Zeta Functions Of Algebraic Cycles Over Finite Fields

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1. Introduction.

In this paper, we generalize the concept of a zeta function from zero cycles to higher dimensional cycles and investigate its meromorphic continuation and order of pole at $t = 1$ by means of a Riemann-Roch approach.

We begin with an example which motivated the present work. A classical problem about finite fields is to find a formula for the number $N'_q(d,n)$ of irreducible polynomials over F_q of degree d in n variables as d varies. This question goes back to Gauss (see the notes of Chapter 3 in [LN]), who first found a formula in the case when there is only one variable and the finite field is a prime field F_p. Dickson generalized Gauss's formula from a prime finite field to a general finite field F_q. Since then, attempts have been made to find a formula in the case of several variables. In 1936, Carlitz ([CA1]) found an analogous formula for the number of irreducible factorizable polynomials of degree d, where a polynomial over F_q is factorizable if it factors as a product of linear factors over the algebraic closure of F_q. This generalizes Gauss's formula in some sense. Later, Carlitz ([CA2-3]) established the asymptotic behavior of $N'_q(d,n)$ (as $d \to \infty$). Some of Carlitz's results were generalized by Cohen ([CO1-2]). Both Carlitz [CA2-3] and Cohen [CO1-2] said that no explicit formula for $N'_q(d,n)$ is known in the several variable case. Using ideas from the theory of zeta functions, we found an explicit but complicated combinatorial formula for $N'_q(d,n)$. The proof actually shows that in the several variable case, there is no combinatorial formula of the Gauss type; rather, there is a nice p-adic formula generalizing Gauss's formula. In some sense, this answers the question of Carlitz and Cohen.

To understand this, we consider homogeneous polynomials. Let $N_q(d,n)$ be the number of homogeneous irreducible polynomials over F_q of degree d in n variables, where two such polynomials are regarded as the same if they differ by a constant. Unique factorization for polynomials implies that

$$\sum_{d=0}^{\infty} \frac{(q^{\binom{n+d-1}{n-1}} - 1)}{(q-1)} t^d = \prod_{P(x)} (1 - t^{\deg P(x)})^{-1} \tag{1.1}$$

$$= \prod_{d=1}^{\infty} (1 - t^d)^{-N_q(d,n)} \tag{1.2}$$

where the first product runs over all irreducible homogeneous polynomials in n variables. The Möbius inversion formula and (1.2) then give an explicit combinatorial formula for $N_q(d,n)$. If $n \leq 2$, the left side of (1.1) is a rational function, so one finds a simple formula in this case (of which Gauss's formula is the affine version). If $n > 2$, the left side of (1.1) not only is not a rational function, but is even not convergent in any neighbourhood of the origin. This explains why for $n > 2$ there is no simple combinatorial formula for $N_q(d,n)$. However, if we view the left side of (1.1) p-adically, we see that it is a p-adic meromorphic function. Thus, there is a nice p-adic formula for $N_q(d,n)$ (see Theorem 2.1).

Note that an irreducible polynomial over F_q is simply a prime divisor over F_q in a projective space. Thus, the question of finding a formula for the number of irreducible polynomials over F_q of degree d has a natural generalization: that is, finding a formula for the number of prime divisors over F_q of degree d in a projective variety X/F_q. Why should one restrict to divisors? Thus, a general question is the following: Given a projective variety X/F_q of dimension n, for each integer $0 \leq r \leq n$, find a formula for the number of prime r-cycles over F_q of degree d as d varies. In the case $r = 0$, this reduces (via Möbius inversion) to the well-known question of finding a formula for the number of F_{q^d}-rational points on X/F_q as d varies, which is an extremely fruitful question in number theory and algebraic geometry, answered in large part by Weil's famous conjectures (Deligne's theorem). We expect that the general question (for all r) will play a similar role.

Recall that the study of the general question for $r = 0$ is equivalent to the study of the zeta function of X. Thus, we should have a similar notion of the zeta function of r-cycles which plays the same role in the general question. This is indeed the case, as we shall see now.

