Combinatorial Congruences and ψ-Operators

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Abstract

The ψ-operator for (φ, Γ)-modules plays an important role in the study of Iwasawa theory via Fontaine's big rings. In this note, we prove several sharp estimates for the ψ-operator in the cyclotomic case. These estimates immediately imply a number of sharp p-adic combinatorial congruences, one of which extends the classical congruences of Fleck (1913) and Weisman (1977).

1 Combinatorial Congruences

Let p be a prime, $n \in \mathbb{Z}_{>0}$. Throughout this paper, let $[x]$ denote the integer part of x if $x \geq 0$ and $[x] = 0$ if $x < 0$. In the author's course lectures [4] on Fontaine's theory and p-adic L-functions given at UC Irvine (spring 2005) and at the Morningside Center of Mathematics (summer 2005), the following two congruences were discovered.

Theorem 1.1. *For integers $r \in \mathbb{Z}$, $j \geq 0$, we have*

$$\sum_{k \equiv r (\mathrm{mod} p)} (-1)^{n-k} \binom{n}{k} \binom{\frac{k-r}{p}}{j} \equiv 0 \ (\mathrm{mod} p^{[\frac{n-1-jp}{p-1}]}).$$

We shall see that the theorem comes from a simple estimate of $\psi(\pi^n)$ for the cyclotomic φ-module.

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Theorem 1.2. *For integer $j \geq 0$, we have*

$$\sum_{\substack{i_0+\cdots+i_{p-1}=n \\ i_1+2i_2+\cdots\equiv r (\mathrm{mod} p)}} \binom{n}{i_0 i_1 \cdots i_{p-1}} \binom{\frac{i_1+2i_2+\cdots-r}{p}}{j} \equiv 0 \ (\mathrm{mod} p^{[\frac{n(p-1)-jp-1}{p-1}]}).$$

As we shall see, this theorem comes from a simple estimate of $\psi(\pi^{-n})$ for the cyclotomic φ-module. Note that when $p = 2$, Theorem 1.2 is equivalent to Theorem 1.1.

The above two congruences can be extended from p to $q = p^a$, where a is a positive integer. To do so, it suffices to estimate the a-th iterate $\psi^a(\pi^n)$. This can be done by induction. The estimate of $\psi^a(\pi^n)$ for $n > 0$ leads to

Theorem 1.3. *For integers $r \in \mathbb{Z}$, $j \geq 0$ and $a > 0$, we have*

$$\sum_{k\equiv r (\mathrm{mod} p^a)} (-1)^{n-k} \binom{n}{k} \binom{\frac{k-r}{p^a}}{j} \equiv 0 (\mathrm{mod} p^{[\frac{n-p^{a-1}-jp^a}{p^{a-1}(p-1)}]}).$$

The estimate of $\psi^a(\pi^n)$ for $n < 0$ leads to

Theorem 1.4. *Let*

$$S_j(n, r, p^a) = \sum_{\substack{i_0+\cdots+i_{(p^a-1)}=n \\ i_1+2i_2+\cdots\equiv r (\mathrm{mod} p^a)}} \binom{n}{i_0 \cdots i_{(p^a-1)}} \binom{(i_1 + 2i_2 + \cdots - r)/p^a}{j}.$$

Then for integer $j \geq 0$, we have

$$S_j(n, r, p^a) \equiv 0 (\mathrm{mod} p^{[\frac{(an-a+1)(p-1)-j(ap-a+1)-1}{p-1}]}).$$

As Z.W. Sun informed me, the special case $j = 0$ of Theorem 1.1.1 was first proved by Fleck [1] in 1913, and the special case of Theorem 1.1.3 for $j = 0$ was first proved by Weisman [5] in 1977. A different extension of Theorem 1.1.1 and Weisman's congruence has been obtained by Z.W. Sun [2] using different combinatorial arguments. Motivated by applications in algebraic topology, Sun-Davis [3] proved yet another extension:

$$\sum_{k\equiv r (\mathrm{mod} p^a)} (-1)^{n-k} \binom{n}{k} \binom{\frac{k-r}{p^a}}{j} \equiv 0 (\mathrm{mod} p^{\left(\mathrm{ord}_p([n/p^{a-1}]!)-j-\mathrm{ord}_p(j!)\right)}).$$

