l)$ be the minimal one satisfying this property. It follows that

$$v(b_{l,m(l)}) \leq v(b_{l,m(l)}T_0^{m(l)}) - \lambda(l)v(T_0) \leq v(c_l(T_0)) - \lambda(l)v(T_0) < \frac{(p^2-1)tv(T_0)}{8} < 1,$$

yielding $b_{l,m(l)} \in \mathbb{Z}_p^\times$. Thus for $m > m(l)$, we get

$$(4.2.3) \quad v(b_{l,m}T^m) > v(b_{l,m}) + m(l)v(T) \geq m(l)v(T) = v(b_{l,m(l)}T^{m(l)}).$$

Moreover, by the minimality of $m(l)$, for $m \in [\lambda(l), m(l) - 1]$, we have

$$(4.2.4) \quad v(b_{l,m}T_0^m) > v(c_l(T_0)) \geq v(b_{l,m(l)}T_0^{m(l)}),$$

yielding $v(b_{l,m}) > v(b_{l,m(l)})$. Hence $b_{l,m} \in p\mathbb{Z}_p$ for those m . Finally, putting (4.2.2), (4.2.3), and (4.2.4) together, we conclude $v(c_l(T_0)) = v(b_{l,m(l)}T_0^{m(l)}) = m(l)v(T_0)$.

Now let $0 < v(T) < \frac{8}{(p^2-1)t+8}$. Since the point $(l, m(l)v(T_0))$ lies strictly below the upper bound polygon for T_0 , by similarity, the point $(l, m(l)v(T))$ lies strictly below the upper bound polygon for T as well. Note that (4.2.1) together with Lemma 4.1 imply that for

$m < \lambda(l)$, the point $(l, v(b_{l,m}T^m))$ lies above the upper bound polygon. Hence $v(b_{l,m}T^m) > m(l)v(T)$ for $m < \lambda(l)$. For $m \in [\lambda(l), m(l) - 1]$, since $b_{l,m} \in p\mathbb{Z}_p$, it follows that

$$v(b_{l,m}T^m) \geq 1 + \lambda(l)v(T) > \frac{(p^2 - 1)tv(T)}{8} + \lambda(l)v(T).$$

Hence $(l, v(b_{l,m}T^m))$ lies above the upper bound polygon by Lemma 4.1, yielding that $v(b_{l,m}T^m) > m(l)v(T)$. For $m > m(l)$, we have $v(b_{l,m}T^m) > m(l)v(T)$ by (4.2.3). We thus conclude that $v(c_l(T)) = v(b_{l,m(l)}T^{m(l)}) = m(l)v(T)$. This proves the claim.

Now let $\mathrm{Sp}c_{D,\omega}^{>\lambda} = \coprod_{i \geq 0} Y_{i,\omega}$ be the desired decomposition, and let

$$\tilde{\alpha}_0(\omega), \tilde{\alpha}_1(\omega), \dots$$

denote the sequence consisting of α_i 's with multiplicity $\deg Y_{i,\omega}$. In the following, we will show that the sequence $\tilde{\alpha}_0(\omega), \tilde{\alpha}_1(\omega), \dots$ is a disjoint union of $p^{M-1}(p-1)t/2$ arithmetic progressions with the same common difference $\frac{\varphi(q)p^M}{2q^2}$.

Let ψ be a character of conductor p^M . We look at weights of the form (k, ψ) for all $k \geq 0$. First, note that $v(T_{(k,\psi)}) = \frac{q}{\varphi(p^M)} = \frac{q}{(p-1)p^{M-1}}$ by the assumption on M ; thus $(k, \psi) \in \mathcal{W}_{\psi|_{\Delta} \cdot \omega_0^k}^{>\lambda}$. Noting the equality $\frac{q}{(p-1)p^{M-1}}\varphi(q) = q^2p^{-M}$, it then follows that the U_p -slopes of $S_{k+2}^D(K^p\mathrm{Iw}_{p^M}, \psi)$ are

$$q^2p^{-M}\tilde{\alpha}_0(\psi|_{\Delta} \cdot \omega_0^k), \dots, q^2p^{-M}\tilde{\alpha}_{(k+1)q^{-1}p^{M-t-1}}(\psi|_{\Delta} \cdot \omega_0^k).$$