Let X be a projective algebraic variety of dimension n over F_q. For each integer r $(0 \leq r \leq n)$ we define the zeta function of r-cycles on X to be the following formal power series

$$Z_r(X,t) = \prod_P (1 - t^{\deg P})^{-1}, \tag{1.3}$$

where P runs over all prime r-cycles of X defined over F_q and deg$P$ is the degree function on the group of r-cycles (for a more precise definition, see section 3). When $r = 0$, $Z_r(X,t)$ is just the classical zeta function of a variety over a finite field. Let $N_r(d)$ be the number of prime r-cycles over F_q of degree d on X. By a theorem of Chow and van der Waerden (Satz 1 in [CW]), the set of effective r-cycles of degree d is parametrized (one-to-one) by an algebraic set (the Chow scheme) in a projective space. Thus, $N_r(d)$ is finite. Let $W_r(d)$ be the following weighted sum (each prime r-cycle of degree k is counted k times)

$$W_r(d) = \sum_{k|d} kN_r(k). \tag{1.4}$$

$N_r(d)$ can be obtained from $W_r(d)$ by Möbius inversion. Then, an alternative expression for $Z_r(X,t)$ is given by

$$\begin{aligned} Z_r(X,t) &= \prod_{d=1}^{\infty} (1 - t^d)^{-N_r(d)} \\ &= \exp\left(\sum_{d=1}^{\infty} \frac{W_r(d)}{d} t^d\right). \end{aligned} \tag{1.5}$$

This function incorporates complete information about the sequences $N_r(d)$ and $W_r(d)$. Thus, our general question is reduced to the study of zeta functions of algebraic cycles.

We expect that these general zeta functions contain rich information about the arithmetic and geometry of X/F_q, as in the case of zeta functions of zero cycles. More precisely, we should expect some similar results or conjectures corresponding to Weil's conjectures ([WE]) on zeta functions of zero cycles and Tate's conjectures ([TA1]) relating the pole of $Z_0(X,t)$ to invariants of algebraic cycles. We now describe two such conjectures for zeta functions of algebraic cycles.

I. First, on the arithmetic side, we conjecture that all $Z_r(X,t)$ are p-adic meromorphic functions. This would imply that there is a p-adic formula for the number of prime F_q-rational r-cycles of degree d, generalizing the well-known formula for the number of F_q-rational points on X, and thus answering our general question raised above. This conjecture generalizes Dwork's theorem on p-adic meromorphicity of $Z_0(X,t)$, which was at the heart of his proof on the rationality of $Z_0(X,t)$ (see, e.g. Chapter 5 in [KO1]).

II. Second, in connection with geometry, we conjecture that for a smooth projective variety X/F_q, the order of pole of $Z_r(X,t)$ at $t = 1$ equals the rank of the group $A_r(X)$ of F_q-rational r-cycles modulo rational equivalence, where the rank of an abelian group is defined to be the maximal number of Z-linearly independent elements. In particular, this would imply that the rank of $A_r(X)$ is finite. Another consequence of the conjecture is that the mysterious Chow ring $A(X/F_q) = \oplus_{r=0}^n A_r(X)$ has finite rank equal to the order of pole at $t = 1$ of the complete zeta function $Z(X,t) = \prod_{r=0}^n Z_r(X,t)$. This conjecture is an analogue of a conjecture of Tate on algebraic cycles over finite fields. Tate's conjecture asserts that the order of pole of $Z_0(X,t)$ at $t = 1/q^r$ equals the rank of the group $A_r^{\text{hom}}(X)$ of F_q-rational r-cycles modulo homological equivalence (see equation (12) in [TA1]).

We shall easily see that both of the conjectures in I and II are true for $r = 0$ (the case of the classical zeta functions) and for $r = n$. Thus, the two conjectures hold for curves. The first new case to consider is the zeta function of curves (1-cycles, or divisors) on a projective surface. A corollary of our results is that both conjectures hold for sufficiently general surfaces which are complete intersections.