2 The operator ψ

Let p be a fixed prime. Let π be a formal variable. Let

$$A^+ = \mathbb{Z}_p[[\pi]]$$

be the formal power series ring over the ring of p-adic integers. Let A be the p-adic completion of $A^+[\frac{1}{\pi}]$, and let $B = A[\frac{1}{p}]$ be the fraction field of A. The rings A^+, A and B correspond to $A^+_{\mathbb{Q}_p}$, $A_{\mathbb{Q}_p}$ and $B_{\mathbb{Q}_p}$ in Fontaine's theory.

We shall not discuss the Galois action on A, which is not needed for our present purpose. The Frobenius map φ acts on the above rings by

$$\varphi(\pi) = (1+\pi)^p - 1.$$

If we let $[\varepsilon] = 1+\pi$, then $\varphi([\varepsilon]) = [\epsilon]^p$. The map φ is injective of degree p. This gives

Proposition 2.1. $\{1, \pi, \cdots, \pi^{p-1}\}$ *(and* $\{1, [\varepsilon], \cdots, [\varepsilon]^{p-1}\}$*) is a basis of* A *over the subring* $\varphi(A)$.

Definition 2.2. The operator $\psi : A \to A$ is defined by

$$\psi(x) = \psi\left(\sum_{i=0}^{p-1} [\varepsilon]^i \varphi(x_i)\right) = x_0 = \frac{1}{p}\varphi^{-1}(\mathrm{Tr}_{A/\varphi(A)}(x)),$$

where $x : A \to A$ denotes the multiplication by x as $\varphi(A)$-linear map.

Example 2.3.

$$\psi([\varepsilon]^n) = \begin{cases} [\varepsilon]^{n/p}, & \text{if } p \mid n; \\ 0, & \text{if } p \nmid n. \end{cases}$$

It is clear that ψ is φ^{-1}-linear:

$$\psi(\varphi(a)x) = a\psi(x) \ \ \forall\, a, x \in A.$$

Example 2.4. *Let* a *be a positive integer relatively prime to* p*. Then*

$$\psi\Big(\frac{1}{(1+\pi)^a - 1}\Big) = \frac{1}{(1+\pi)^a - 1}.$$

In fact,

$$\begin{aligned} \psi\left(\frac{1}{[\varepsilon]^a - 1}\right) &= \psi\left(\frac{1}{[\varepsilon]^{ap} - 1} \cdot \frac{[\varepsilon]^{ap} - 1}{[\varepsilon] - 1}\right) \\ &= \frac{1}{[\varepsilon]^a - 1}\psi\left(1 + [\varepsilon]^a + \cdots + [\varepsilon]^{(p-1)a}\right) \\ &= \frac{1}{[\varepsilon]^a - 1} = \frac{1}{(1+\pi)^a - 1}. \end{aligned}$$

By p-adic continuity, the above example holds for any p-adic unit $a \in \mathbb{Z}_p{}^*$. In the general theory of (φ, Γ)-modules, it is important to find the fix points of ψ for applications to p-adic L-functions and Iwasawa theory. In the simplest cyclotomic case, we have the following description for the fixed points (see [4]).

Proposition 2.5.

$$A^{\psi=1} = \frac{1}{\pi}\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \left\{ \sum_{k=0}^{\infty} \varphi^k(x) \;\middle|\; x \in \bigoplus_{i=1}^{p-1} [\varepsilon]^i \varphi(a_i + \pi \mathbb{Z}_p[[\pi]]), \sum_{i=1}^{p-1} a_i = 0 \right\},$$

where $a_i \in \mathbb{Z}_p$.

For example, if a is a positive integer relatively prime to p, then the element

$$\frac{a}{(1+\pi)^a - 1} - \frac{1}{\pi} \in (A^+)^{\psi=1}$$

gives the cyclotomic units and the Euler system. This element is the Amice transform of a p-adic measure which produces the p-adic zeta function of $\mathbb{Q}$. This type of connections is conjectured to be a general phenomenon for (φ, Γ)-modules coming from global p-adic Galois representations.

