Artin Conjecture for $p$-adic Galois Representations of Function Fields

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Abstract

For a global function field $K$ of positive characteristic $p$, we show that Artin’s entireness conjecture for L-functions of geometric $p$-adic Galois representations of $K$ is true in a non-trivial $p$-adic disk but is false in the full $p$-adic plane. In particular, we prove the non-rationality\footnote{Acknowledgements. It is a pleasure to thank Jean-Yves Etesse whose persistent interest in the non-rationality of the geometric unit root L-function motivated us to complete this paper. The second author would like to thank BICMR for its hospitality while this paper was written.} of the geometric unit root L-functions.

1 Introduction

Let $\mathbb{F}_q$ be the finite field of $q$ elements with characteristic $p$. Let $C$ be a smooth projective geometrically connected curve defined over $\mathbb{F}_q$ with function field $K$. Let $U$ be a Zariski open dense subset of $C$ with inclusion map $j : U \hookrightarrow C$. Let $G_K = \text{Gal}(K^{\text{sep}}/K)$ denote the absolute Galois group of $K$. For example, we can take $C = \mathbb{P}^1$, $U = \mathbb{P}^1 - \{0, \infty\}$ and $K = \mathbb{F}_q(t)$.

Let $\pi_1^{\text{arith}}(U)$ denote the arithmetic fundamental group of $U$. That is,

$$\pi_1^{\text{arith}}(U) = G_K / \langle I_x \rangle_{x \in |U|},$$
where the denominator denotes the closed normal subgroup generated by the inertial subgroups $I_x$ as $x$ runs over the closed points $|U|$ of $U$. Let $D_x$ denote the decomposition group of $G_K$ at $x$. One has the following exact sequence

$$1 \to I_x \to D_x \to \text{Gal} \left( \bar{k}_x/k_x \right) \to 1,$$

where $k_x$ denotes the residue field of $K$ at $x$. The Galois group $\text{Gal} \left( \bar{k}_x/k_x \right)$ is topologically generated by the geometric Frobenius element $\text{Frob}_x$ which is characterized by the property:

$$\text{Frob}^{-1}_x : \alpha \to \alpha^{#k_x}.$$

Let $P_x$ denote the $p$-Sylow subgroup of $I_x$. Then we have the following exact sequence

$$1 \to P_x \to I_x \to I_{\text{tame}}^x = \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \to 1.$$

Let $F_\ell$ be a finite extension of $\mathbb{Q}_\ell$, where $\ell$ is a prime number which may or may not equal to $p$. Let $V$ be a finite dimensional vector space over $F_\ell$. Let

$$\rho : G_K \longrightarrow GL(V)$$

be a continuous $\ell$-adic representation of $G_K$ unramified on $U$. Equivalently,

$$\rho : \pi_1^{\text{arith}}(U) \longrightarrow GL(V)$$

is a continuous representation of $\pi_1^{\text{arith}}(U)$. The representation $\rho$ is called geometric if it comes from an $\ell$-adic cohomology of a smooth proper variety over $U$. The geometric representations are the most interesting ones in applications.

Given a representation $\rho$, its L-function is defined by

$$L(U, \rho, T) = \prod_{x \in |U|} \frac{1}{\det(I - \rho(\text{Frob}_x)T^{\deg(x)}|V)} \in 1 + TR_\ell[[T]],$$

where $R_\ell$ is the ring of integers in $F_\ell$. It is clear that this L-function is trivially $\ell$-adic analytic in the open unit disc $|T|_\ell < 1$.

We are interested in further analytic properties of this L-function $L(U, \rho, T)$, especially for those representations which come from geometry. More precisely, we want to know

**Question 1.1** (Meromorphic continuation). When and where the L-function $L(U, \rho, T)$ is $\ell$-adic meromorphic?

**Question 1.2** (Artin’s conjecture). Assume that $\rho$ has no geometrically trivial component. When and where the L-function $L(U, \rho, T)$ is $\ell$-adic entire (no poles or analytic)?

The answer depends very much on whether $\ell$ equals to $p$ or not. In the easier case $\ell \neq p$, the Grothendieck [3] trace formula gives the following complete answer.

