

SLOPES FOR HIGHER RANK ARTIN–SCHREIER–WITT TOWERS

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ABSTRACT. We fix a monic polynomial $\bar{f}(x) \in \mathbb{F}_q[x]$ over a finite field of characteristic p , and consider the \mathbb{Z}_p^ℓ -Artin–Schreier–Witt tower defined by $\bar{f}(x)$; this is a tower of curves $\cdots \rightarrow C_m \rightarrow C_{m-1} \rightarrow \cdots \rightarrow C_0 = \mathbb{A}^1$, whose Galois group is canonically isomorphic to \mathbb{Z}_p^ℓ , the degree ℓ unramified extension of \mathbb{Z}_p , which is abstractly isomorphic to $(\mathbb{Z}_p)^\ell$ as a topological group. We study the Newton slopes of zeta functions of this tower of curves. This reduces to the study of the Newton slopes of L-functions associated to characters of the Galois group of this tower. We prove that, when the conductor of the character is large enough, the Newton slopes of the L-function asymptotically form a finite union of arithmetic progressions. As a corollary, we prove the spectral halo property of the spectral variety associated to the \mathbb{Z}_p^ℓ -Artin–Schreier–Witt tower. This extends the main result in [DWX] from rank one case $\ell = 1$ to the higher rank case $\ell \geq 1$.

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1. INTRODUCTION

The topic we study in this paper reflects interests from two related areas. We shall first introduce our theorem from the p -adic and Iwasawa theoretic perspective of L -functions of varieties, and then explain the (philosophical) implication on spectral halo of eigenvarieties.

For a positive integer ℓ , a \mathbb{Z}_p^ℓ -Witt tower over a finite field \mathbb{F}_q of characteristic p is a sequence of finite étale Galois covers over \mathbb{F}_q ,

$$\cdots \rightarrow C_m \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 = \mathbb{A}^1,$$

whose total Galois group is isomorphic to \mathbb{Z}_p^ℓ . The integer ℓ is called the rank of the tower. All such Witt towers, uncountably many, can be constructed explicitly from Witt vectors, and their genera can be read off from an explicit formula, see [DKW]. A main interest in arithmetic geometry is to understand the zeros of the zeta-functions of the curves C_m over \mathbb{F}_q . In the context of Witt towers and the spirit of Iwasawa theory, a natural question is: what are the p -adic valuations (slopes) of the zeros of the zeta-function of C_m , especially what is the asymptotic behavior as $m \rightarrow \infty$? This is an emerging new field of study, which

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is expected to be quite fruitful and yet rather complicated in general, as there are too many Witt towers and most of them behave very badly. In order for the valuation sequence to have a strong stable property as m grows, it is reasonable (and necessary) to assume that the genus sequence has a stable property. Fortunately, Witt towers with a stable genus formula can be classified, and this is recently done in [DKW]. It is then natural to investigate the deeper slope stable property for the zeta function sequence of a genus stable Witt tower.

The first nontrivial case is when the tower is defined by the Teichmüller lift of a polynomial over \mathbb{F}_q (see the next paragraph), called the Artin–Schreier–Witt tower, which does satisfy the genus stable property. When the Artin–Schreier–Witt tower has the Galois group \mathbb{Z}_p (rank one case), the slope stability question has been successfully answered in [DWX], where it is shown that the valuations of the zeros are given by a finite union of arithmetic progressions. This implies a strong stable property for the slopes when $m \rightarrow \infty$. Our goal of this paper is to generalize the results in [DWX] to the higher rank case, that is, to Artin–Schreier–Witt towers whose Galois groups are canonically identified with \mathbb{Z}_{p^ℓ} which is the unramified extension of \mathbb{Z}_p of degree ℓ , by a suitable adaptation of the methods in [DWX]. The argument turns out to be more difficult because the space of characters is now multi-dimensional (see the discussion after Theorem 1.4).

Let us be more precise. Fix a prime number p . Let \mathbb{F}_q be a finite extension of \mathbb{F}_p of degree a so that $q = p^a$. Let ℓ be an integer which divides a . For an element $\bar{b} \in \overline{\mathbb{F}_q}^\times$, let $\omega(\bar{b})$ denote its Teichmüller lift in \mathbb{Z}_q (the unramified extension of \mathbb{Z}_p with the residue field \mathbb{F}_q); we put $\omega(0) = 0$. Let σ denote (the lift of) the arithmetic p -Frobenius on \mathbb{F}_q and \mathbb{Z}_q .

We fix a monic polynomial $\bar{f}(x) = x^d + \bar{a}_{d-1}x^{d-1} + \cdots + \bar{a}_0 \in \mathbb{F}_q[x]$ whose degree d is *not* divisible by p . We write $\bar{a}_d = 1$, and $a_i := \omega(\bar{a}_i)$ for $i = 0, \dots, d$. Let $f(x)$ denote the polynomial $x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Z}_q[x]$, called the *Teichmüller lift* of the polynomial $\bar{f}(x)$. The \mathbb{Z}_{p^ℓ} -Artin–Schreier–Witt tower associated to $f(x)$ is the sequence of curves C_m over \mathbb{F}_q defined by

$$C_m : \quad \underline{y}_m^{F^\ell} - \underline{y}_m = \sum_{i=0}^d (\bar{a}_i x^i, 0, 0, \dots),$$

where $\underline{y}_m = (y_m^{(1)}, y_m^{(2)}, \dots)$ are viewed as Witt vectors of length m , and \bullet^{F^ℓ} means raising each Witt coordinate to the p -th power. In explicit terms, this means that C_1 is the usual Artin–Schreier curve given by $y^{p^\ell} - y = \bar{f}(x)$, and C_2 is the curve above C_1 given by an additional equation (over \mathbb{F}_q)

$$z^{p^\ell} - z + \frac{y^{p^{\ell+1}} - y^p - (y^{p^\ell} - y)^p}{p} = \frac{f^\sigma(x^p) - (f(x))^p}{p} \pmod{p},$$

where σ is the Frobenius automorphism and $f^\sigma(x) := x^d + \sigma(a_{d-1})x^{d-1} + \cdots + \sigma(a_0)$.

The Galois group of the tower may be identified with \mathbb{Z}_{p^ℓ} , such that $\underline{a} \in \mathbb{Z}_{p^\ell}$ sends \underline{y}_m to $\underline{y}_m + \underline{a}_m$, where \underline{a}_m denotes the m -th truncated Witt vector of \underline{a} . Each curve C_m has a zeta function defined by

$$Z(C_m, s) = \exp \left(\sum_{k \geq 1} \frac{s^k}{k} \cdot \#C_m(\mathbb{F}_{q^k}) \right) = \frac{P(C_m, s)}{1 - qs},$$

where $P(C_m, s) \in 1 + s\mathbb{Z}[s]$ is a polynomial of degree $2g(C_m)$, pure of q -weight 1, and $g(C_m)$ denotes the genus of C_m .

Write \mathbb{C}_p for the completion of an algebraic closure of \mathbb{Q}_p , and let $\mathcal{O}_{\mathbb{C}_p}$ denote its valuation ring with maximal ideal $\mathfrak{m}_{\mathbb{C}_p}$. Using the Galois group, we may factor $Z(C_m, s)$ into a product of L-functions:

$$Z(C_m, s) = \prod_{\chi: \mathbb{Z}_{p^\ell}/p^m \mathbb{Z}_{p^\ell} \rightarrow \mathbb{C}_p^\times} L_f(\chi, s),$$

where for each character χ , $L_f(\chi, s)$ is the L -function on $\mathbb{A}_{\mathbb{F}_q}^1$ given by

$$(1.0.1) \quad L_f(\chi, s) = \prod_{x \in |\mathbb{A}^1|} \frac{1}{1 - \chi\left(\mathrm{Tr}_{\mathbb{Q}_q^{\mathrm{deg}(x)}/\mathbb{Q}_{p^\ell}}(f(\omega(x)))\right) s^{\mathrm{deg}(x)}},$$

where $|\mathbb{A}^1|$ denotes the set of closed points of $\mathbb{A}_{\mathbb{F}_q}^1$ and $\omega(x)$ denotes the Teichmüller lift of any of the conjugate geometric points in the closed point x . For $\chi = 1$, the L -function $L_f(1, s)$ is simply the trivial factor $1/(1 - qs)$, which is the zeta function of the affine line.

The goal of this paper is to understand the p -adic valuation of the zeros of these L -functions for all non-trivial finite characters χ . For this purpose, we will also need to consider the characters which are not finite and put them in a family. In this paper, all characters $\chi: \mathbb{Z}_{p^\ell} \rightarrow \mathbb{C}_p^\times$ are assumed to be continuous. For a finite character χ , let m_χ be the nonnegative integer so that the image of χ has cardinality p^{m_χ} ; we call m_χ the *conductor* of χ . Our normalization on Newton polygons is as follows: given a valuation ring R and an element ϖ of positive valuation, the ϖ -adic Newton polygon of a power series $c_0 + c_1 s + \cdots \in R[[s]]$ is the lower convex hull of the points $(k, \mathrm{val}_\varpi(c_k))$ ($k \in \mathbb{Z}_{\geq 0}$), where the valuation $\mathrm{val}_\varpi(-)$ is normalized so that $\mathrm{val}_\varpi(\varpi) = 1$.