3 Sharp estimates for ψ

The ring A is a topological ring with respect to the (p, π)-topology. A basis of neighborhoods of 0 is the sets $p^k A + \pi^n A^+$, where $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. The operator ψ is uniformly continuous. This continuity will give rise to combinatorial congruences.

For $s \in A^+$, one checks that

$$\begin{aligned} \psi(\pi^p s) &= \psi(([\varepsilon] - 1)^p s) \\ &= \psi(([\varepsilon]^p - 1)s + pss_1) \\ &= \pi\psi(s) + p\psi(ss_1) \in (p, \pi)\psi(sA^+). \end{aligned}$$

In particular,

$$\psi(\pi^p A^+) \subset (p, \pi) A^+.$$

Thus, by iteration, we get

Proposition 3.1 (Weak Estimate). *Let $n \geq 0$. Then*

$$\psi(\pi^n A^+) \subset (p,\pi)^{[n/p]} A^+ = \sum_{j=0}^{[n/p]} \pi^j p^{[n/p]-j} A^+.$$

Since the exponent $[(n-jp)/p]$ is decreasing in j, this proposition implies that for $x \in \pi^n A^+$, we have

$$\psi(x) = \sum_{j=0}^{\infty} a_j \pi^j,\ a_j \in \mathbb{Z}_p,\ \mathrm{ord}_p(a_j) \geq [(n-jp)/p].$$

This already gives a non-trivial combinatorial congruence. Let r be an integer. Let us calculate $\psi(\pi^n[\varepsilon]^{-r})$ in a different way.

Lemma 3.2.

$$\psi(\pi^n[\varepsilon]^{-r}) = \sum_{j\geq 0} \pi^j \sum_{k\equiv r(\mathrm{mod} p)} (-1)^{n-k} \binom{n}{k} \binom{(k-r)/p}{j}.$$

Proof. Since $\pi = [\varepsilon]-1$ and $[\varepsilon] = 1+\pi$, we have

$$\begin{aligned}
\psi(\pi^n[\varepsilon]^{-r}) &= \psi(([\varepsilon]-1)^n[\varepsilon]^{-r}) \\
&= \psi\left(\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}[\varepsilon]^{k-r}\right) \\
&= \sum_{k\equiv r(\mathrm{mod} p)} (-1)^{n-k}\binom{n}{k}[\varepsilon]^{(k-r)/p} \\
&= \sum_{k\equiv r(\mathrm{mod} p)} (-1)^{n-k}\binom{n}{k}\sum_{j\geq 0}\binom{(k-r)/p}{j}\pi^j \\
&= \sum_{j\geq 0}\pi^j \sum_{k\equiv r(\mathrm{mod} p)} (-1)^{n-k}\binom{n}{k}\binom{(k-r)/p}{j}.
\end{aligned}$$

□

Comparing the coefficients of π^j in this equation and the weak estimate, we get

Corollary 3.3 (Weak Congruence). *Let $n \geq 0$. We have*

$$\sum_{k\equiv r(\mathrm{mod} p)} (-1)^{n-k}\binom{n}{k}\binom{(k-r)/p}{j} \equiv 0(\mathrm{mod} p^{[(n-jp)/p]}).$$

The above simple estimate is crude and certainly not optimal since we ignored a factor of π. We now improve on it.