Hence, by Atkin–Lehner theory (Proposition 3.22), in the U_p -slope sequence on $S_{k+2}^D(K^p\mathrm{Iw}_{p^M}; \psi^{-1})$, from the $(kq^{-1}p^Mt + 1)$ st to the $(k+1)q^{-1}p^Mt$ th is given by

$$k+1 - q^2p^{-M}\tilde{\alpha}_{qp^{M-t-1}}(\psi|_{\Delta} \cdot \omega_0^k), \dots, k+1 - q^2p^{-M}\tilde{\alpha}_0(\psi|_{\Delta} \cdot \omega_0^k).$$

This implies the relations

$$\begin{aligned} \tilde{\alpha}_{(k+1)q^{-1}p^{M-t-1-i}}(\psi^{-1}|_{\Delta} \cdot \omega_0^k) &= q^{-2}p^M(k+1 - q^2p^{-M}\tilde{\alpha}_i(\psi|_{\Delta} \cdot \omega_0^k)) \\ &= (k+1)q^{-2}p^M - \tilde{\alpha}_i(\psi|_{\Delta} \cdot \omega_0^k) \end{aligned}$$

for $0 \leq i \leq q^{-1}p^Mt - 1$. Replacing ψ by $\psi\omega_0^{-1}$ and k by $k+1$, we get

$$\tilde{\alpha}_{(k+2)q^{-1}p^{M-t-1-i}}(\psi^{-1}|_{\Delta} \cdot \omega_0^{k+2}) = (k+2)q^{-2}p^M - \tilde{\alpha}_i(\psi|_{\Delta} \cdot \omega_0^k).$$

We thus deduce that

$$(4.2.5) \quad \tilde{\alpha}_{(k+2)q^{-1}p^{M-t-1-i}}(\psi^{-1}|_{\Delta} \cdot \omega_0^{k+2}) = \tilde{\alpha}_{(k+1)q^{-1}p^{M-t-1-i}}(\psi^{-1}|_{\Delta} \cdot \omega_0^k) + q^{-2}p^M.$$

We conclude the theorem by (4.2.5). In fact, for any character ω of Δ and $j \in \mathbb{Z}_{\geq 0}$, write $j = (k+1)q^{-1}p^Mt - 1 - i$ for some $k \in \mathbb{Z}_{\geq 0}$ and $i \in [0, q^{-1}p^Mt - 1]$. Choose ψ so that $\psi|_{\Delta} \cdot \omega_0^k = \omega$. It then follows from (4.2.5) that

$$(4.2.6) \quad \tilde{\alpha}_{j+q^{-1}p^Mt}(\omega\omega_0^2) = \tilde{\alpha}_j(\omega) + q^{-2}p^M.$$

In particular, since $\omega_0^{\varphi(q)} = \omega_0^{\frac{q(p-1)}{p}} = 1$, we have

$$\tilde{\alpha}_{j+(p-1)p^{M-1}t/2}(\omega) = \tilde{\alpha}_j(\omega) + \frac{\varphi(q)p^M}{2q^2}.$$

Therefore, the sequence $\tilde{\alpha}_0(\omega), \tilde{\alpha}_1(\omega), \dots$ is the disjoint union of arithmetic progressions $\tilde{\alpha}_j(\omega), \tilde{\alpha}_{j+(p-1)p^{M-1}t/2}(\omega), \dots$ for

$$0 \leq j \leq (p-1)p^{M-1}t/2 - 1,$$

which have common difference $\frac{\varphi(q)p^M}{2q^2}$.

Remark 4.3. The argument for the second part of the proof, namely, assuming the Claim to prove the slope ratios being the unions of arithmetic progressions, works equally well to the case of modular curves, as independently proved by Bergdall and Pollack [BP15⁺].

4.4. Proof of Corollary 1.8. (1) Specialize Theorem 1.3 to the weight character $x^k\psi$ and note that $v(T_{x^k\psi}) = \frac{q}{(p-1)p^{m-1}}$ by the assumption on m . If we use $\beta_0^\dagger(k, \psi), \beta_1^\dagger(k, \psi), \dots$ to denote the sequence of slopes of U_p -action on $S_{x^k\psi}^{D, \dagger}$, then by (3.23.3) and (3.23.4) we have the following inequalities

$$q^2p^{-m} \lfloor n/qt \rfloor \leq \beta_n^\dagger(k, \psi) \leq q^2p^{-m} (\lfloor n/qt \rfloor + 1) \quad \text{for all } n \geq 0.$$

By the classicality result Proposition 2.15, $\beta_i(k, \psi) = \beta_i^\dagger(k, \psi)$ for $i = 0, \dots, q^{-1}p^m(k+1)t-1$. This proves (1).