Theorem 1.1 (Main Theorem). *For any nontrivial finite character χ with conductor m_χ , $L_f(\chi, s)$ is a polynomial of degree $dp^{m_\chi-1} - 1$. Write*

$$L_f(\chi, s) = \sum_{k=0}^{dp^{m_\chi-1}-1} c_k s^k.$$

We have the following:

- (i) For any $0 \leq n \leq p^{m_\chi-1}$, we have $\mathrm{val}_q(c_{nd-1}) = \frac{n(nd-1)}{2p^{m_\chi-1}}$ and $\mathrm{val}_q(c_{nd}) = \frac{n(nd+1)}{2p^{m_\chi-1}}$.
- (ii) For any $0 \leq n \leq p^{m_\chi-1}$, the q -adic Newton polygon of $L_f(\chi, s)$ passes through the points $(nd-1, \frac{n(nd-1)}{2p^{m_\chi-1}})$ and $(nd, \frac{n(nd+1)}{2p^{m_\chi-1}})$.
- (iii) The q -adic Newton polygon of $L_f(\chi, s)$ has slopes (in increasing order)

$$\bigcup_{i=1}^{p^{m_\chi-1}} \{\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{id}\} - \{0\},$$

where

$$\begin{cases} \alpha_{ij} = \frac{i-1}{p^{m_\chi-1}} & \text{for } j = 1, \\ \frac{i-1}{p^{m_\chi-1}} < \alpha_{ij} < \frac{i}{p^{m_\chi-1}} & \text{for } j > 1. \end{cases}$$

Remark 1.2. We do not know how to get the arithmetic progression property as in [DWX], which is uniform in χ (depending only on the large conductor m_χ , not on the choice of χ with the given conductor m_χ). However, for $j = 1$, the slopes $\alpha_{i1} = \frac{i-1}{p^{m_\chi-1}}$ do form an arithmetic progression, which depends only on the conductor m_χ . For any fixed $j > 1$, part (iii) only proves that the slopes α_{ij} are approximately an arithmetic progression.

If we restrict to those characters χ that factors through a fixed quotient $\eta: \mathbb{Z}_{p^\ell} \rightarrow \mathbb{Z}_p$, then the slopes α_{ij} form a union of finitely many arithmetic progressions (independent of the character χ but a priori depending on the quotient η), as the problem reduces to the case of usual \mathbb{Z}_p -tower but with non-Teichmüller polynomials considered in [Li]. It is unclear whether these arithmetic progressions depend on the choice of the quotient $\eta: \mathbb{Z}_{p^\ell} \rightarrow \mathbb{Z}_p$.

It would be more convenient for us to consider the p -adic function defined by

$$(1.2.1) \quad C_f^*(\chi, s) = L_f^*(\chi, s)L_f^*(\chi, qs)L_f^*(\chi, q^2s) \cdots,$$

where $L_f^*(\chi, s) := (1 - \chi(\text{Tr}_{\mathbb{Q}_q/\mathbb{Q}_{p^\ell}}(f(0)))s)L_f(\chi, s)$ is the L-function of χ over the torus $\mathbb{G}_m = \mathbb{A}^1 - 0$.

From $C_f^*(\chi, s)$, one may recover $L_f^*(\chi, s)$ as $L_f^*(\chi, s) = \frac{C_f^*(\chi, s)}{C_f^*(\chi, qs)}$. Hence Theorem 1.1 is essentially a corollary of the following

Theorem 1.3. *Given a nontrivial finite character χ with conductor m_χ , write*

$$C_f^*(\chi, s) = \sum_{k=0}^{\infty} w_k(\chi) s^k.$$

Then for all $k \geq 0$, we have

$$\text{val}_q(w_k(\chi)) \geq \frac{k(k-1)}{2dp^{m_\chi-1}} \quad \text{and,}$$

$$\text{For } k = nd \text{ or } nd + 1, \quad \text{val}_q(w_k(\chi)) = \frac{k(k-1)}{2dp^{m_\chi-1}}.$$

In particular, the q -adic Newton polygon of $C_f^*(\chi, s)$ passes through the points $(nd, \frac{n(nd-1)}{2p^{m_\chi-1}})$ and $(nd+1, \frac{n(nd+1)}{2p^{m_\chi-1}})$ for all $n \geq 0$.

We will show that both Theorem 1.1 and 1.3 follow from Theorem 1.4 below, in 2.6.

To effectively prove Theorem 1.3, it is important to consider all characters in a big family. We fix a basis $\{c_1, \dots, c_\ell\}$ of \mathbb{Z}_{p^ℓ} as a free \mathbb{Z}_p -module; we write $\bar{c}_j = c_j \bmod p$ for each j . The Galois group \mathbb{Z}_{p^ℓ} of the tower can be identified with \mathbb{Z}_p^ℓ explicitly as

$$\begin{aligned} \mathbb{Z}_{p^\ell} &\xrightarrow{\cong} \mathbb{Z}_p^\ell \\ x &\longmapsto (\text{Tr}_{\mathbb{Q}_{p^\ell}/\mathbb{Q}_p}(xc_1), \dots, \text{Tr}_{\mathbb{Q}_{p^\ell}/\mathbb{Q}_p}(xc_\ell)). \end{aligned}$$

We consider the *universal character* of \mathbb{Z}_{p^ℓ} :

$$\begin{aligned} \chi_{\text{univ}} : \quad \mathbb{Z}_{p^\ell} &\longrightarrow \mathbb{Z}_p[[\underline{T}]]^\times := \mathbb{Z}_p[[T_1, \dots, T_\ell]]^\times \\ x &\longmapsto (1 + T_1)^{\text{Tr}_{\mathbb{Q}_{p^\ell}/\mathbb{Q}_p}(xc_1)} \cdots (1 + T_\ell)^{\text{Tr}_{\mathbb{Q}_{p^\ell}/\mathbb{Q}_p}(xc_\ell)}. \end{aligned}$$

(When $\ell = 1$, we simply write T for \underline{T} .) Any continuous character $\chi : \mathbb{Z}_{p^\ell} \rightarrow \mathbb{C}_p^\times$ can be recovered from χ_{univ} by evaluating each T_j at $\chi(c_j^*) - 1$, where $c_1^*, \dots, c_\ell^* \in \mathbb{Z}_{p^\ell}$ are elements such that $\text{Tr}_{\mathbb{Q}_{p^\ell}/\mathbb{Q}_p}(c_i^*c_j)$ is equal to 1 if $i = j$ and is equal to 0 if $i \neq j$.

Similar to the finite character case, we can define the power series

$$C_f^*(\chi_{\text{univ}}, s) = C_f^*(\underline{T}, s) = 1 + w_1(\underline{T})s + w_2(\underline{T})s^2 + \cdots \in 1 + s\mathbb{Z}_p[[\underline{T}]][[s]],$$

for the universal character χ_{univ} ; for details, see Section 3. This power series interpolates $C_f^*(\chi, s)$ for all (finite) characters $\chi : \mathbb{Z}_{p^\ell} \rightarrow \mathbb{C}_p^\times$ via the formula

$$C_f^*(\chi, s) = C_f^*(\underline{T}, s)|_{T_j = \chi(c_j^*) - 1 \text{ for all } j}.$$

Theorem 1.4. *Let I denote the ideal $(T_1, \dots, T_\ell) \subseteq \mathbb{Z}_p[[\underline{T}]]$. For $k \in \mathbb{Z}_{\geq 0}$, we put $\lambda_k = \frac{ak(k-1)(p-1)}{2d}$. Then we have the following.*

(1) *For any $k > 0$, we have*

$$(1.4.1) \quad w_k(\underline{T}) \in I^{\lceil \lambda_k \rceil}.$$

(2) When $k = nd$ or $nd + 1$, we have

$$(1.4.2) \quad w_k(\underline{T}) \equiv u_k \cdot \mathfrak{S}(\underline{T})^{\lambda_k/\ell} \pmod{(pI^{\lambda_k} + I^{\lambda_k+1})^1}$$

for some unit $u_k \in \mathbb{Z}_p$, where $\mathfrak{S}(\underline{T})$ is the following polynomial

$$(1.4.3) \quad \mathfrak{S}(\underline{T}) := \prod_{i=1}^{\ell} \left(\sum_{j=1}^{\ell} \sigma^i(c_j) T_j \right).$$

Theorem 1.4 is the main technical result of this paper. Part (1) is proved at the end of Section 4; part (2) is proved at the end of Section 5, relying on the key Theorem 5.1.

Let us now explain the philosophical meaning of Theorem 1.4. The first estimate (1.4.1) uses a standard argument to establish certain Hodge bound. It implies, for example, when $k = nd$ or $nd + 1$, the “leading term” (if nonzero) of $w_k(\underline{T})$ must be a homogeneous polynomial of degree λ_k in \underline{T} .

- (i) When $\ell = 1$, this leading term has to be a monomial in T ; so specializing to any continuous non-trivial character χ of \mathbb{Z}_p , this “leading term” (if its coefficient is a p -adic unit) is also the “leading term” of $w_k(\chi)$. Theorem 1.4(2) is proved in [DWX, Proposition 3.4], which is the key of the proof of [DWX, Theorem 1.2].
- (ii) In clear contrast, when $\ell > 1$, this “leading term”, even if its coefficients are p -adic units, may not continue to have smaller valuation than higher degree terms after certain specialization. In particular, the naïve generalization of Theorem 1.3 to *all* non-trivial characters of \mathbb{Z}_{p^ℓ} is *false*. It is thus of crucial importance to understand: what does the “leading term” of $w_k(\underline{T})$ look like? This is exactly answered by (1.4.2), which shows that the “leading term” of $w_k(\underline{T})$ is, up to a p -adic unit, a power of a particular polynomial $\mathfrak{S}(\underline{T})$ independent of k and the Teichmüller polynomial f .

We also point out that the polynomial $\mathfrak{S}(\underline{T})$ modulo $pI^\ell + I^{\ell+1}$ is canonically independent of the choice of the basis $\{c_1, \dots, c_\ell\}$ (Lemma 2.1). Moreover, $\mathfrak{S}(\underline{T})$ is in some sense “elliptic” as its zero *avoids* all the evaluations of the T_j ’s corresponding to *finite* continuous non-trivial characters of \mathbb{Z}_{p^ℓ} (Lemma 2.5).

While Theorem 1.4 is known when $\ell = 1$ by [DWX], its proof for general ℓ is quite different. The idea lies in a careful study of the matrix whose characteristic power series gives rise to $C_f^*(\underline{T}, s)$. We do this in two steps. The first step is to show that the leading term of $w_k(\underline{T})$ comes from the determinant of the upper left $k \times k$ -submatrix. The second step is to show that the determinant of the mod p reduction of the upper left $k \times k$ -submatrix is “independent of ℓ ”, in the sense that it is the same matrix for the $\ell = 1$ case except replacing $T = T_1$ by the polynomial $\sum_{j=1}^{\ell} c_j T_j$; see Theorem 5.1.² In this way, we reduce the proof of Theorem 1.4 for general ℓ to the known case of $\ell = 1$.