Theorem 3.4 (Sharp Estimate I). *For $n \geq 0$, we have*

$$\psi(\pi^n A^+) \subset \sum_{j=0}^{[n/p]} \pi^j p^{[\frac{n-1-jp}{p-1}]} A^+.$$

Proof. We prove the theorem by induction. The theorem is trivial if $n \leq p-1$. Write

$$\varphi(\pi) = (1+\pi)^p - 1 = \pi^p - p\pi s_1,\ s_1 \in A^+.$$

Then,

$$\psi(\pi^p s) = \psi((\varphi(\pi) + p\pi s_1)s) = \pi\psi(s) + p\psi(\pi s_1 s).$$

This proves that the theorem is true for $n = p$. Let $n > p$. Assume the theorem holds for $\leq n-1$. It follows that

$$\psi(\pi^n A^+) = \psi(\pi^p \pi^{n-p} A^+) \subseteq \pi\psi(\pi^{n-p} A^+) + p\psi(\pi^{n+1-p} A^+).$$

By the induction hypothesis, the right side is contained in

$$\pi \sum_{j=0}^{[(n-p)/p]} \pi^j p^{[\frac{n-p-1-jp}{p-1}]} A^+ + p \sum_{j=0}^{[(n+1-p)/p]} \pi^j p^{[\frac{n-p-jp}{p-1}]} A^+$$

$$= \sum_{j=1}^{[n/p]} \pi^j p^{[\frac{n-1-jp}{p-1}]} A^+ + \sum_{j=0}^{[(n+1-p)/p]} \pi^j p^{[\frac{n-1-jp}{p-1}]} A^+.$$

□

The function $[(n-1-jp)/(p-1)]$ is decreasing in j and vanishes for $j \geq [n/p]$. Comparing the coefficients of π^j in the lemma and the above sharp estimate, we deduce

Corollary 3.5 (Sharp Congruence I). *Let $r \in \mathbb{Z}$.*

$$\sum_{k \equiv r (\mathrm{mod} p)} (-1)^{n-k} \binom{n}{k} \binom{(k-r)/p}{j} \equiv 0 (\mathrm{mod} p^{[\frac{n-1-jp}{p-1}]}),$$

where $j \geq 0$ is a non-negative integer.

Theorem 3.6 (Sharp Estimate II). *For $n > 0$, we have*

$$\psi\left(\frac{1}{\pi^n}A^+\right) \subseteq \sum_{j=0}^{[n(p-1)/p]} \frac{1}{\pi^{n-j}} p^{[\frac{n(p-1)-jp-1}{p-1}]} A^+.$$

Proof. Note that

$$\varphi(\pi)/\pi = \pi^{p-1} + \binom{p}{1}\pi^{p-2} + \cdots + \binom{p}{p-1} \in (\pi^{p-1}, p),$$

so $(\varphi(\pi)/\pi)^n \in (\pi^{p-1}, p)^n$. Then

$$\begin{aligned} \psi\left(\frac{1}{\pi^n}A^+\right) &= \psi\left(\frac{1}{\varphi(\pi)^n}\left(\frac{\varphi(\pi)}{\pi}\right)^n A^+\right) \\ &= \frac{1}{\pi^n}\psi\left(\left(\frac{\varphi(\pi)}{\pi}\right)^n A^+\right) \\ &\subseteq \frac{1}{\pi^n}\sum_{i=0}^{n} p^{n-i}\psi(\pi^{i(p-1)}A^+). \end{aligned}$$

By Sharp Estimate I, we have

$$\psi(\pi^{i(p-1)}A^+) \subseteq \sum_{j=0}^{[i(p-1)/p]} \pi^j p^{[\frac{i(p-1)-1-jp}{p-1}]} A^+.$$

Then,

$$\begin{aligned} \psi\left(\frac{1}{\pi^n}A^+\right) &\subseteq \sum_{j=0}^{[n(p-1)/p]} \frac{1}{\pi^{n-j}} \sum_{[jp/(p-1)]\leq i\leq n} p^{n-i+[\frac{i(p-1)-jp-1}{p-1}]} A^+ \\ &\subseteq \sum_{j=0}^{[n(p-1)/p]} \frac{1}{\pi^{n-j}} p^{[\frac{n(p-1)-jp-1}{p-1}]} A^+. \end{aligned}$$

□

Corollary 3.7 (Sharp Congruence II). *Let*

$$S_j(n,r,p) = \sum_{\substack{i_0+\cdots+i_{p-1}=n \\ i_1+2i_2+\cdots\equiv r(\mathrm{mod} p)}} \binom{n}{i_0\cdots i_{p-1}} \binom{(i_1+2i_2+\cdots-r)/p}{j}.$$