(2) Recall once again that $v(T_\psi) = \frac{q}{p^{M-1}(p-1)}$. By specializing Theorem 1.5 to the weight character ψ that lifts ω , and using the classicality result (Proposition 2.15), we see that

$$\tilde{\alpha}_i(\omega) = p^M q^{-2} \beta_i(\omega),$$

for $i = 0, \dots, q^{-1}p^M t - 1$. By (4.2.6), we have

$$\tilde{\alpha}_{i+nq^{-1}p^M t}(\omega) = \tilde{\alpha}_i(\omega\omega_0^{-2n}) + np^M q^{-2} = p^M q^{-2} \beta_i(\omega\omega_0^{-2n}) + np^M q^{-2}.$$

Thus specializing Theorem 1.5 to a general classical character $x^k\psi_m$ with $m \geq M$, we see the U_p -slopes on $S_{k+2}^{D, \dagger}(\psi_m)$ are exactly given by $q^2p^{-m}\tilde{\alpha}_0(\psi_m|_{\Delta}\omega_0^k), q^2p^{-m}\tilde{\alpha}_1(\psi_m|_{\Delta}\omega_0^k), \dots$, or equivalently the set

$$\bigcup_{n \geq 0} \{p^{M-m}(\beta_0(\psi_m|_{\Delta}\omega_0^{k-2n}) + n), \dots, p^{M-m}(\beta_{q^{-1}p^M t - 1}(\psi_m|_{\Delta}\omega_0^{k-2n}) + n)\}.$$

By classicality result (Proposition 2.15) again, we see that the slopes on $S_{k+2}^D(\psi_m)$ are those in the union with $n \in \{0, \dots, p^{m-M}(k+1) - 1\}$. This concludes the proof of the corollary.

5. INTEGRAL MODELS OF THE SPACE OF OVERCONVERGENT AUTOMORPHIC FORMS

As mentioned before, the U_p -action on S_{int}^D is unlikely to be compact. This subtlety was carefully circumvented in the proof of our main theorem (e.g. the statement of Proposition 2.17). But we feel that it might be beneficial to introduce a variant construction, for which the U_p -action is compact. We carry out this construction in this section.

Definition 5.1. Let R be a complete noetherian ring, with ideal of definition \mathfrak{m}_R . Let M be a topological R -module isomorphic to

$$\widehat{\bigoplus}_{i \in \mathbb{Z}_{\geq 0}} Re_i := \varprojlim_n \left(\bigoplus_{i \in \mathbb{Z}_{\geq 0}} (R/\mathfrak{m}_R^n) e_i \right),$$

equipped with a continuous R -linear action of an operator U . We refer to $(e_i)_{i \in \mathbb{Z}_{\geq 0}}$ as an *orthonormal basis*. We say that the U -action on M is *compact* if the induced action on

$M/\mathfrak{m}_R^n M$ has finitely generated image for any $n \in \mathbb{Z}_{\geq 0}$. This definition does not depend on the choice of the orthonormal basis of M .

When the U -action is compact, if P denotes the infinite matrix for the U -action with respect to the basis $(e_i)_{i \in \mathbb{N}}$, the *characteristic power series* of the U -action:

$$\text{Char}(U; M) := \det(I_\infty - XP) = \lim_{n \rightarrow \infty} \det(I_\infty - X(P \bmod \mathfrak{m}_R^n)) \in R[[X]]$$

is well defined: note that its r -th coefficient is the trace of the action of U on the r -th wedge product of M , which is well defined by first modulo \mathfrak{m}_R^n and then taking the limit. Moreover, it does not depend on the choice of the orthonormal basis.

Example 5.2. We give an example where the operator is not compact; this example may be served as a toy model of the U_p -action on S_{int}^D .