1.5. Analogy with the Igusa tower of modular curves. An important philosophical implication of Theorem 1.4 is through the close analogy between the Artin–Schreier–Witt tower and the Igusa tower of modular varieties:

- the Galois group \mathbb{Z}_p of the \mathbb{Z}_p -Artin–Schreier–Witt tower ($\ell = 1$) is the additive version of the Galois group \mathbb{Z}_p^\times of the Igusa tower of modular curves,
- the big Banach module $\tilde{\mathbf{B}}$ in (3.7.1) is analogous to the space of overconvergent modular forms,
- the linear operator ψ defined in (3.7.2) is analogous to the U_p -operator, and
- the power series $C_f^*(T, s)$ is analogous to the characteristic power series of U_p .

¹Under the hypothesis $p \nmid d$, $\lambda_{nd}/\ell = \frac{an(nd-1)(p-1)}{2\ell}$ and $\lambda_{nd+1}/\ell = \lambda_{nd}/\ell + \frac{an(p-1)}{\ell}$ are always integers.

²The product over the Frobenius twists of c_j is a result of the setup of the Dwork’s trace formula; see Corollary 3.9.

Inspired by this analogy, we define the *Artin–Schreier–Witt eigenvariety* to be the zero locus of the universal multi-variable power series $C_f^*(\underline{T}, s)$ inside $\mathbb{G}_m^{\text{rig}} \times (\mathcal{W} - \{\underline{0}\})$,³ where \mathcal{W} is the *weight space* $\text{Max}(\mathbb{Z}_p[[\underline{T}]][\frac{1}{p}])$. As shown in the diagram below, this eigenvariety \mathcal{E}_f admits a *weight map* wt to the weight space, and an “ a_p -map” to $\mathbb{G}_{m,\text{rig}}$ remembering the value of s^{-1} . The p -adic valuation of the image of the “ a_p -map” is called the *slope* of the point.

$$\begin{array}{ccc} & \text{slope} & \\ & \curvearrowright & \\ \mathcal{E}_f & \xrightarrow{a_p} & \mathbb{C}_p^\times \xrightarrow{\text{val.}} \mathbb{Q}. \\ & \downarrow \text{wt} & \\ & \mathcal{W} - \{\underline{0}\} & \end{array}$$

This gives a full picture analogous to the case of eigencurves or more generally eigenvarieties.

A key component of the analogy is that the proof of the decomposition of the \mathbb{Z}_p -Artin–Schreier–Witt eigencurve ([DWX, Theorem 4.2]) is very similar to the proof of the decomposition of the Coleman–Mazur eigencurve over the boundary of the weight space ([LWX, Theorem 1.3]), where the Hodge estimate (see Definition 4.3) is analogous to [LWX, Proposition 3.12(1)], and the numerics provided by the Poincaré duality of L-functions corresponds to the numerics given by the Atkin–Lehner involution. The only difference is that the Hodge lower bound in [LWX] is obtained by a slightly different mechanism.⁴

The state-of-art technique on the study of spectral halo (based on [LWX]) is intrinsic to $\text{GL}_2(\mathbb{Q}_p)$.⁵ To extend [LWX] beyond this case, say to $\text{GL}_2(\mathbb{Q}_{p^\ell})$, it is natural to first study under the analogous Artin–Schreier–Witt setup. More precisely, the corresponding question concerns a \mathbb{Z}_{p^ℓ} -Artin–Schreier–Witt tower of varieties over the ℓ -dimensional base $(\mathbb{G}_m)^\ell$ for $\ell > 1$. There are now two distinct difficulties we encounter:

- (a) the weight space has become multi-dimensional, and
- (b) the base of the variety has become multi-dimensional.

Interestingly, for automorphic eigenvarieties, these two difficulties appear simultaneously, whereas on the Artin–Schreier–Witt side, we can tackle them one at a time.

This paper addresses the difficulty (a). The solution we propose is the following: it might be too much to ask for a decomposition of the eigenvariety over the entire weight space such that the slopes on each component are determined by the weight map, exactly because of the issue explained in (ii) after Theorem 1.4.⁶ Instead, we study a subspace of \mathcal{W} , the *admissible locus*, defined by

$$\mathcal{W}^{\text{adm}} := \{\underline{t} \in \mathcal{W}(\mathbb{C}_p) - \{\underline{0}\} \mid \text{val}_q(\mathfrak{S}(\underline{t})) = \ell \cdot \min\{\text{val}_q(t_1), \dots, \text{val}_q(t_\ell)\}\}.$$

This is an increasing union of affinoid subdomains of \mathcal{W} , which is independent of the polynomial f , and is canonically independent of the choice of basis $\{c_1, \dots, c_\ell\}$ (Corollary 2.4).

³For Artin–Schreier–Witt tower, the trivial character behaves slightly differently.

⁴In [LWX], we looked at the Betti realization instead of the de Rham realization to circumvent the technical difficulties caused by the geometry of the base modular curve.

⁵Recently, C. Johansson and J. Newton [JN] generalized the Hodge estimate of [LWX], but unfortunately the naïve generalization of the numerical coincidence no longer holds. They can still define certain extension of the eigenvariety to the “adic boundary” of the weight space. But much less is known regarding to this extension, compare to the $\text{GL}_2(\mathbb{Q}_p)$ -case.

⁶An alternative way to explain this is: the “adic boundary” of the weight space is (non-canonically isomorphic to) $\mathbb{P}_{\mathbb{F}_p}^{\ell-1}$; so when $\ell = 1$, there is only one direction approaching the boundary. But when $\ell > 1$, we may have to give up on some “bad directions” approaching the boundary. Theorem 1.4 says that the bad direction is exactly the

hypersurface defined by the polynomial $\prod_{i=1}^{\ell} \left(\sum_{j=1}^{\ell} \sigma^i(c_j) T_j \right) \bmod p$.

Moreover, \mathcal{W}^{adm} contains all points corresponding to *finite* non-trivial characters of \mathbb{Z}_p^ℓ (Lemma 2.5).

One corollary of Theorem 1.4 is the following.

Theorem 1.6. *Put $\mathcal{E}_f^{\text{adm}} := \text{wt}^{-1}(\mathcal{W}^{\text{adm}})$. Then $\mathcal{E}_f^{\text{adm}}$ is the disjoint union*

$$\mathcal{E}_f^{\text{adm}} = X_0 \coprod X_{(0,1)} \coprod X_1 \coprod X_{(1,2)} \coprod \cdots,$$

of infinitely many rigid subspaces, such that for each interval $J = [n, n]$ or $(n, n + 1)$,

- the map $\text{wt} : X_J \rightarrow \mathcal{W}^{\text{adm}}$ is finite and flat, and
- for each point $x \in X_J$, we have

$$\frac{\text{val}_q(a_p(x))}{\text{val}_q(\mathfrak{S}(\text{wt}(x)))} \in \frac{a(p-1)}{\ell} \cdot J.$$

This is Theorem 6.1.

Remark 1.7. One may interpret Theorem 1.6 as the pull-back of the following diagram:

$$\begin{array}{ccc} \text{decomposition pattern of } \mathcal{E}_f^{\text{adm}} & \longrightarrow & \text{decomposition pattern of } \mathcal{E}_f(\mathbb{Z}_p) \\ \downarrow & & \downarrow \\ \mathcal{W}^{\text{adm}} & \xrightarrow{T \mapsto T = \mathfrak{S}(T)} & \mathcal{W}(\mathbb{Z}_p) - \{0\}, \end{array}$$

where the right hand side is the corresponding theorem ([DWX, Theorem 4.2]) for the case $\ell = 1$.

Roadmap of the paper. In Section 2, we give several basic facts regarding the polynomial $\mathfrak{S}(\underline{T})$, and show that Theorems 1.1 and 1.3 follow from Theorem 1.4. Starting from Section 3, we use another set of variables $\underline{\pi}$ instead of \underline{T} . We define the characteristic power series $C_f^*(\underline{\pi}, s)$ in Section 3, and give a lower bound for its I -adic Newton polygon in Section 4. Section 5 is devoted to the proof of Theorem 1.4, by showing that its validity is independent of ℓ and hence reduce to the known case $\ell = 1$. Section 6 interprets everything in the language of eigenvarieties. In the appendix, we include several errata for the paper [DWX].

2. WEIGHT SPACE

We collect some basic facts regarding the weight space and characters of \mathbb{Z}_p^ℓ .

Lemma 2.1. (1) *The ideal $I = (T_1, \dots, T_\ell) \subseteq \mathbb{Z}_p[[\underline{T}]] \cong \mathbb{Z}_p[[\mathbb{Z}_p^\ell]]$ is canonically independent of the choice of the basis $\{c_1, \dots, c_\ell\}$.*

(2) *The polynomial $\mathfrak{S}(\underline{T}) \bmod pI^\ell + I^{\ell+1}$ is independent of the choice of the basis $\{c_1, \dots, c_\ell\}$.*

Proof. (1) Note that a change of basis of \mathbb{Z}_p^ℓ over \mathbb{Z}_p results in a change of variables of $\{T_1, \dots, T_\ell\}$ in a way that χ_{univ} is well-defined. In fact, I is the augmentation ideal, or equivalently the kernel of $\mathbb{Z}_p[[\mathbb{Z}_p^\ell]] \rightarrow \mathbb{Z}_p$. So it is canonically independent of the choice of the basis.

(2) The group of all possible change of basis matrices $\text{GL}_\ell(\mathbb{Z}_p)$ is generated by the following three types:

- (a) only swapping c_i with c_j ;
- (b) for a unique fixed i , scaling c_i to $u_i c_i$ for $u_i \in \mathbb{Z}_p^\times$;
- (c) only changing c_1 to $c_1 + u c_2$ for $u \in \mathbb{Z}_p$.