Our main object of study will be the zeta function of divisors. Our first main result states that if $A_{n-1}(X)$ is finitely generated with rank one, then the zeta function of divisors is p-adic meromorphic with a simple pole at $t = 1$ (a more general result is given in Theorem 6.7). Thus, both conjectures hold in this case. Examples are: any nonsingular complete intersection of dimension at least three (Grothendieck's theorem) and a sufficiently general surface which is a complete intersection (Deligne's theorem). Our proof generalizes Weil's rationality proof for the zeta function of zero cycles on a smooth curve and uses the Riemann-Roch theorem (the general form). This is one of the first arithmetic applications of the higher dimensional Riemann-Roch theorem (see the introduction in [BL]). In the special case when the homogeneous coordinate ring of X is a unique factorization domain, we shall give a much easier proof using the Hilbert-Serre theorem.

Our second main result is as follows: If the monoid of effective divisor classes in $A_{n-1}(X)$ is a finitely generated monoid (by a theorem of Lang and Néron, $A_{n-1}(X)$ itself is a finitely generated abelian group), then the zeta function of divisors is p-adic meromorphic on the closed unit disk and has a pole at $t = 1$ of order equal to the rank of $A_{n-1}(X)$. Thus, the second conjecture is true in this case.

A by-product of our proof is a formula for the special value of $Z_{n-1}(t)$ at $t = 1$. The formula is of the form $-\frac{1}{(q^\lambda-1)}hR(X)$, where h is the order of torsion in $A_{n-1}(X)$, $R(X)$ is a certain regulator, and $q^\lambda - 1$ is the number of roots of unity in the quotient field of the homogeneous coordinate ring $S(X)$. This is a higher dimensional analogue in characteristic p of the residue formula for the Dedekind zeta function or the Birch-Swinnerton-Dyer conjecture for the Hasse-Weil zeta function of an elliptic curve.

This result suggests that one should look at the special value of $Z_r(X, t)$ at $t = 1$. Hopefully, one can formulate a conjectural formula for the special value. Unfortunately, this has not even been done for the classical zeta function $Z_0(X, t)$, although Artin and Tate did have a conjecture for the special value of the middle dimensional part of $Z_0(X, t)$ at $t = 1/q$ in the case when X is a surface over F_q (see Conjecture (C) in [TA2]). The conjectural Artin-Tate formula involves the Brauer group of X and the order of torsion in the Néron-Severi group (the group of algebraic cycles modulo algebraic equivalence). Our results suggest that the special value of $Z_r(X, t)$ at $t = 1$ should at least involve the order of torsion of $A_r(X)$.

In the last section, we attempt to generalize our results from divisors to cycles of higher codimension. For this to work, it is necessary to generalize the Riemann-Roch problem to higher codimensions. This suggests another interesting problem in algebraic geometry. For an effective r-cycle C, we define $L(C)$ to be the set of effective r-cycles rationally equivalent to C. This is a connected algebraic set, which is close to being linear. The generalized Riemann-Roch problem is then to study the behaviour of the sequence of varieties $L(dC)$ for d large. In case $C = D$ is a divisor, we reduce to the ordinary Riemann-Roch problem, that is, the behaviour of the dimensions $l(dD)$ for d large. For our purpose over a finite field, a satisfactory answer to the generalized Riemann-Roch problem would give information about the number of F_q-rational points on the sequence of varieties $L(dC)$, which is what we need in higher codimensions. Assuming that the monoid of effective r-cycle classes in $A_r(X)$ is finitely generated and that certain properties hold for the higher codimensional Riemann-Roch problem (properties which hold in the case of divisors), we can then prove that $Z_r(X, t)$ is meromorphic on the closed unit disk and has a pole of order $\rho_r = \mathrm{rank}(A_r(X))$ at $t = 1$ with special value $c_r h_r R_r(X)$, where h_r is the order of torsion of $A_r(X)$, $R_r(X)$ is a regulator, and c_r is a rational number under our assumptions.

If X/F_q is a quasi-projective variety in $\mathbf{P}^m$, but is not projective, one can define the zeta function $Z_r(X, t)$ of algebraic cycles in a similar way. We may suppose that $X = Y - Z$, where $Z \subset Y$ are projective. For a prime r-cycle C on X, define its degree to be the degree of its closure in Y. The zeta function $Z_r(X, t)$ is defined in a way similar to (1.3). One has the factorization $Z_r(X, t) = Z_r(Y, t)/Z_r(Z, t)$. Thus it suffices to study $Z_r(X, t)$ for X projective.