**Theorem 1.3.** Assume that $\ell \neq p$. The L-function $L(U, \rho, T)$ is a rational function in $F_\ell(T)$. If $\rho$ has no geometrically trivial component, then $L(U, \rho, T)$ is a polynomial in $F_\ell[T]$. 

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In the case $\ell = p$, the situation is much more subtle. A general conjecture of Katz \cite{Katz} as proved by Emerton-Kisin \cite{EmertonKisin} says that the above two questions still have a complete positive answer if we restrict to the closed unit disc. That is, we have

**Theorem 1.4.** Assume that $\ell = p$. The $L$-function $L(U, \rho, T)$ is $p$-adic meromorphic on the closed unit disc $|T|_p \leq 1$. If $\rho$ has no geometrically trivial component, then the $L$-function $L(U, \rho, T)$ is $p$-adic analytic (no poles) on the closed unit disc $|T|_p \leq 1$.

The extension of the above results to larger $p$-adic disc is more subtle. For any given $\epsilon > 0$, there are examples \cite{Examples} showing that the $L$-function $L(U, \rho, T)$ is not $p$-adic meromorphic in the disc $|T|_p < 1 + \epsilon$, disproving another conjecture of Katz \cite{Katz}. However, if $\rho$ comes from geometry, then Dwork’s conjecture \cite{Dwork} as proved by the second author \cite{SecondAuthor} shows the $L$-function is indeed a good $p$-adic function:

**Theorem 1.5.** Assume that $\ell = p$. If $\rho$ comes from geometry, then the $L$-function $L(U, \rho, T)$ is $p$-adic meromorphic in the whole $p$-adic plane $|T|_p < \infty$.

The aim of this paper is to study Artin’s entireness conjecture for such $L$-functions of geometric $p$-adic representations. Our main result is the following theorem.

**Theorem 1.6.** Assume that $\ell = p$ and $\rho$ comes from geometry with no geometrically trivial components. Then, there is a positive constant $c(p, \rho)$ such that the $L$-function $L(U, \rho, T)$ is $p$-adic analytic (no poles) in the larger disc $|T|_p < 1 + c(p, \rho)$. Furthermore, there are geometrically non-trivial rank one geometric $p$-adic representations $\rho$ such that $L(U, \rho, T)$ is not $p$-adic analytic (in fact having infinitely many poles) in $|T|_p < \infty$.

The second part of the theorem shows that Artin’s conjecture is false in the entire plane $|T|_p < \infty$. It shows that the first part of the theorem is best one can hope for, and Artin’s conjecture is true in a larger disk than the closed unit disk for geometric $p$-adic representations. An interesting further question is how big the constant $c(p, \rho)$ can be. Our proof gives an explicit positive constant depending only on $p$ and some embedding rank of $\rho$. In the simpler ordinary case with $R_p = \mathbb{Z}_p$, one can take $c(p, \rho) = p - 1$ which is independent of $\rho$.

## 2 $\ell$-adic case: $\ell \neq p$

Since $\ell \neq p$, the restriction of the $\ell$-adic representation $\rho$ to $P_\ell$ is of finite order and thus the representation $\rho$ is almost tame. In fact, by class field theory, $\rho$ itself has finite order up to a twist if $\rho$ has rank one. Thus, there are not too many such $\ell$-adic representations. The $L$-function $L(U, \rho, T)$ is always a rational function. This follows from Grothendieck’s trace formula \cite{Grothendieck}:

**Theorem 2.1.** Let $\mathcal{F}_\rho$ denote the lisse $\ell$-adic sheaf on $U$ associated with $\rho$. Then, there are finite dimensional vector spaces $H^i_c(U \otimes \overline{\mathbb{F}}_q, \mathcal{F}_\rho)$ $(i = 0, 1, 2)$ over $F_\ell$ such that

$$L(U, \rho, T) = \prod_{i=0}^{2} \det(I - \text{Frob}_q T|H^i_c(U \otimes \overline{\mathbb{F}}_q, \mathcal{F}_\rho))^{(-1)^{i-1}} \in F_\ell(T).$$
If $U$ is affine, then $H^0_c = 0$. If $\rho$ does not contain a geometrically trivial component, then $H^2_c = 0$. Thus, in most cases, it is $H^1_c$ that is the most interesting.