Consider $M = \mathcal{C}(\mathbb{Z}_p; \mathbb{Z}_p)$. The operator U sends a continuous function $f(z)$ to

$$h(z) = f(pz) + f(pz + 1) + \cdots + f(pz + p - 1).$$

One can use Lemma 3.12 to control some of the entries of the infinite matrix of U with respect to the Mahler basis. But the U -action is not compact. First note that the infinite matrix is going to be upper triangular because of the shape of the the operator U and the trivial degree bound in Lemma 3.7(3). Next, we look at the image of $\binom{z}{p^m}$ under U for $m \geq 2$, which will appear on the $(p^m + 1)$ st column of the infinite matrix.

$$U\left(\binom{z}{p^m}\right) = \binom{pz}{p^m} + \binom{pz+1}{p^m} + \cdots + \binom{pz+p-1}{p^m}.$$

Note that evaluating the right hand side at $z = p^{m-1}$, we get $\sum_{i=0}^{p-1} \binom{p^m+i}{p^m}$, which is congruent to p modulo p^m . On the other hand, it is clear that $U\left(\binom{z}{p^m}\right)|_{z=i}$ is equal to zero for $i = 0, 1, \dots, p^{m-1} - 1$. It follows that the Mahler coefficient of $\binom{z}{p^{m-1}}$ is not divisible by p^2 . This implies that the operator U cannot be compact.

Remark 5.3. The non-compactness of U_p may cause technical difficulties in applications. Our fix to this problem is to introduce a subspace stable under the action of the monoid \mathbf{M}_1 . But we first explain that another apparently easier fix: developing a more general compact operator theory, would not easily work.

As shown in Theorem 3.16, the U_p -action on S_{int}^D satisfies the following property which is slightly weaker than being compact: there exists an orthonormal basis such that, the associated infinite matrix, modulo \mathfrak{m}_λ^n for each n , is strictly upper triangular except for the first $d(n) \times d(n)$ -minor for some $d(n) \in \mathbb{N}$ depending on n . It still makes sense to define characteristic power series for such an infinite matrix by taking the limit over the characteristic power series of its minors. Unfortunately, this power series defined in this generality depends on the choice of the orthonormal basis (even if restricting to those bases satisfying the condition above). Here is an example: consider $M = \widehat{\bigoplus}_{i \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_p e_i$ equipped with the action of U , sending e_0 to 0 and e_i to e_{i-1} for $i \in \mathbb{N}$. Then for this choice of orthonormal basis, the corresponding characteristic power series is just $1 \in \mathbb{Z}_p[[X]]$, as the infinite matrix for U is strict upper triangular. Now if we consider another orthonormal basis

of M :

$$\begin{aligned} e'_0 &= e_0 + pe_1 + p^2e_2 + p^3e_3 + \cdots; \\ e'_1 &= e_1 + pe_2 + p^2e_3 + p^3e_4 + \cdots; \\ e'_2 &= e_2 + pe_3 + p^2e_4 + p^3e_5 + \cdots; \\ &\dots \quad \dots \end{aligned}$$

Then we have $U(e'_0) = pe'_0$ and $U(e'_i) = e'_{i-1}$ for $i \in \mathbb{N}$. So the corresponding infinite matrix for U is $p, 0, 0, 0, \dots$ on the main diagonal, all 1 at the entries just above the diagonal, and 0 everywhere else. In particular, the corresponding power series is $1 - pX \in \mathbb{Z}_p[[X]]$. So, in general, the characteristic power series for this type of operators might depend on the choice of the orthonormal basis.

5.4. Integral models of overconvergent automorphic forms. Let $[-]': \mathbb{Z}_p^\times \rightarrow (\Lambda^{>1/p})^\times$ denote the universal character of \mathbb{Z}_p^\times . Recall that (2.3.1) gives an isomorphism between $\text{Ind}_{B(\mathbb{Z}_p)}^{\text{Iw}_q}([-]')$ and $\mathcal{C}(\mathbb{Z}_p; \Lambda^{>1/p})$; the latter admits an orthonormal basis (over $\Lambda^{>1/p}$) given by the functions $\left(\binom{z}{n}\right)_{n \in \mathbb{Z}_{\geq 0}}$. We consider a closed subspace

$$(5.4.1) \quad \text{Ind}_{B(\mathbb{Z}_p)}^{\text{Iw}_q}([-]')^{\text{mod}} = \widehat{\bigoplus_{n \geq 0} T^n \Lambda^{>1/p}} \cdot \binom{z}{n}.$$