Then integer $j \geq 0$, we have

$$S_j(n,r,p) \equiv 0 (\mathrm{mod} p^{[\frac{n(p-1)-1-jp}{p-1}]}).$$

Proof.

$$\begin{aligned}
& \psi\left(\frac{1}{\pi^n}[\varepsilon]^{-r}\right) \\
= & \frac{1}{\pi^n}\psi\left(\left(\frac{[\varepsilon]^p-1}{[\varepsilon]-1}\right)^n[\varepsilon]^{-r}\right) \\
= & \frac{1}{\pi^n}\psi((1+[\varepsilon]+\cdots+[\varepsilon]^{p-1})^n\cdot[\varepsilon]^{-r}) \\
= & \frac{1}{\pi^n}\sum_{\substack{i_0+\cdots+i_{p-1}=n \\ i_1+2i_2+\cdots\equiv r(\mathrm{mod} p)}}[\varepsilon]^{(i_1+2i_2+\cdots-r)/p}\binom{n}{i_0\cdots i_{p-1}} \\
= & \frac{1}{\pi^n}\sum_{\substack{i_0+\cdots+i_{p-1}=n \\ i_1+2i_2+\cdots\equiv r(\mathrm{mod} p)}}\sum_{j\geq 0}\pi^j\binom{n}{i_0\cdots i_{p-1}}\binom{(i_1+2i_2+\cdots-r)/p}{j} \\
= & \sum_{j=0}^{\infty}\frac{1}{\pi^{n-j}}S_j(n,r,p).
\end{aligned}$$

The function $[(n(p-1)-jp-1)/(p-1)]$ is decreasing in j and vanishes for $j\geq[n(p-1)/p]$. Comparing the coefficients of $\frac{1}{\pi^{n-j}}$, the congruence follows. $\square$

4 Sharp estimates for ψ^a

Let a be a positive integer. In this section, we extend the sharp estimates for ψ to ψ^a.

Theorem 4.1 (Sharp Estimate I). *For $n\geq 0$, we have*

$$\psi^a(\pi^n A^+)\subseteq\sum_{j=0}^{[n/p^a]}\pi^j p^{\left[\frac{n-p^{a-1}-jp^a}{p^{a-1}(p-1)}\right]}A^+.$$

Proof. We prove the theorem by induction on a. The theorem is true if $a=1$. Assume now $a\geq 2$ and assume that the theorem holds for $a-1$.

Then,

$$\begin{aligned}
\psi^a(\pi^n A^+) &= \psi(\psi^{a-1}\pi^n A^+) \\
&\subseteq \psi(\sum_{i=0}^{[n/p^{a-1}]} \pi^i p^{[\frac{n-p^{a-2}-ip^{a-1}}{p^{a-2}(p-1)}]} A^+) \\
&\subseteq \sum_{i=0}^{[n/p^{a-1}]} \sum_{j=0}^{[i/p]} \pi^j p^{[\frac{n-p^{a-2}-ip^{a-1}}{p^{a-2}(p-1)}]+[\frac{i-1-jp}{p-1}]} A^+ \\
&\subseteq \sum_{j=0}^{[n/p^a]} \pi^j \sum_{pj\le i\le [n/p^{a-1}]} p^{[\frac{n-p^{a-2}-ip^{a-1}}{p^{a-2}(p-1)}]+[\frac{i-1-jp}{p-1}]} A^+.
\end{aligned}$$

The exponent of p for a fixed j is decreasing in i and hence the minimun exponent of p is attained when $i = [n/p^{a-1}]$. The minimun exponent is computed to be

$$[\frac{n-p^{a-2}-[n/p^{a-1}]p^{a-1}}{p^{a-1}-p^{a-2}}] + [\frac{[n/p^{a-1}]-1-jp}{p-1}] = [\frac{n-p^{a-1}-jp^a}{p^{a-1}(p-1)}].$$

□

The proof of the lemma gives more general

Lemma 4.2.