**Corollary 2.2.** Let $U$ be affine. Assume that $\rho$ does not contain a geometrically trivial component. Then, the $L$-function

$$L(U, \rho, T) = \det(I - \text{Frob}_q T | H^1_c(U \otimes \overline{\mathbb{F}}_q, \mathcal{F}_\rho))$$

is a polynomial.

This is the $\ell$-adic function field analogue of Artin’s entireness conjecture.

Fix an embedding $\iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$. A representation $\rho$ is called $\iota$-pure of weight $w \in \mathbb{R}$ if each eigenvalue of $\text{Frob}_x$ acting on $V$ has absolute value $q^{\deg(x)w/2} / 2$ for all $x \in |U|$. A representation $\rho$ is called $\iota$-mixed of weights at most $w$ if each irreducible subquotient of $\rho$ is $\iota$-pure of weights at most $w$. If $\rho$ is $\iota$-pure of weight $w$ for every embedding $\iota$, then $\rho$ is called pure of weight $w$. Similarly, if $\rho$ is $\iota$-mixed of weights at most $w$ for every $\iota$, then $\rho$ is called mixed of weights at most $w$. The fundamental theorem of Deligne \cite{3} on the Weil conjectures implies

**Theorem 2.3.** If $\rho$ is geometric, then $\rho$ is mixed with integral weights. Furthermore, if $\rho$ is mixed of weights at most $w$, then $H^i_c(U \otimes \overline{\mathbb{F}}_q, \mathcal{F}_\rho)$ is mixed of weights at most $w + i$.

The $\ell$-adic function field Langlands conjecture for $\text{GL}(n)$, which was established by Lafforgue \cite{12}, implies

**Theorem 2.4.** If $\rho$ is irreducible, then $\rho$ is geometric up to a twist and hence pure of some weight.

Thus, in the $\ell$-adic case with $\ell \neq p$, essentially all $\ell$-adic representations are geometric from the viewpoint of $L$-functions.

### 3 $p$-adic case

In the case $\ell = p$, the restriction of the $p$-adic representation $\rho$ to $P_\mathbb{Z}$ can be infinite and thus $\rho$ can be very wildly ramified. The $L$-function $L(U, \rho, T)$ is naturally more complicated and cannot be rational in general. One can ask for its $p$-adic meromorphic continuation. The function $L(U, \rho, T)$ is trivially $p$-adic analytic in the open unit disc $|T|_p < 1$ as the coefficients are in the ring $R_p$. It was shown in \cite{14} that $L(U, \rho, T)$ is not $p$-adic meromorphic in general, disproving a conjecture of Katz \cite{10}. However, one can show that $L(U, \rho, T)$ is $p$-adic meromorphic on the closed unit disc $|T|_p \leq 1$. Its zeros and poles on the closed unit disc are controlled by $p$-adic étale cohomology of $\rho$. This was proved by Emerton-Kisin \cite{7}, confirming a conjecture of Katz \cite{10}. That is,

**Theorem 3.1.** For any $p$-adic representation $\rho$ of $\pi_{1\text{arith}}(U)$, the quotient

$$L(U, \rho, T) \prod_{i=0}^{2} \det(I - \text{Frob}_q T | H^i_c(U \otimes \overline{\mathbb{F}}_q, \mathcal{F}_\rho))(-1)^{i+1}$$

has no zeros and poles on the closed unit disc $|T|_p \leq 1$. 


In the case that $\rho$ has rank one, this was first proved by Crew [2]. Note that $H^2_c(U \otimes \overline{\mathbb{F}}_q, \mathcal{F}_\rho) = 0$ since $U$ is a curve and $\ell = p$. If $U$ is affine, then $H^0_c(U \otimes \overline{\mathbb{F}}_q, \mathcal{F}_\rho) = 0$. This gives

**Corollary 3.2.** Let $U$ be affine. Then, the L-function $L(U, \rho, T)$ is $p$-adic analytic on the closed unit disc $|T|_p \leq 1$.