It suffices to check the independence of $\mathfrak{S}(\underline{T}) \bmod pI^\ell + I^{\ell+1}$ under these three changes of basis. Case (a) will result in swapping T_i with T_j . The independence of $\mathfrak{S}(\underline{T})$ follows from the definition. Case (b) will result in changing T_i to $(1 + T_i)^{u_i^{-1}} - 1$. The invariance of $\mathfrak{S}(\underline{T}) \bmod pI^\ell + I^{\ell+1}$ of this change of basis follows from the congruence

$$c_i T_i \equiv u_i c_i ((1 + T_i)^{u_i^{-1}} - 1) \pmod{pI + I^2}.$$

Case (c) will result in changing T_2 to $(1 + T_2)(1 + T_1)^{-u} - 1$, and keeping all the other variables unchanged. Then the invariance of $\mathfrak{S}(\underline{T}) \bmod pI^\ell + I^{\ell+1}$ of $\mathfrak{S}(\underline{T})$ follows from the congruence

$$c_1 T_1 + c_2 T_2 \equiv (c_1 + u c_2) T_1 + c_2 ((1 + T_2)(1 + T_1)^{-u} - 1) \pmod{pI + I^2}. \quad \square$$

Remark 2.2. In view of Lemma 2.1, the validity of Theorem 1.4 is independent of $\{c_1, \dots, c_\ell\}$; so it suffices to prove it for a particular choice of basis $\{c_1, \dots, c_\ell\}$.

2.3. Weight space. Using the variables T_1, \dots, T_ℓ , we can explicitly present the weight space as

$$\mathcal{W} := \text{Max}(\mathbb{Z}_p[[\mathbb{Z}_p^\ell]][\frac{1}{p}]) = \{(t_1, \dots, t_\ell) \in \mathbb{C}_p \mid \text{val}_q(t_j) > 0 \text{ for all } j\}.$$

Since $\mathfrak{S}(\underline{T})$ is a homogeneous polynomial of degree ℓ , we have

$$\text{val}_q(\mathfrak{S}(\underline{t})) \geq \ell \cdot \min\{\text{val}_q(t_1), \dots, \text{val}_q(t_\ell)\}.$$

Our theory will apply to the case when the above inequality is an equality, namely over the *admissible locus*

$$\mathcal{W}^{\text{adm}} := \{(t_1, \dots, t_\ell) \in \mathcal{W} - \{0\} \mid \text{val}_q(\mathfrak{S}(\underline{t})) = \ell \cdot \min\{\text{val}_q(t_1), \dots, \text{val}_q(t_\ell)\}\}.$$

Corollary 2.4. *The admissible locus $\mathcal{W}^{\text{adm}} \subset \mathcal{W}$ is independent of the choice of the basis $\{c_1, \dots, c_\ell\}$.*

Proof. This follows from Lemma 2.1(2) and the definition of the admissible locus. \square

Lemma 2.5. *Let $\chi : \mathbb{Z}_p^\ell \rightarrow \mathbb{C}_p^\times$ be a finite non-trivial character. Its coordinate on the weight space is given by $t_{j,\chi} := \chi(c_j^*) - 1$ for $j = 1, \dots, \ell$. Then this point lies on the admissible locus \mathcal{W}^{adm} .*

Proof. The coordinate of χ is clearly as given. Let m_χ denote the conductor of χ , so that the image of χ lies in $\mathbb{Z}_p[\zeta_{p^{m_\chi}}]$. In particular, each $t_{j,\chi} \in \mathbb{Z}_p[\zeta_{p^{m_\chi}}]$.

Note that c_1, \dots, c_ℓ form a basis of \mathbb{Z}_p^ℓ over \mathbb{Z}_p . So they also form an *orthonormal basis* of $\mathbb{Z}_p^\ell[\zeta_{p^{m_\chi}}]$ over $\mathbb{Z}_p[\zeta_{p^{m_\chi}}]$. It then follows that

$$\text{val}_q\left(\sum_{j=1}^{\ell} c_j t_{j,\chi}\right) = \min\{\text{val}_q(t_{1,\chi}), \dots, \text{val}_q(t_{\ell,\chi})\}.$$

Taking the norm from $\mathbb{Z}_p^\ell[\zeta_{p^{m_\chi}}]$ to $\mathbb{Z}_p[\zeta_{p^{m_\chi}}]$ shows that

$$\text{val}_q(\mathfrak{S}(t_{1,\chi}, \dots, t_{\ell,\chi})) = \ell \cdot \min\{\text{val}_q(t_{1,\chi}), \dots, \text{val}_q(t_{\ell,\chi})\}.$$

This means that the point corresponding to χ lies in \mathcal{W}^{adm} . \square

2.6. Proof of Theorem 1.4 \Rightarrow Theorem 1.1 and 1.3. For a finite non-trivial character χ with coordinates $t_{j,\chi} = \chi(c_j^*) - 1$, we know that

$$\begin{aligned} \min \{ \text{val}_q(t_{1,\chi}), \dots, \text{val}_q(t_{\ell,\chi}) \} &= \min \{ \text{val}_q(\chi(c_1^*) - 1), \dots, \text{val}_q(\chi(c_\ell^*) - 1) \} \\ &= \frac{1}{a} \cdot \frac{1}{p^{m_\chi - 1}(p - 1)}. \end{aligned}$$

Hence Theorem 1.4(1) implies

$$(2.6.1) \quad \text{val}_q(w_k(\chi)) \geq \lambda_k \cdot \frac{1}{a} \cdot \frac{1}{p^{m_\chi - 1}(p - 1)} = \frac{k(k - 1)}{2dp^{m_\chi - 1}}.$$

Moreover, by Lemma 2.5, the point corresponding to this finite character χ lies on the admissible locus \mathcal{W}^{adm} . So Theorem 1.4(2) implies that the equality in (2.6.1) holds for $k = nd$ or $nd + 1$.

From this, we deduce that the q -adic Newton polygon of $C_f^*(\chi, s)$ lies above the polygon with vertices $(k, \frac{k(k-1)}{2dp^{m_\chi-1}})$, and so it must pass through the points $(nd, \frac{n(nd-1)}{2p^{m_\chi-1}})$ and $(nd + 1, \frac{n(nd+1)}{2p^{m_\chi-1}})$ given by $(k, \text{val}_q(w_k(\chi)))$ for $k = nd$ and $nd + 1$ with $n \in \mathbb{Z}_{\geq 0}$. This completes the proof of Theorem 1.3.

For Theorem 1.1, we observe that $L_f^*(\chi, s) = \frac{C_f^*(\chi, s)}{C_f^*(\chi, qs)}$ is a polynomial of degree $dp^{m_\chi - 1}$, and the set $\{ \alpha \in \mathbb{C}_p \mid \alpha^{-1} \text{ is a root of } L_f^*(\chi, s) = 0 \}$ is the same as the set

$$\{ \beta \in \mathbb{C}_p \mid \beta^{-1} \text{ is a root of } C_f^*(\chi, s) = 0 \text{ and } \text{val}_q(\beta) \in [0, 1] \}.$$

So Theorem 1.1 follows from Theorem 1.3 directly as $L_f(\chi, s)$ is obtained from $L_f^*(\chi, s)$ by removing its unique linear factor with slope zero. \square

Remark 2.7. The same argument proves the analog of Theorem 1.3 for all continuous characters χ of \mathbb{Z}_p^ℓ whose corresponding point on the weight space lies in \mathcal{W}^{adm} .

3. I -ADIC EXPONENTIAL SUMS

We fix the polynomial \bar{f} and its Teichmüller lift f as in the introduction.

Notation 3.1. We first recall that the *Artin–Hasse exponential series* is defined by

$$(3.1.1) \quad E(\pi) = \exp \left(\sum_{i=0}^{\infty} \frac{\pi^{p^i}}{p^i} \right) = \prod_{p \nmid i, i \geq 1} (1 - \pi^i)^{-\mu(i)/i} \in 1 + \pi + \pi^2 \mathbb{Z}_p \llbracket \pi \rrbracket.$$

Setting $T = E(\pi) - 1$ defines an isomorphism $\mathbb{Z}_p \llbracket \pi \rrbracket \cong \mathbb{Z}_p \llbracket T \rrbracket$.

For the rest of the paper, it will be more convenient to set $T_j = E(\pi_j) - 1$ for each j and use π_1, \dots, π_ℓ as the parameters for the ring $\mathbb{Z}_p \llbracket \underline{T} \rrbracket \cong \mathbb{Z}_p \llbracket \underline{\pi} \rrbracket$. In particular, we have

$$I = (T_1, \dots, T_\ell) = (\pi_1, \dots, \pi_\ell).$$

Definition 3.2. For a positive integer k , the *I -adic exponential sum* of f over $\mathbb{F}_{q^k}^\times$ is

$$S^*(k, \underline{\pi}) := \sum_{x \in \mathbb{F}_{q^k}^\times} \prod_{j=1}^{\ell} E(\pi_j)^{\text{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p} [c_j f(\omega(x))]} \in \mathbb{Z}_p \llbracket \underline{\pi} \rrbracket.^7$$

⁷This sum agrees with $S_f(k, T)$ in [LW] (in the one-dimensional case).