$$\psi^a(\pi^n[\varepsilon]^{-r}) = \sum_{j\ge 0}\pi^j \sum_{k\equiv r(\mathrm{mod} p^a)} (-1)^{n-k}\binom{n}{k}\binom{(k-r)/p^a}{j}.$$

Comparing the coefficients of π^j in the lemma and the sharp estimate for ψ^a, we get

Corollary 4.3 (Sharp Congruence I). *Let $r \in \mathbb{Z}$. Then*

$$\sum_{k\equiv r(\mathrm{mod} p^a)} (-1)^{n-k}\binom{n}{k}\binom{(k-r)/p^a}{j} \equiv 0(\mathrm{mod} p^{[\frac{n-p^{a-1}-jp^a}{p^{a-1}(p-1)}]}),$$

where $j \ge 0$ is a non-negative integer.

Theorem 4.4 (Sharp Estimate II). *For $n > 0$ and $a > 0$, we have*

$$\psi^a\left(\frac{1}{\pi^n}A^+\right) \subseteq \sum_{j=0}^{[\frac{(an-a+1)(p-1)}{ap-a+1}]} \frac{1}{\pi^{n-j}} p^{[\frac{(an-a+1)(p-1)-j(ap-a+1)-1}{p-1}]} A^+.$$

Proof. The theorem is true for $a=1$. Assume now that $a>1$ and assume that the theorem is true for $a-1$. Then

$$\begin{aligned}\psi^a\left(\frac{1}{\pi^n}A^+\right) &= \psi\left(\psi^{a-1}\left(\frac{1}{\pi^n}A^+\right)\right)\\ \subseteq\ &\psi\left(\sum_{j=0}^{[\frac{(a-1)n-a+2)(p-1)}{(a-1)p-a+2}]}\frac{1}{\pi^{n-j}}p^{[\frac{((a-1)n-a+2)(p-1)-j((a-1)p-a+2)-1}{p-1}]}A^+\right)\\ \subseteq\ &\sum_j\sum_i\frac{1}{\pi^{n-j-i}}p^{[\frac{((a-1)n-a+2)(p-1)-j((a-1)p-a+2)-1}{p-1}]+[\frac{(n-j)(p-1)-ip-1}{p-1}]}A^+,\end{aligned}$$

where the indices i and j satisfy

$$0\leq j\leq[\frac{(a-1)n-a+2)(p-1)}{(a-1)p-a+2}],\ 0\leq i\leq[(n-j)(p-1)/p].$$

For fixed $i+j=k$, the exponent of p is decreasing in j and the minimun value is attained when $j=k$ and $i=0$. It follows that

$$\begin{aligned}\psi^a\left(\frac{1}{\pi^n}A^+\right) &\subseteq \sum_{k\geq 0}\frac{1}{\pi^{n-k}}p^{[\frac{((a-1)n-a+2)(p-1)-k((a-1)p-a+2)-1}{p-1}]+[n-k-1]}A^+\\ &\subseteq \sum_{k=0}^{[\frac{(an-a+1)(p-1)}{ap-a+1}]}\frac{1}{\pi^{n-k}}p^{[\frac{(an-a+1)(p-1)-k(ap-a+1)-1}{p-1}]}A^+,\end{aligned}$$

where we stop at $k=[\frac{(an-a+1)(p-1)}{ap-a+1}]$ in the summation as the exponent of p is zero if $k\geq[\frac{(an-a+1)(p-1)}{ap-a+1}]$. □

Corollary 4.5 (Sharp Congruence II). *Let*

$$S_j(n,r,p^a)=\sum_{\substack{i_0+\cdots+i_{(p^a-1)}=n\\ i_1+2i_2+\cdots\equiv r(\mathrm{mod} p^a)}}\binom{n}{i_0\cdots i_{(p^a-1)}}\binom{(i_1+2i_2+\cdots-r)/p^a}{j}.$$

Then for integer $j\geq 0$, we have

$$S_j(n,r,p^a)\equiv 0(\mathrm{mod} p^{[\frac{(an-a+1)(p-1)-j(ap-a+1)-1}{p-1}]}).$$

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