We claim that this subspace is stable under the action of the monoid \mathbf{M}_1 . Indeed, by Proposition 3.14(2), for the action of $\delta_p \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{M}_1$ on the Mahler basis, the coefficient $P_{m,n}(\delta_p)$ belongs to $\mathfrak{m}_\Lambda^{\max\{m-n, 0\}} \Lambda^{>1/p} = T^{\max\{m-n, 0\}} \Lambda^{>1/p}$. Then, with respect to the basis $\left(T^n \binom{z}{n}\right)_{n \in \mathbb{Z}_{\geq 0}}$, the (m, n) -entry of the infinite matrix has coefficients in

$$T^{n-m} \cdot T^{\max\{m-n, 0\}} \Lambda^{>1/p} = T^{\max\{0, n-m\}} \Lambda^{>1/p}.$$

This concludes the proof of the claim.

Moreover, Proposition 3.14(1) says that for $\delta_p \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} p\mathbb{Z}_p & \mathbb{Z}_p \\ q\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$, namely those δ_p appearing in the expression of U_p , the coefficient $P_{m,n}(\delta_p)$ belongs to $\mathfrak{m}_\Lambda^{\max\{m-\lfloor n/p \rfloor, 0\}}$. So under the new basis $\left(T^n \binom{z}{n}\right)_{n \in \mathbb{Z}_0}$, the (m, n) -entry of the infinite matrix has coefficients in

$$T^{n-m} \cdot \mathfrak{m}_\Lambda^{\max\{m-\lfloor n/p \rfloor, 0\}} \Lambda^{>1/p} = T^{\max\{n-\lfloor n/p \rfloor, n-m\}} \Lambda^{>1/p}.$$

Now, we define the space of *integral 1-overconvergent automorphic forms* to be

$$S_{\text{int}}^{D, \dagger, 1} := \left\{ \varphi : D^\times \backslash (D \otimes \mathbb{A}_f)^\times / K^P \rightarrow \text{Ind}_{B(\mathbb{Z}_p)}^{\text{Iw}_q}([-]')^{\text{mod}} \mid \varphi(xu_p) = \varphi(x) \Big|_{u_p}^{[-]}, \text{ for } u_p \in \text{Iw}_q \right\}.$$

It is a topological module over $\Lambda^{>1/p}$ isomorphic to $\widehat{\bigoplus_{i \in \mathbb{Z}_{\geq 0}} \Lambda^{>1/p} e_i}$. Viewing the U_p -action on $S_{\text{int}}^{D, \dagger, 1}$ with respect to the basis

$$1_0, \dots, 1_{t-1}, Tz_0, \dots, Tz_{t-1}, T^2 \binom{z_0}{2}, \dots, T^2 \binom{z_{t-1}}{2}, T^3 \binom{z_0}{3}, \dots,$$

the corresponding infinite matrix has its entry of its n th column in

$$T^{\lfloor n/t \rfloor - \lfloor n/pt \rfloor} \Lambda^{>1/p}.$$

In particular, the action of U_p is compact, and the characteristic power series $\text{Char}(U_p; S_{\text{int}}^{D, \dagger, 1})$ agrees with the ones in Proposition 2.17.⁶

Remark 5.5. Similar constructions will give integral models of the space of r -overconvergent automorphic forms (with weights in $\Lambda^{>1/p}$) for $r > 0$, on which the U_p -action is compact.

Remark 5.6. The above construction may be regarded as the étale realization of integral models of overconvergent automorphic forms. We are curious about the possibility of comparing our construction with the integral models constructed in [AIP15⁺] by understanding the comparison theorem on this level.

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⁶This construction was recently generalized by Johansson and Newton to general overconvergent cohomology [JN16⁺].

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RUOCHUAN LIU, BEIJING INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH, PEKING UNIVERSITY, 5 YI HE YUAN ROAD, BEIJING, 100871, CHINA.

E-mail address: liuruochuan@math.pku.edu.cn

DAQING WAN, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, IRVINE, 340 ROWLAND HALL, DEPARTMENT OF MATHEMATICS, IRVINE, CA 92697.

E-mail address: dwan@math.uci.edu

LIANG XIAO, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, 341 MANSFIELD ROAD, UNIT 1009, STORRS, CT 06269-1009.

E-mail address: liang.xiao@uconn.edu