Note that the sum is taken over $\mathbb{F}_{q^k}^\times$. The superscript $*$ reminds us that we are working over the torus \mathbb{G}_m . We define the *I-adic characteristic power series associated to f* to be

$$(3.2.1) \quad \begin{aligned} C_f^*(\underline{\pi}, s) &:= \exp\left(\sum_{k=1}^{\infty} \frac{1}{1-q^k} S^*(k, \underline{\pi}) \frac{s^k}{k}\right)^8 \\ &= \sum_{k=0}^{\infty} w_k(\underline{\pi}) s^k \in \mathbb{Z}_p[[\underline{\pi}, s]]. \end{aligned}$$

The *I-adic L-series of f* is defined by

$$L_f^*(\underline{\pi}, s) = \exp\left(\sum_{k=1}^{\infty} S^*(k, \underline{\pi}) \frac{s^k}{k}\right).$$

These two series determine each other, and are related by the relation

$$C_f^*(\underline{\pi}, s) = L_f^*(\underline{\pi}, s) L_f^*(\underline{\pi}, qs) L_f^*(\underline{\pi}, q^2 s) \cdots.$$

It is clear that for a finite character $\chi : \mathbb{Z}_p^\ell \rightarrow \mathbb{C}_p^\times$,

$$L_f^*(\chi, s) = L_f^*(\underline{\pi}, s)|_{E(\pi_j)=\chi(c_j^*) \text{ for all } j}, \quad C_f^*(\chi, s) = C_f^*(\underline{\pi}, s)|_{E(\pi_j)=\chi(c_j^*) \text{ for all } j}.$$

Here the subscript means to evaluate the power series at $\pi_j \in \mathfrak{m}_{\mathbb{C}_p}$ for which $E(\pi_j) = \chi(c_j^*)$ (the elements c_j^* are defined just before Theorem 1.4).

Hypothesis 3.3. From now till the end of Section 5, assume the chosen basis $\{c_1, \dots, c_\ell\}$ consists of Teichmüller lifts, i.e. $c_j = \omega(\bar{c}_j)$ for $j = 1, \dots, \ell$.

Notation 3.4. For our given polynomial $f(x) = \sum_{i=0}^d a_i x^i \in \mathbb{Z}_q[x]$, we put

$$(3.4.1) \quad E_f(x)_\pi := \prod_{i=0}^d E(a_i \pi x^i) \in \mathbb{Z}_q[[\pi]][[x]].$$

So $E_{c_j f}(x)_{\pi_j}$ would mean $\prod_{i=0}^d E(c_j a_i \pi_j x^i)$. If σ denotes arithmetic p -Frobenius automorphism which acts naturally on \mathbb{Q}_q , and trivially on π and x , then we have, for every $j \in \mathbb{Z}_{\geq 0}$,

$$E_f^{\sigma^j}(x)_\pi = \prod_{i=0}^d E(a_i^{\sigma^j} \pi x^i) \in \mathbb{Z}_q[[\pi]][[x]].$$

Lemma 3.5. (1) *If we write $E_f(x)_\pi = \sum_{n=0}^{\infty} b_n(\pi) x^n \in \mathbb{Z}_q[[\pi]][[x]]$, then $b_n(\pi) \in \pi^{\lceil n/d \rceil} \mathbb{Z}_q[[\pi]]$.*

(2) *If we write $\prod_{j=1}^{\ell} E_{c_j f}(x)_{\pi_j} = \sum_{n=0}^{\infty} e_n(\underline{\pi}) x^n \in \mathbb{Z}_q[[\underline{\pi}]] [[x]]$, then $e_n(\underline{\pi}) \in I^{\lceil n/d \rceil}$ and $e_0 = 1$.*

Proof. Note that the i th factor of $E_f(x)_\pi$ in (3.4.1) is a power series in πx^i for $1 \leq i \leq d$; so every term in their product is a sum of products of $\pi, \pi x, \dots, \pi x^d$. (1) is clear from this. (2) follows from (1) immediately. \square

Convention 3.6. In this paper, the row and column indices of matrices start with zero.

⁸Our $C_f^*(\underline{\pi}, s)$ agrees with the $C_f(T, s)$ in [LW] (in the one-dimensional case); we will not introduce a version $C(T, s)$ without the star since it will not be used in our proof.

3.7. Dwork's trace formula. Consider the following “Banach module” over $\mathbb{Z}_q[[\pi]]$ with “orthonormal basis” $\Gamma := \{1, x, x^2, \dots\}$:⁹

$$(3.7.1) \quad \tilde{\mathbf{B}} := \mathbb{Z}_q[[\pi]]\langle x \rangle = \left\{ \sum_{n=0}^{\infty} d_n(\pi)x^n \mid d_n(\pi) \in \mathbb{Z}_q[[\pi]] \text{ and } \lim_{n \rightarrow \infty} d_n = 0 \right\}.^{10}$$

Let ψ_p denote the operator on $\tilde{\mathbf{B}}$ defined by

$$\psi_p \left(\sum_{n \geq 0} d_n(\pi)x^n \right) := \sum_{n \geq 0} d_{pn}(\pi)x^n,$$

and let ψ be the composite linear operator

$$(3.7.2) \quad \psi := \psi_p \circ \prod_{j=1}^{\ell} E_{c_j f}(x)_{\pi_j} : \tilde{\mathbf{B}} \longrightarrow \tilde{\mathbf{B}},$$

where $\prod_{j=1}^{\ell} E_{c_j f}(x)_{\pi_j}(g) := \prod_{j=1}^{\ell} E_{c_j f}(x)_{\pi_j} \cdot g$ for any $g \in \tilde{\mathbf{B}}$. One can easily check that

$$\psi(x^n) = \sum_{m=0}^{\infty} e_{mp-n}(\pi)x^m,$$

where $e_n = e_n(\pi)$ is as defined in Lemma 3.5(2). Explicitly, the matrix of ψ with respect to the basis $\Gamma := \{1, x, x^2, \dots\}$ is given by

$$(3.7.3) \quad N = (e_{mp-n})_{m,n \geq 0} = \begin{pmatrix} e_0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots \\ e_p & e_{p-1} & \cdots & e_0 & 0 & \cdots & 0 & \cdots \\ e_{2p} & e_{2p-1} & \cdots & e_p & e_{p-1} & \cdots & e_0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots \\ e_{mp} & e_{mp-1} & \cdots & e_{mp-p} & e_{mp-p-1} & \cdots & e_{mp-2p} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}.$$

The operator $\sigma^{-1} \circ \psi$ is σ^{-1} -linear, but its a -th iteration $(\sigma^{-1} \circ \psi)^a$ is linear since σ^a acts trivially on $\mathbb{Z}_q[[\pi]]$. For the same reason, $\sigma^a(N) = N$.

Theorem 3.8 (Dwork Trace Formula). *For every $k > 0$, we have*

$$S^*(k, \pi) = (q^k - 1) \text{Tr}_{\tilde{\mathbf{B}}/\mathbb{Z}_q[[\pi]]} ((\sigma^{-1} \circ \psi)^{ak}).$$

Proof. The proof is the same as in [LW, Lemma 4.7]. The key point is that the Dwork trace formula is universally true, see [W] for a thorough understanding of the universal Dwork trace formula. \square

Corollary 3.9. *The theorem above has an equivalent multiplicative form:*

$$(3.9.1) \quad C_f^*(\pi, s) = \det(I - s\sigma^{a-1}(N) \cdots \sigma(N)N).$$

⁹Since $\mathbb{Z}_q[[\pi]]$ is not a Banach algebra, $\tilde{\mathbf{B}}$ is not a Banach space in the literal sense.

¹⁰This $\tilde{\mathbf{B}}$ is different from the space B considered in [DWX, Section 2], where the extra rescaling factors $\pi^{i/d}$ are used to simplify the notation of the proof. We cannot do such simplification over a multi-dimensional weight space.

Proof. It follows from the following list of equalities

$$\begin{aligned}
C_f^*(\pi, s) &= \exp\left(\sum_{k=1}^{\infty} \frac{1}{1-q^k} S^*(k, \pi) \frac{s^k}{k}\right) \\
&= \exp\left(\sum_{k=1}^{\infty} -\mathrm{Tr}_{\tilde{\mathbf{B}}/\mathbb{Z}_q[[\pi]]}((\sigma^{-1} \circ \psi)^{ak}) \frac{s^k}{k}\right) \\
&= \det\left(I - (\sigma^{-1} \circ \psi)^a s \mid \tilde{\mathbf{B}}\right) \\
&= \det\left(I - s\sigma^{-1}(N)\sigma^{-2}(N) \cdots \sigma^{-a}(N)\right) \\
&= \det\left(I - s\sigma^{a-1}(N) \cdots \sigma(N)N\right). \quad \square
\end{aligned}$$

4. A HODGE BOUND FOR $C_f^*(\underline{T}, s)$

In this section, we prove Theorem 1.4(1), which will follow from the key estimate of a certain (variant of) Hodge polygon bound in Proposition 4.6. We continue to assume Hypothesis 3.3.

Notation 4.1. We define a “valuation function”

$$\begin{aligned}
\mathrm{val}_I : \mathbb{Z}_q[[\pi]] &\longrightarrow \mathbb{Z} \cup \{\infty\}, \\
\mathrm{val}_I(x) &= \begin{cases} n & \text{if } x \in I^n \text{ and } x \notin I^{n+1}, \\ \infty & \text{if } x = 0. \end{cases}
\end{aligned}$$

Note that $\mathrm{val}_I(ab) = \mathrm{val}_I(a) + \mathrm{val}_I(b)$ for $a, b \in \mathbb{Z}_q[[\pi]]$.

Remark 4.2. Using this “valuation function”, we can similarly define the I -adic Newton polygon of a power series $\sum_{k \geq 0} c_k(\pi)s^k \in \mathbb{Z}_q[[\pi, s]]$ to be the lower convex hull of the points $(k, \mathrm{val}_I(c_k(\pi)))$. Then Theorem 1.4 says that the I -adic Newton polygon of $C_f^*(\pi, s)$ lies above the polygon with vertices (k, λ_k) with $\lambda_k = \frac{ak(k-1)(p-1)}{2d}$, and it passes through the points (nd, λ_{nd}) and $(nd+1, \lambda_{nd+1})$ for all $n \in \mathbb{Z}_{\geq 0}$.

Definition 4.3. Let $M_{\infty}(\mathbb{Z}_q[[\pi]])$ denote the set of matrices with entries in $\mathbb{Z}_q[[\pi]]$, whose rows and columns are indexed by $\mathbb{Z}_{\geq 0}$ (recall from Convention 3.6 that all row and column indices start from 0).