The (compatible) $p$-adic analogue of a lisse $\ell$-adic sheaf (or $\ell$-adic representation) on $U$ for $\ell \neq p$ is an overconvergent $F$-isocrystal over $U$, which is not a $p$-adic representation. Its pure slope parts, under the Newton-Hodge decomposition, are $p$-adic representations up to twists (unit root $F$-isocrystals, no longer overconvergent in general). $P$-adic representations arising in this way are also called geometric, as they are a natural generalization of the geometric representations we defined before. For geometric $p$-adic representations, the following meromorphic continuation was conjectured by Dwork [5] and proved by the second author [15] [16].

**Theorem 3.3.** If the $p$-adic representation $\rho$ is geometric, then the L-function $L(U, \rho, T)$ is $p$-adic meromorphic everywhere.

**Remark 3.4.** It would be interesting to know if a sub-quotient of a geometric $p$-adic representation remains geometric in terms of our general definition.

Unlike the $\ell$-adic case, most $p$-adic representations are not geometric. It seems very difficult to classify geometric $p$-adic representations, even in the rank one case. This may be viewed as the $p$-adic Langlands program for function fields of characteristic $p$, which is still wide open, even in the rank one case.

Our first new result of this paper is to show that the Artin entireness conjecture fails for $L$-functions $L(U, \rho, T)$ of geometric $p$-adic representations, even for non-trivial rank one $\rho$.

**Theorem 3.5.** There are geometrically non-trivial rank one geometric $p$-adic representations $\rho$ on certain affine curves $U$ over $\mathbb{F}_p$ such that the $L$-function $L(U, \rho, T)$ is $p$-adic meromorphic on $|T|_p < \infty$, but having infinitely many poles.

**Proof.** Let $p > 2$ be an odd prime and $N \geq 4$ be a positive integer prime to $p$. Let $Y$ be the component of ordinary non-cuspidal locus of the modulo $p$ reduction of the compactified modular curve $X_1(Np)$. This is an affine curve over the finite field $\mathbb{F}_p$. Let $E_1(Np)$ be the universal elliptic curve over $Y$. Its relative $p$-adic étale cohomology is a rank one geometric $p$-adic representation $\rho$ of $\pi_1^{\text{arith}}(Y)$. For a non-zero integer $k$, the $k$-th tensor power $\rho^\otimes k$ is again a geometric $p$-adic representation of $\pi_1^{\text{arith}}(Y)$. The Monsky trace formula gives the following relation

$$L(Y, \rho^\otimes k, T) = \frac{D(k + 2, T)}{D(k, pT)},$$

where $D(k, T)$ is the characteristic power series of the $U_p$-operator acting on the space of overconvergent $p$-adic cusp forms of weight $k$ and tame level $N$. The series $D(k, T)$ is a $p$-adic entire function. Equation (1) implies that the $L$-function $L(Y, \rho^\otimes k, T)$ is
$p$-adic meromorphic in $T$, which was first proved by Dwork in \cite{Dwork} via Monsky’s trace formula, see also \cite{Wiles} and \cite{Serre}.

We want to show that the L-function $L(Y, \rho \otimes k, T)$ is not $p$-adic entire for infinitely many integers $k$. For this purpose, we need to describe the coefficients of the L-function in more detail, following Coleman \cite{Coleman} Appendix I.

For an order $\mathcal{O}$ in a number field, let $h(\mathcal{O})$ denote the class number of $\mathcal{O}$. If $\gamma$ is an algebraic integer, let $\mathcal{O}_\gamma$ be the set of orders in $\mathbb{Q}(\gamma)$ containing $\gamma$. For a positive integer $m$, let $W_{p,m}$ denote the finite set of $p$-adic units $\gamma \in \mathbb{Q}_p$ such that $\mathbb{Q}(\gamma)$ is an imaginary quadratic field, $\gamma$ is an algebraic integer and

$$\text{Norm}_{\mathbb{Q}(\gamma)}(\gamma) = p^m.$$ 

By Coleman \cite{Coleman} Theorem II, for all integers $k$, we have

$$D(k, T) = \exp\left(\sum_{m=1}^{\infty} A_m(k) \frac{T^m}{m}\right),$$

where

$$A_m(k) = \sum_{\gamma \in W_{p,m}} \sum_{\mathcal{O} \in \mathcal{O}_\gamma} h(\mathcal{O}) B_N(\mathcal{O}, \gamma) \frac{\gamma^k}{\gamma^2 - p^m},$$

and $B_N(\mathcal{O}, \gamma)$ is the number of elements of $\mathcal{O}/\mathcal{O}\gamma$ of order $N$ fixed under multiplication by $p^m/\gamma$. This is really another form of the Monsky trace formula. It follows that

$$L(Y, \rho \otimes k, T) = \exp\left(\sum_{m=1}^{\infty} C_m(k) \frac{T^m}{m}\right),$$

where

$$C_m(k) = A_m(k + 2) - A_m(k) p^m = \sum_{\gamma \in W_{p,m}} \sum_{\mathcal{O} \in \mathcal{O}_\gamma} h(\mathcal{O}) B_N(\mathcal{O}, \gamma) \gamma^k.$$

It is clear that $C_m(k)$ is an algebraic number in $\mathbb{Q}_p$. We need the following key property.

**Lemma 3.6.** If $6|k$, then the field generated by all the algebraic numbers $C_m(k)$ in $\mathbb{Q}_p$ is equal to the compositum of all imaginary quadratic fields in $\mathbb{Q}_p$ in which $p$ splits. In particular, this field is an infinite algebraic extension of $\mathbb{Q}$ in $\mathbb{Q}_p$.

**Proof.** Since $\gamma$ is a $p$-adic unit and $\text{Norm}_{\mathbb{Q}}(\gamma) = p^m$, we see that $p$ splits in $\mathbb{Q}(\gamma)$. Thus $C_m(k)$ is contained in the compositum of all imaginary quadratic fields in $\mathbb{Q}_p$ in which $p$ splits. Conversely, let $K$ be any imaginary quadratic field in $\mathbb{Q}_p$ in which $p$ splits. Write $p\mathcal{O}_K = \mathfrak{p}\mathfrak{p}$. Without loss of generality, we may suppose $\mathfrak{p} = p\mathbb{Z}_p \cap \mathcal{O}_K$. For $m = h(\mathcal{O}_K)$, $\mathfrak{p}^m = (\gamma)$ is a principal ideal. Thus $\mathfrak{p}^m = (\gamma)$. It follows that $\text{Norm}_K(\gamma) = \gamma\mathfrak{p} = p^m$. By replacing $\gamma$ with $\gamma^n$ and $m$ with $mn$ for some suitable positive integer $n$, we may further suppose that $\bar{\gamma} \equiv 1 \mod N\mathcal{O}_K$. In particular, we have $B_N(\mathcal{O}_K, \gamma) > 0$. Now for any $\gamma' \in K \cap W_{p,m}$, since $\text{Norm}_K(\gamma') = \text{Norm}_K(\gamma) = p^m$, we may write $\gamma' = u\gamma$ for some $u \in \mathcal{O}_K^\times$. Since $K$ is imaginary quadratic, it is well-known that $|\mathcal{O}_K^\times|$ divides 6. Thus $\gamma'^k = \gamma^k$, yielding

$$C_m(k) = \left(\sum_{\gamma' \in K \cap W_{p,m}} \sum_{\mathcal{O} \in \mathcal{O}_{\gamma'}} h(\mathcal{O}) B_N(\mathcal{O}, \gamma')\gamma^k + \alpha\right.$$
σ is a sum of elements contained in quadratic fields different from K. We therefore deduce that $K = \mathbb{Q}(\gamma^k)$ is contained in the field generated by $C_m(k)$. This yields the lemma.