We say a matrix $N = (h_{mn})_{m, n \geq 0} \in M_{\infty}(\mathbb{Z}_q[[\pi]])$ is *twisted I -adically incremental (in d steps)* if $\mathrm{val}_I(h_{mn}) \geq \frac{mp-n}{d}$ (or equivalently $\mathrm{val}_I(h_{mn}) \geq \lceil \frac{mp-n}{d} \rceil$) for all integers $m, n \geq 0$.¹¹ By Lemma 3.5(2) and (3.7.3), we see that the matrix N and more generally $\sigma^i(N)$ is twisted I -adically incremental for every i .

Proposition 4.6 below allows us to control the I -adic Newton polygon of $C_f^*(\pi, s)$ using the twisted I -adic incrementing properties of these $\sigma^i(N)$'s.

Notation 4.4. For a matrix M , we write

$$\begin{bmatrix} m_0 & m_1 & \cdots & m_{k-1} \\ n_0 & n_1 & \cdots & n_{k-1} \end{bmatrix}_M$$

for the $k \times k$ -matrix formed by elements whose row indices belong to $\{m_0, m_1, \dots, m_{k-1}\}$ and whose column indices belong to $\{n_0, n_1, \dots, n_{k-1}\}$.

¹¹We invite the readers to compare this with [LWX, Proposition 3.12(1)], which is the estimate before the conjugation by a diagonal matrix.

Lemma 4.5. *Let $M = (h_{mn}) \in M_\infty(\mathbb{Z}_q[[\pi]])$ be a twisted I -adically incremental matrix, then for indices m_0, \dots, m_{k-1} and n_0, \dots, n_{k-1} , we have*

$$\text{val}_I \left(\det \begin{bmatrix} m_0 & m_1 & \cdots & m_{k-1} \\ n_0 & n_1 & \cdots & n_{k-1} \end{bmatrix}_M \right) \geq \sum_{i=0}^{k-1} \frac{pm_i - n_i}{d}.$$

Proof. In fact, we show that the val_I of each term in the determinant above is greater than or equal to $\sum_{i=0}^{k-1} \frac{pm_i - n_i}{d}$. Indeed, for each permutation $\sigma \in \text{Aut}(\{0, \dots, k-1\})$, we have

$$\text{val}_I(h_{m_0 n_{\sigma(0)}} \cdots h_{m_{k-1} n_{\sigma(k-1)}}) \geq \frac{pm_0 - n_{\sigma(0)}}{d} + \cdots + \frac{pm_{k-1} - n_{\sigma(k-1)}}{d} \geq \sum_{i=0}^{k-1} \frac{pm_i - n_i}{d}.$$

The lemma follows. \square

Proposition 4.6. *Let $M_0, M_1, \dots, M_{a-1} \in M_\infty(\mathbb{Z}_q[[\pi]])$ be twisted I -adically incremental matrices, and let $\det(I - sM_{a-1} \cdots M_1 M_0) = \sum_{k=0}^{\infty} (-1)^k r_k(\pi) s^k$ denote the characteristic power series of their product, then for every integer $k \geq 0$, we have*

$$\text{val}_I(r_k(\pi)) \geq \frac{ak(k-1)(p-1)}{2d}, \quad \text{and}$$

$$r_k(\pi) \equiv \prod_{j=0}^{a-1} \left(\det \begin{bmatrix} 0 & 1 & \cdots & k-1 \\ 0 & 1 & \cdots & k-1 \end{bmatrix}_{M_j} \right) \pmod{I^{\lceil \frac{ak(k-1)(p-1)+(p-1)}{2d} \rceil}}.$$

Proof. From the definition of characteristic power series, we see

$$\begin{aligned} (4.6.1) \quad r_k(\pi) &= \sum_{0 \leq m_0 < m_1 < \cdots < m_{k-1} < \infty} \det \begin{bmatrix} m_0 & m_1 & \cdots & m_{k-1} \\ m_0 & m_1 & \cdots & m_{k-1} \end{bmatrix}_{M_{a-1} \cdots M_1 M_0} \\ &= \sum_{\substack{0 \leq m_{0,0} < m_{0,1} < \cdots < m_{0,k-1} < \infty \\ \dots \\ 0 \leq m_{a-1,0} < m_{a-1,1} < \cdots < m_{a-1,k-1} < \infty}} \det \left(\prod_{j=0}^{a-1} \begin{bmatrix} m_{j+1,0} & m_{j+1,1} & \cdots & m_{j+1,k-1} \\ m_{j,0} & m_{j,1} & \cdots & m_{j,k-1} \end{bmatrix}_{M_j} \right) \\ &= \sum_{\substack{0 \leq m_{0,0} < m_{0,1} < \cdots < m_{0,k-1} < \infty \\ \dots \\ 0 \leq m_{a-1,0} < m_{a-1,1} < \cdots < m_{a-1,k-1} < \infty}} \prod_{j=0}^{a-1} \left(\det \begin{bmatrix} m_{j+1,0} & m_{j+1,1} & \cdots & m_{j+1,k-1} \\ m_{j,0} & m_{j,1} & \cdots & m_{j,k-1} \end{bmatrix}_{M_j} \right). \end{aligned}$$

Here and after, we set $m_{a,i} = m_{0,i}$ for all $0 \leq i \leq k-1$.

Since every M_i is twisted I -adically incremental, we can control each term in (4.6.1) using Lemma 4.5:

$$\begin{aligned} (4.6.2) \quad & \text{val}_I \left(\prod_{j=0}^{a-1} \left(\det \begin{bmatrix} m_{j+1,0} & m_{j+1,1} & \cdots & m_{j+1,k-1} \\ m_{j,0} & m_{j,1} & \cdots & m_{j,k-1} \end{bmatrix}_{M_j} \right) \right) \\ & \geq \sum_{j=0}^{a-1} \sum_{i=0}^{k-1} \frac{pm_{j+1,i} - m_{j,i}}{d} = \frac{p-1}{d} \sum_{j=0}^{a-1} \sum_{i=0}^{k-1} m_{j,i} \geq \frac{ak(k-1)(p-1)}{2d}. \end{aligned}$$

This verifies the first statement.

Notice that the last inequality of (4.6.2) is an equality if and only if $m_{j,i} = i$ for all $0 \leq j \leq a-1$ and $0 \leq i \leq k-1$; and when it is not an equality, (4.6.2) is greater than or equal to $\frac{ak(k-1)(p-1)+(p-1)}{2d}$. Therefore, we have

$$r_k(\underline{\pi}) \equiv \prod_{j=0}^{a-1} \left(\det \begin{bmatrix} 0 & 1 & \cdots & k-1 \\ 0 & 1 & \cdots & k-1 \end{bmatrix}_{M_j} \right) \pmod{I^{\lceil \frac{ak(k-1)(p-1)+(p-1)}{2d} \rceil}}. \quad \square$$

Proof of Theorem 1.4(1). By Corollary 3.9, $C_f^*(\underline{\pi}, s)$ is the characteristic power series of the product $\sigma^{a-1}(N) \cdots \sigma(N)N$. But each $\sigma^i(N)$ is twisted I -adically incremental, which implies Theorem 1.4(1) by applying Proposition 4.6.¹² \square

5. THE PROOF OF THEOREM 1.4(2)

As a reminder, Hypothesis 3.3 is still in force in this section. This section is devoted to prove Theorem 1.4(2), whose proof will appear at the end of this section. Its key ingredient is the following

Theorem 5.1. *Put $\bar{\mathfrak{X}} := \sum_{j=1}^{\ell} \bar{c}_j \pi_j \in \mathbb{F}_q[[\underline{\pi}]]$, then*

$$\det \begin{bmatrix} 0 & 1 & \cdots & k-1 \\ 0 & 1 & \cdots & k-1 \end{bmatrix}_N \pmod{p}$$

viewed as an element of $\mathbb{F}_q[[\underline{\pi}]]$, lies in $\mathbb{F}_q[[\bar{\mathfrak{X}}]]$. Moreover, the coefficients of this determinant as a power series in $\bar{\mathfrak{X}}$ does not depend on ℓ .

Proof. We write $\bar{e}_n := e_n \pmod{p} \in \mathbb{F}_q[[\underline{\pi}]]$. Consider the following $(kp-p+1) \times (kp-p+1)$ matrix

$$\bar{N}_k^T = \begin{bmatrix} \overbrace{\begin{matrix} \bar{e}_0 & \bar{e}_p & \bar{e}_{2p} & \cdots & \bar{e}_{(k-1)p} \\ 0 & \bar{e}_{p-1} & \bar{e}_{2p-1} & \cdots & \bar{e}_{(k-1)p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \bar{e}_0 & \bar{e}_p & \cdots & \bar{e}_{(k-1)p-p} \\ 0 & 0 & \bar{e}_{p-1} & \cdots & \bar{e}_{(k-1)p-p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & * & \cdots & \bar{e}_{(k-1)(p-1)} \\ 0 & 0 & * & \cdots & \bar{e}_{(k-1)(p-1)-1} \\ 0 & 0 & * & \cdots & \bar{e}_{(k-1)(p-1)-2} \\ 0 & 0 & * & \cdots & \bar{e}_{(k-1)(p-1)-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \bar{e}_0 \end{matrix}}^k & \overbrace{\begin{matrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{matrix}}^{(k-1)(p-1)} \end{bmatrix}.$$

¹²Although this section assumes Hypothesis 3.3, as pointed out in Remark 2.2, the validity of Theorem 1.4(1) does not depend on the choice of the basis $\{c_1, \dots, c_\ell\}$. So our proof is complete.