We now return to the proof of the theorem. Let $k \geq 2$ be a positive integer divided by 6. Let $\mathcal{F}$ denote the relative rigid cohomology of $E_1(N^p)$ over $Y$, which is an ordinary overconvergent $F$-isocrystal over $Y$ of rank two, self-dual and pure of weight 1. The rank one $p$-adic representation $\rho$ is precisely the unit root part of $\mathcal{F}$. It follows that the $L$-function of the $k$-th Adams operation of $\mathcal{F}$ is

$$L(Y, \rho^{\otimes k}, T)L(Y, \rho^{\otimes (-k)}, p^k T) = \frac{L(Y, \text{Sym}^k \mathcal{F}, T)}{L(Y, \text{Sym}^{k-2} \mathcal{F}, p T)}.$$  

The right side is a rational function with integer coefficients. If both $L(Y, \rho^{\otimes k}, T)$ and $L(Y, \rho^{\otimes (-k)}, T)$ had a finite number of poles, then the above left side would be a $p$-adic meromorphic function with a finite number of poles, and it is at the same time a rational function. It would then follow that both $L(Y, \rho^{\otimes k}, T)$ and $L(Y, \rho^{\otimes (-k)}, T)$ would be rational functions. This implies that the coefficients of $L(Y, \rho^{\otimes k}, T)$ and $L(Y, \rho^{\otimes (-k)}, T)$ generate a finite algebraic extension of $\mathbb{Q}$ in $\mathbb{Q}_p$, contradicting to the lemma. We conclude that at least one of the two functions $L(Y, \rho^{\otimes k}, T)$ and $L(Y, \rho^{\otimes (-k)}, T)$ has infinitely many poles. The theorem is proved.

**Remark 3.7.** For any positive integer $k \geq 2$, we believe that both functions $L(Y, \rho^{\otimes k}, T)$ and $L(Y, \rho^{\otimes (-k)}, T)$ have infinitely many poles. But we do not know how to prove it.

**Remark 3.8.** In the analogous setting of the family of Kloosterman sums, the unit root $L$-function is again expected to be non-rational, but this remains unknown at present, see page 4 in [9].

Our second result of this paper is to show that for a geometric $p$-adic representation $\rho$ on a smooth affine curve $U$ over $\mathbb{F}_p$, the $L$-function $L(U, \rho, T)$ is $p$-adic analytic (no poles) in the larger disc $|T|_p < 1 + c(p, \rho)$ for some positive constant $c(p, \rho)$. In fact, we shall prove a more general theorem in the context of $\sigma$-modules as in [15][16]. For simplicity of notation, we use $L(\rho, T)$ to denote $L(U, \rho, T)$.

**Theorem 3.9.** Let $U$ be a smooth affine curve over $\mathbb{F}_q$. Let $\rho$ be a unit root $\sigma$-module which arises as a pure slope part of an overconvergent $\sigma$-module on $U$. Then, there is a positive constant $c(p, \rho)$ such that the $L$-function $L(\rho, T)$ is $p$-adic analytic (no poles) in the larger disc $|T|_p < 1 + c(p, \rho)$.

**Proof.** Let $\phi$ be an overconvergent $\sigma$-module on $U$ with coefficients in $R_p$ with uniformizer $\pi$. Since $\phi$ is overconvergent, Corollary 3.2 in [16] shows that its $L$-function $L(\phi, T)$ is $p$-adic meromorphic everywhere. As $U$ is a smooth affine curve, Corollary 3.3 in [16] further shows that $L(\phi, T)$ is $p$-adic analytic in the disk $|T|_p < |\pi^{-1}|_p$. Note that $|\pi^{-1}|_p$ is a constant greater than 1. For example, in the case $\pi = p$, we have $|\pi^{-1}|_p = p$.

We first assume that $\phi$ is ordinary. For an integer $i \geq 0$, let $\phi_i$ denote the unit root $\sigma$-module on $U$ coming from the slope $i$-part in the Hodge-Newton decomposition of $\phi$. 

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It is no longer overconvergent in general. We need to show that the unit root σ-module L-function \( L(\phi_i, T) \) is \( p \)-adic analytic in the disk \( |T|_p < |\pi^{-1}|_p \). By the definition of \( \phi_i \) and our ordinarity assumption, we have the decomposition

\[
L(\phi, T) = \prod_{i \geq 0} L(\phi_i, \pi^iT) = L(\phi_0, T) \prod_{i \geq 1} L(\phi_i, \pi^iT).
\]

As mentioned above, the left side is \( p \)-adic analytic in the disk \( |T|_p < |\pi^{-1}|_p \). For each \( i \geq 1 \), the right side factor \( L(\phi_i, \pi^iT) \) is trivially \( p \)-adic analytic with no zeros and poles in the disk \( |T|_p < |\pi^{-1}|_p \). We deduce that the first right side factor \( L(\phi_0, T) \) is also \( p \)-adic analytic in the disk \( |T|_p < |\pi^{-1}|_p \). This proves the theorem in the case \( i = 0 \).

For \( i > 0 \), we need to use the proof of Dwork’s conjecture in [15][16]. Let \( \psi = \phi_i \). We need to prove that \( L(\psi, T) \) is \( p \)-adic analytic in the disk \( |T|_p < |\pi^{-1}|_p \). Let \( r_i \) denote the rank of \( \phi_i \). Define

\[
\tau = \wedge^{r_0} \phi_0 \otimes \wedge^{r_1} \phi_1 \otimes \cdots \otimes \wedge^{r_{i-1}} \phi_{i-1}.
\]

This is a rank one unit root \( \sigma \)-module on \( U \), not overconvergent in general. Define

\[
\varphi = \pi^{-r_1-\cdots-(i-1)r_{i-1}-i} \wedge^{r_0+r_{i-1}+1} \phi.
\]

Since \( \phi \) is ordinary and overconvergent, it follows that \( \varphi \) is also ordinary and overconvergent. For an integer \( j \geq 0 \), let \( \varphi_j \) denote the unit root \( \sigma \)-module on \( U \) coming from the slope \( j \)-part in the Hodge-Newton decomposition of \( \varphi \). Then, it is easy to check that we have the following decomposition (see equation (5.1) in [16]).

\[
L(\varphi \otimes \tau^{-1}, T) = L(\psi, T) \prod_{j \geq 1} L(\varphi_j \otimes \tau^{-1}, \pi^iT).
\]

For each \( j \geq 1 \), the factor \( L(\varphi_j \otimes \tau^{-1}, \pi^iT) \) is trivially \( p \)-adic analytic with no zeros and poles in the disk \( |T|_p < |\pi^{-1}|_p \). To prove that \( L(\psi, T) \) is also \( p \)-adic analytic in the disk \( |T|_p < |\pi^{-1}|_p \), it suffices to prove that the left side factor \( L(\varphi \otimes \tau^{-1}, T) \) is \( p \)-adic analytic in the disk \( |T|_p < |\pi^{-1}|_p \).

Now, the rank one unit root \( \sigma \)-module \( \tau \) is the slope zero part of the following ordinary and overconvergent \( \sigma \)-module

\[
\Phi = \pi^{-r_1-\cdots-(i-1)r_{i-1}} \wedge^{r_0+r_{i-1}+1} \phi.
\]

By Theorem 7.8 in [16], we deduce that there is a sequence of nuclear overconvergent \( \sigma \)-modules \( \Phi_{\infty,-k} \ (k \geq 2) \) such that

\[
L(\varphi \otimes \tau^{-1}, T) = \prod_{k \geq 1} L(\varphi \otimes \Phi_{\infty,-1-k} \otimes \wedge^{k}\Phi, T)^{(-1)^{k-1}}.
\]

Since \( \Phi \) is ordinary and its slope zero part has rank one, \( \wedge^{k}\Phi \) is divisible by \( \pi^{k-1} \). It follows that for \( k \geq 2 \), the \( L \)-function \( L(\varphi \otimes \Phi_{\infty,-1-k} \otimes \wedge^{k}\Phi, T) \) is trivially \( p \)-adic analytic with no zeros and poles in the disk \( |T|_p < |\pi^{-1}|_p \). For the remaining case \( k = 1 \), we apply the one dimensional case of the following \( n \)-dimensional integrality result and deduce that \( L(\varphi \otimes \Phi_{\infty,-2} \otimes \Phi, T) \) is \( p \)-adic analytic in the disk \( |T|_p < |\pi^{-1}|_p \). The theorem is proved in the ordinary case.
Lemma 3.10. Let $U$ be a smooth affine variety of equi-dimension $n$ over $\mathbb{F}_q$. Let $\phi$ be an overconvergent nuclear $\sigma$-module on $U$. Then, the $L$-function $L(\phi, T)^{(−1)^{n−1}}$ is $p$-adic analytic (no poles) in the disc $|T|_p < |\pi|_p^{−1}$.