Note that the upper left $k \times k$ -block of \bar{N}_k^T is the transpose of $\begin{bmatrix} 0 & 1 & \cdots & k-1 \\ 0 & 1 & \cdots & k-1 \end{bmatrix}_N$ modulo p ,¹³ so we have an equality in $\mathbb{F}_q[[\pi]]$:

$$(5.1.1) \quad \det \begin{bmatrix} 0 & 1 & \cdots & k-1 \\ 0 & 1 & \cdots & k-1 \end{bmatrix}_N \pmod{p} = \det(\bar{N}_k^T).$$

To study \bar{N}_k^T , we need the following

Lemma 5.2. *We have the following equality and congruence.*

$$(5.2.1) \quad ne_n = \sum_{i=1}^d \sum_{r=0}^{\infty} i \cdot e_{n-ip^r} a_i^{p^r} \left(\sum_{j=1}^{\ell} (c_j \pi_j)^{p^r} \right)$$

$$(5.2.2) \quad \equiv \sum_{i=1}^d \sum_{r=0}^{\infty} i \cdot e_{n-ip^r} a_i^{p^r} \left(\sum_{j=1}^{\ell} c_j \pi_j \right)^{p^r} \pmod{p}.$$

Proof. Taking the derivative of $\prod_{j=1}^{\ell} E_{c_j f}(x)_{\pi_j}$ gives

$$\left(\prod_{j=1}^{\ell} E_{c_j f}(x)_{\pi_j} \right)' = \left(\prod_{j=1}^{\ell} E_{c_j f}(x)_{\pi_j} \right) \left(\sum_{i=1}^d \sum_{r=0}^{\infty} \left(\sum_{j=1}^{\ell} (c_j \pi_j)^{p^r} \right) i x^{ip^r-1} a_i^{p^r} \right).$$

Replacing $\prod_{j=1}^{\ell} E_{c_j f}(x)_{\pi_j}$ by $\sum_{n=0}^{\infty} e_n x^n$, the above equality becomes

$$\sum_{n=0}^{\infty} n e_n x^{n-1} = \left(\sum_{n=0}^{\infty} e_n x^n \right) \left(\sum_{i=1}^d \sum_{r=0}^{\infty} \left(\sum_{j=1}^{\ell} (c_j \pi_j)^{p^r} \right) i x^{ip^r-1} a_i^{p^r} \right).$$

Then (5.2.1) follows by comparing the x^{n-1} -coefficients. The congruence (5.2.2) follows from the easy fact that $\left(\sum_{j=1}^{\ell} c_j \pi_j \right)^{p^r} \equiv \sum_{j=1}^{\ell} (c_j \pi_j)^{p^r} \pmod{p}$. \square

We now continue with the proof of Theorem 5.1. Let $\bar{N}_{k,1}^T$ be the matrix consisting of the first k columns of \bar{N}_k^T . Then the (m, n) -entry of $\bar{N}_{k,1}^T$ is just \bar{e}_{np-m} . Applying Lemma 5.2 to $np-m$ in place of n (and then taking the reduction modulo p), we deduce

$$-m \bar{e}_{np-m} = (np-m) \bar{e}_{np-m} = \sum_{i=1}^d \sum_{r=0}^{\infty} i \cdot \bar{e}_{np-(m+ip^r)} \bar{a}_i^{p^r} \bar{\mathfrak{X}}^{p^r}$$

in $\mathbb{F}_q[[\pi]]$. Note that the coefficients in the above congruence does not involve the column index n . So if we use $\bar{R}_m(k)$ to denote the m th row of $\bar{N}_{k,1}^T$, we get

$$(5.2.3) \quad m \cdot \bar{R}_m(k) + \sum_{i=1}^d \sum_{r=0}^{\infty} i \cdot \bar{R}_{m+ip^r}(k) \bar{a}_i^{p^r} \bar{\mathfrak{X}}^{p^r} = 0$$

for all $0 \leq m \leq kp-p$. In other words, the m th row of $\bar{N}_{k,1}^T$ with $m \not\equiv 0 \pmod{p}$ can be written as a linear combination of the rows below it, and the coefficients of this linear combination belong to $\mathbb{F}_q[[\bar{\mathfrak{X}}]]$ (as opposed to $\mathbb{F}_q[[\pi]]$).

¹³Here, we made a tough choice to consider the transpose instead, so that the display of N_{nd}^T is much nicer.

To take advantage of this linear relation among the rows $\bar{R}_m(k)$, we define the (upper triangular) matrix $\bar{A}_k(\bar{\mathfrak{X}}) \in M_{kp-p+1}(\mathbb{F}_q[[\bar{\mathfrak{X}}]])$ so that, if we write $\bar{R}_m(k)'$ to denote the m th row of $\bar{P}_k := \bar{A}_k(\bar{\mathfrak{X}})\bar{N}_k^T$, then we have

$$(5.2.4) \quad \bar{R}_m(k)' = \begin{cases} m\bar{R}_m(k) + \sum_{i=1}^d \sum_{r=0}^{\infty} i \cdot \bar{R}_{m+ip^r}(k) \bar{a}_i^{p^r} \bar{\mathfrak{X}}^{p^r}, & \text{if } p \nmid m, \\ \bar{R}_m(k), & \text{if } p \mid m. \end{cases}$$

Explicitly, if we write $\bar{A}_k(\bar{\mathfrak{X}}) = (\bar{a}_{mn})_{m,n \in \mathbb{Z}_{\geq 0}}$, we have

$$\bar{a}_{mn} = \begin{cases} i\bar{a}_i^{p^r} \bar{\mathfrak{X}}^{p^r} & \text{when } n - m = ip^r \text{ with } 1 \leq i \leq d \text{ and } p \nmid i, \\ 1 & \text{when } m = n \text{ and } p \mid m, \\ m & \text{when } m = n \text{ and } p \nmid m, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, in the first case, there is only one term, because other terms with $p \mid i$ is zero modulo p .

By the recurrence relations of $\{e_n\}$ in (5.2.3), the matrix $\bar{P}_k := \bar{A}_k(\bar{\mathfrak{X}})\bar{N}_k^T$ takes the following form

$$\bar{P}_k = \left(\bar{p}_{mn} \right)_{\substack{0 \leq m \leq kp-p \\ 0 \leq n \leq kp-p}} = \begin{matrix} & & & & \overbrace{\hspace{4cm}}^k & & \overbrace{\hspace{4cm}}^{(k-1)(p-1)} & & \\ & & & & \bar{e}_0 & \bar{e}_p & \bar{e}_{2p} & \cdots & \bar{e}_{(k-1)p} & * & * & * & \cdots & * \\ & & & & 0 & 0 & 0 & \cdots & 0 & * & * & * & \cdots & * \\ & & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & 0 & \bar{e}_0 & \bar{e}_p & \cdots & \bar{e}_{(k-2)p} & * & * & * & \cdots & * \\ & & & & 0 & 0 & 0 & \cdots & 0 & * & * & * & \cdots & * \\ & & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & 0 & 0 & \bar{e}_0 & \cdots & \bar{e}_{(k-3)p} & * & * & * & \cdots & * \\ & & & & 0 & 0 & 0 & \cdots & 0 & * & * & * & \cdots & * \\ & & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & 0 & 0 & 0 & \cdots & 0 & * & * & * & \cdots & * \\ & & & & 0 & 0 & 0 & \cdots & \bar{e}_0 & * & * & * & \cdots & * \end{matrix},$$

where for $n \geq k$, \bar{p}_{mn} is a function of $\bar{\mathfrak{X}}$ given by

$$(5.2.5) \quad \bar{p}_{mn} = \begin{cases} i\bar{a}_i^{p^r} \bar{\mathfrak{X}}^{p^r} & \text{when } n - m = ip^r \text{ with } 1 \leq i \leq d \text{ and } p \nmid i, \\ 1 & \text{when } m = n \text{ and } p \mid m, \\ m & \text{when } m = n \text{ and } p \nmid m, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\bar{A}_k(\bar{\mathfrak{X}})$ is upper triangular, we have

$$(5.2.6) \quad \det(\bar{A}_k(\bar{\mathfrak{X}})) = \prod_{i=1}^{p-1} i^{k-1} = (-1)^{k-1} \quad \text{in } \mathbb{F}_q[[\bar{\mathfrak{X}}]].$$

For a similar reason (and $e_0 = 1$), we have

$$\det(\bar{P}_k) = \det \begin{bmatrix} 1 & 2 & \cdots & \lfloor \frac{i}{p} \rfloor + i & \cdots & kp - p - 1 \\ k & k+1 & \cdots & k+i-1 & \cdots & kp - p \end{bmatrix}_{\bar{P}_k}.$$

Combining these two, we deduce

$$\begin{aligned} (-1)^{k-1} \det(\bar{N}_k^T) &= \det(\bar{A}_k(\bar{\mathfrak{T}})) \det(\bar{N}_k^T) = \det(\bar{P}_k) \\ &= \det \left[\begin{array}{cccccc} 1 & 2 & \cdots & \lfloor \frac{i}{p} \rfloor + i & \cdots & kp - p - 1 \\ k & k+1 & \cdots & k+i-1 & \cdots & kp - p \end{array} \right]_{\bar{P}_k}. \end{aligned}$$

The key observation here is that the entries of the (sub)matrix

$$\left[\begin{array}{cccccc} 1 & 2 & \cdots & \lfloor \frac{i}{p} \rfloor + i & \cdots & kp - p - 1 \\ k & k+1 & \cdots & k+i-1 & \cdots & kp - p \end{array} \right]_{\bar{P}_k}$$

all lies in the subring $\mathbb{F}_q[[\bar{\mathfrak{T}}]]$ of $\mathbb{F}_q[[\pi]]$, as seen in its explicit form (5.2.5). Moreover, the coefficients on these entries are independent of ℓ . It follows that

$$(5.2.7) \quad \det(\bar{N}_k^T) \in \mathbb{F}_q[[\bar{\mathfrak{T}}]]$$

is a power series whose coefficients are independent of ℓ . The Theorem follows from this and the equality (5.1.1). \square

Now, we deduce Theorem 1.4(2) from Theorem 5.1.

Proof of Theorem 1.4(2). For $k = nd$ or $nd + 1$, we note that $\lambda'_k := \lambda_k/a = \frac{n(nd-1)(p-1)}{2}$ or $\frac{n(nd+1)(p-1)}{2}$ are integers because $p \nmid d$.