Proof. In the case that $\phi$ has finite rank, this integrality is already proved in Corollary 3.3 in [16]. In this case, the finite rank Monsky trace formula (Theorem 3.1 in [16]) states that

$$L(\phi, T)^{(−1)^{n−1}} = \prod_{i=0}^{n} \det(I - \phi^*_iT|\mathbf{M}_i^* \otimes R K)^{(−1)^i},$$

where $R = R_p$ in our current notation and $\det(I - \phi^*_iT|\mathbf{M}_i^* \otimes R K) \in 1 + TR[[T]]$ is a $p$-adic entire function. Now, the divisibility $\phi^*_i \equiv 0( \mod \pi^i)$ (equation (3.4) in [16]) shows that

$$\det(I - \phi^*_iT|\mathbf{M}_i^* \otimes R K) \in 1 + \pi^iT R[[\pi^iT]].$$

This implies the integrality in the finite rank case. For infinite rank nuclear overconvergent $\sigma$-modules, the proof is the same. One simply uses the infinite rank nuclear overconvergent trace formula (Theorem 5.8 in [15]):

$$L(\phi, T)^{(−1)^{n−1}} = \prod_{i=0}^{n} \det(I - \Theta_iT|\mathbf{M}_i^* \otimes R K)^{(−1)^i}.$$

Note that there is a misprint of indices in Theorem 5.8 in [15]: $\det(I - \Theta_iT|\mathbf{M}_i^* \otimes R K)$ there should be $\det(I - \Theta_{n−i}T|\mathbf{M}_{n−i}^* \otimes R K)$, compare the finite rank case (Theorem 3.1 in [16]). Now, one uses the same divisibility $\Theta_i \equiv 0( \mod \pi^i)$, which follows from the definition of $\Theta_i$ (Definition 5.5 in [15]) and the fact that $\phi_i = \phi \otimes \sigma_i$ is divisible by $\pi^i$.

In the general non-ordinary case, by a similar argument, we may apply the methods in [15][16] for non-ordinary case to give an explicit positive constant $c(p, \rho)$ depending on $\pi$ and the rank of $\phi$ such that $L(\rho, T)$ is $p$-adic analytic in the disk $|T|_p < 1 + c(p, \rho)$. The constant $c(p, \rho)$ depends very badly on the rank of $\phi$, and so we would not bother to write it down explicitly.

The above theorem has a higher dimensional generalization. We state this generalization below.

Remark 3.11. Let $U$ be a smooth affine variety of equi-dimension $n$ over $\mathbb{F}_q$. Let $\rho$ be a unit root $\sigma$-module on $U$. Then, the $L$-function $L(\rho, T)^{(−1)^{n−1}}$ is $p$-adic analytic on the closed unit disk $|T|_p \leq 1$. If $\rho$ arises as a pure slope part of an overconvergent $\sigma$-module on $U$. Then, there is a positive constant $c(p, \rho)$ such that the $L$-function $L(\rho, T)^{(−1)^{n−1}}$ is $p$-adic analytic (no poles) in the larger disc $|T|_p < 1 + c(p, \rho)$.

The first part follows from Emerton-Kisin’s theorem on the Katz conjecture and standard properties of $p$-adic étale cohomology. The proof of the second part is the same as the above theorem, and use the results in [15][16]. We expect that both parts remain true if $U$ is an equi-dimensional complete intersection (possibly singular) in a smooth affine variety $X$ over $\mathbb{F}_q$. This possible generalization is motivated by the characteristic $p$ entireness result in [13].
Remark 3.12. In the special case that $U$ is the compliment of a hypersurface and the Frobenius lifting is the $q$-th power lifting of the coordinates, the result of Dwork-Sperber [6] can be used to prove the $p$-adic analytic continuation of $L(\rho,T)^{(-1)^{n-1}}$ for geometric ordinary $\rho$ in the open disc $\text{ord}_p(T) > -(p-1)/(p+1)$. This result is weaker than our result since the disc $\text{ord}_p(T) > -(p-1)/(p+1)$ is smaller than the disc $\text{ord}_p(T) > -1$ obtained in our approach.

References