Since N is twisted I -adically incremental, Lemma 4.5 implies that

$$\det \left[\begin{array}{cccc} 0 & 1 & \cdots & k-1 \\ 0 & 1 & \cdots & k-1 \end{array} \right]_N \in I^{\frac{p(0+1+\cdots+(k-1))-(0+1+\cdots+(k-1))}{d}} = I^{\lambda'_k}.$$

Combining this with Theorem 5.1, we see that

$$\det \left[\begin{array}{cccc} 0 & 1 & \cdots & k-1 \\ 0 & 1 & \cdots & k-1 \end{array} \right]_N \bmod p = \bar{v}_{\lambda'_k} \bar{\mathfrak{T}}^{\lambda'_k} + \bar{v}_{\lambda'_k+1} \bar{\mathfrak{T}}^{\lambda'_k+1} + \cdots \in \bar{\mathfrak{T}}^{\lambda'_k} \mathbb{F}_q[[\bar{\mathfrak{T}}]],$$

where $\bar{v}_{\lambda'_k} \in \mathbb{F}_q$ is independent of ℓ . Thus,

$$(5.2.8) \quad \det \left[\begin{array}{cccc} 0 & 1 & \cdots & k-1 \\ 0 & 1 & \cdots & k-1 \end{array} \right]_N \equiv \bar{v}_{\lambda'_k} \bar{\mathfrak{T}}^{\lambda'_k} \pmod{pI^{\lambda'_k} + I^{\lambda'_k+1}}.$$

Applying Proposition 4.6 to the series of product $\sigma^{a-1}(N) \cdots \sigma(N)N$ (whose characteristic power series defines $C_f^*(\pi, s)$), we get

$$w_k(\pi) \equiv \prod_{i=0}^{a-1} \left(\det \left[\begin{array}{cccc} 0 & 1 & \cdots & k-1 \\ 0 & 1 & \cdots & k-1 \end{array} \right]_{\sigma^i(N)} \right) \pmod{I^{\lambda'_k+1}}.$$

Combining this with (5.2.8), we deduce

$$(5.2.9) \quad w_k(\pi) \equiv \prod_{i=0}^{a-1} \bar{v}_{\lambda'_k}^{p^i} \cdot \prod_{i=0}^{a-1} \sigma^i(\bar{\mathfrak{T}})^{\lambda'_k} \equiv \prod_{i=0}^{a-1} \bar{v}_{\lambda'_k}^{p^i} \cdot \mathfrak{S}(\underline{T})^{\lambda_k/a} \pmod{pI^{\lambda'_k} + I^{\lambda'_k+1}},$$

where the second congruence made use of the following congruence

$$\mathfrak{S}(\underline{T}) := \prod_{i=1}^{\ell} \left(\sum_{j=1}^{\ell} c_j^{p^i} T_j \right) \equiv \prod_{i=1}^{\ell} \left(\sum_{j=1}^{\ell} c_j^{p^i} \pi_j \right) \equiv \prod_{i=1}^{\ell} \sigma^i(\bar{\mathfrak{T}}) \pmod{pI^{\ell} + I^{\ell+1}}.$$

¹⁴Here we are allowed to write $\bar{v}_{\lambda'_k}$ because only this element modulo p affects the congruence relation.

From (5.2.9), we see that Theorem 5.1(2) is equivalent to $\prod_{i=0}^{a-1} \bar{v}_{\chi_k}^{p^i} \in \mathbb{F}_p^\times$. But as pointed out above, this element is *independent* of ℓ . We know that Theorem 1.4(2) holds when $\ell = 1$, as proved in [DWX, Propostion 3.4], so it holds for all ℓ .¹⁵ \square

6. ARTIN–SCHREIER–WITT EIGENVARIETIES

We now interpret Theorem 1.4 using the language of eigenvarieties. Recall the weight space \mathcal{W} and its admissible locus \mathcal{W}^{adm} from Section 2. We remind the readers that \mathcal{W}^{adm} is independent of the choice of the basis $\{c_1, \dots, c_\ell\}$ (Corollary 2.4) and contains all the points corresponding to finite non-trivial characters of \mathbb{Z}_{p^ℓ} (Lemma 2.5).

The *eigenvariety* \mathcal{E}_f associated to the Artin–Schreier–Witt tower for $\bar{f}(x)$ is defined as the zero locus of $C_f^*(\underline{T}, s)$, viewed as a rigid analytic subspace of $(\mathcal{W} - \{0\}) \times \mathbb{G}_{m, \text{rig}}$, where s is the coordinate of the second factor.¹⁶ Denote the natural projection to the first factor by $\text{wt} : \mathcal{E}_f \rightarrow \mathcal{W} - \{0\}$; and denote the *inverse* of the natural projection to the second factor by

$$\alpha : \mathcal{E}_f \xrightarrow{\text{Pr}_2} \mathbb{G}_m^{\text{rig}} \xrightarrow{x \mapsto x^{-1}} \mathbb{G}_m^{\text{rig}}.$$

We use $\mathcal{E}_f^{\text{adm}} := \text{wt}^{-1}(\mathcal{W}^{\text{adm}})$ to denote the preimage of the admissible locus of the eigenvariety.

Theorem 6.1. *The admissible locus of the eigenvariety $\mathcal{E}_f^{\text{adm}}$ is an infinite disjoint union*

$$X_0 \coprod X_{(0,1)} \coprod X_1 \coprod X_{(1,2)} \coprod \cdots$$

of rigid analytic spaces such that for each interval $J = [n, n]$ or $(n, n+1)$ with $n \in \mathbb{Z}_{\geq 0}$,

- *the map $\text{wt} : X_J \rightarrow \mathcal{W}^{\text{adm}}$ is finite and flat of degree 1 if J represents a point and of degree $d-1$ if J represents a genuine interval, and*
- *for each point $x \in X_J$, we have*

$$\frac{\text{val}_q(\alpha(x))}{\text{val}_q(\mathfrak{S}(\text{wt}(x)))} \in \frac{a(p-1)}{\ell} \cdot J.$$

Proof. Similar arguments have appeared multiple times in the literature; see [BK, Theorem A], [LWX, Theorem 1.3], or [DWX, Theorem 4.2]. So we only sketch the proof here.

For a continuous character χ of \mathbb{Z}_{p^ℓ} whose corresponding points lies on the admissible locus \mathcal{W}^{adm} , Theorem 1.3 (see Remark 2.7) implies that the q -valuations of the zeros of $C_f^*(\chi, s)$ consists of

- for all n , exactly one zero has valuation $-\text{val}_q(\mathfrak{S}(\chi)) \cdot \frac{an(p-1)}{\ell}$, and
- for all n , exactly $d-1$ zeros (counted with multiplicity) have valuations in the interval

$$-\text{val}_q(\mathfrak{S}(\chi)) \cdot \frac{a(p-1)}{\ell} \cdot [n + \frac{1}{d}, n + \frac{d-1}{d}].$$

From this, we see that $\mathcal{E}_f^{\text{adm}}$ is the disjoint union of the following subspaces

$$X_{[n,n]} := \mathcal{E}_f^{\text{adm}} \cap \{(t, a_p) \in \mathcal{W}^{\text{adm}} \times \mathbb{G}_m^{\text{rig}} \mid \text{val}_q(a_p) = \text{val}_q(\mathfrak{S}(t)) \cdot \frac{an(p-1)}{\ell}\}, \text{ and}$$

$$\begin{aligned} X_{(n,n+1)} &:= \mathcal{E}_f^{\text{adm}} \cap \{(t, a_p) \in \mathcal{W}^{\text{adm}} \times \mathbb{G}_m^{\text{rig}} \mid \text{val}_q(a_p) \in \text{val}_q(\mathfrak{S}(t)) \cdot \frac{a(p-1)}{\ell} \cdot (n, n+1)\} \\ &= \mathcal{E}_f^{\text{adm}} \cap \{(t, a_p) \in \mathcal{W}^{\text{adm}} \times \mathbb{G}_m^{\text{rig}} \mid \text{val}_q(a_p) \in \text{val}_q(\mathfrak{S}(t)) \cdot \frac{a(p-1)}{\ell} \cdot [n + \frac{1}{d}, n + \frac{d-1}{d}]\}. \end{aligned}$$

¹⁵Once again, Remark 2.2 allows us to prove Theorem 1.4(2) under Hypothesis 3.3.

¹⁶Here we removed the zero point of the weight space, because when $\underline{T} = \underline{0}$, $C_f^*(0, s) = 1 - s$ which is very different from other points of the weight space.

Note that, restricting to every open affinoid subdomain of $\mathcal{E}_f^{\text{adm}}$, the above decomposition is a union of affinoid subdomains. So $\mathcal{E}_f^{\text{adm}} = \coprod_{n=0}^{\infty} (X_{[n,n]} \sqcup X_{(n,n+1)})$ is a decomposition into an infinite disjoint union of rigid subspaces. The degree of each X_J follows from the description of the number of zeros above. \square

APPENDIX A. ERRATA FOR [DWX]

• (pointed out to us by Hui Zhu) upper half of page 7 (or lower half of page 1458 in the published version) the displayed formula

$$E_f(x) = \sum_{j=0}^{\infty} u_j \pi^{j/d} x^j \in B, \quad \text{for } u_j \in \mathbb{Z}_p.$$

should have $u_j \in \mathbb{Z}_p[[\pi^{1/d}]]$ instead.

• (pointed out to us by Hui Zhu) **Theorem 3.8** on Line 2 of the second paragraph of its proof, we took λ'_i to be the minimal integer satisfying certain properties. There might not be such λ'_i , in which case we should simply take λ'_i to be infinity. This will not affect the proof, as all we care are those λ'_i 's that are “close” to the lower bound polygon.

• **Theorem 4.2(2)** The statement that each $\mathcal{C}_{f,i}$ is finite and flat over \mathcal{W} is not literally true because $\mathcal{C}_{f,i}$ often misses the point over $T = 0$ in \mathcal{W} , as the slopes at points on $\mathcal{C}_{f,i}$ tend to ∞ as T approaches to 0. So one should replace the \mathcal{W} and \mathcal{C}_f in the statement with $\mathcal{W}^\circ := \mathcal{W} \setminus \{0\}$ and $\mathcal{C}_f^\circ := \mathcal{C}_f \setminus \text{wt}^{-1}(0)$.

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