The above is mostly concerned with the similarities between the classical zeta function $Z_0(X, t)$ and the generalized zeta function $Z_r(X, t)$. We now say a few words about the differences between them. First, the duality and the Riemann hypothesis in Weil's conjectures do not seem to generalize to zeta functions of higher cycles. The best duality one can hope for would be that the order of pole of $Z_r(X, t)$ at $t = 1$ equals the order of pole of $Z_{n-r}(X, t)$ at $t = 1$. Assuming Conjecture II, this is equivalent to equality between the rank of $A_r(X)$ and the rank of $A_{n-r}(X)$. This is indeed true for $r = 0$ and $r = n$. However, Katz's version of Griffith's famous example (Exposé XX in [DK]) implies that for a certain three-fold X over a finite field F_q, the rank of $A_1(X)$ is greater than the rank of $A_2(X)$. In this sense, we cannot expect duality for the $Z_r(X, t)$. The Riemann hypothesis does not generalize, because $Z_r(X, t)$ is expected to be a p-adic meromorphic function, and we shall show that $Z_r(X,t)$ is never rational for $0<r<n$. In fact, these functions are never complex analytic in any neighborhood of the origin. A p-adic question would be to determine the Newton polygon of the reciprocal zeros and poles of $Z_r(X,t)$, for example, giving an analogue of the Katz conjecture (Mazur's theorem, see [MA]).

Concerning the various methods to study zeta functions over finite fields, we recall that there are three quite independent approaches to proving rationality of the zeta function of zero cycles on a curve. The first is Weil's Riemann-Roch approach (see [MO]). The second is the p-adic method (Dwork). The third is the l-adic approach. Both the p-adic approach and the l-adic approach work for the zeta function of zero cycles on higher dimensional X. Each has its own advantages. It has been somewhat mysterious why the Riemann-Roch approach does not generalize. The present work provides some explanations and shows that there is some hope that the Riemann-Roch approach will work for all zeta functions $Z_r(X,t)$, at least in certain important cases. An advantage of the Riemann-Roch approach is that it gives not only meromorphic continuation but also the special value at $t=1$. However, we expect more difficulties for $Z_r(X,t)$ $(0<r<n)$ than for $Z_0(X,t)$.

The idea of defining zeta functions of algebraic cycles can be adapted to diophantine problems over other fields, not just the finite field F_q. For example, for a projective variety X over a number field K, one can consider the sequence of numbers a_h of K-rational points on X of height h, where the degree and dimension of the points are fixed and the height function is a suitable generalization (via Chow varieties) to algebraic cycles of the ordinary height function. One can define the zeta function to be the Dirichlet series associated with the sequence a_h. Fundamental questions are then meromorphic continuation, the order of pole and the special value. Results in analytic number theory, such as Hardy-Littlewood's the circle method, can be applied to study these questions. One can also define zeta functions of algebraic cycles over a p-adic field, generalizing Igusa's local zeta function ([IG]) and Serre's zeta function on the reductions of a p-adic analytic space (see Section 3 in [SE]). These zeta functions are expected to be rational functions. We hope to examine these questions in more detail in the future.

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2. The Number of Irreducible Polynomials

Let F_q be a finite field of q elements. Fix a positive integer n. For each nonnegative integer d, let M_d be the number of homogeneous polynomials over F_q of degree d in n variables ($M_0=1$). Let N_d be the number of irreducible ones ($N_0=0$). Two polynomials are identified if they differ by a constant. The purpose of this section is to prove a p-adic formula and a combinatorial formula for N_d. The special case treated in this section also serves to motivate the later more general results. It is clear that $N_1=(q^n-1)/(q-1)$. We need only consider N_d for $d>1$.

Theorem 2.1 (p-adic formula). If $n>2$, there is a sequence of p-adic integers α_i approaching zero such that for all positive integers $d>1$,

$$N_d=-\frac{1}{d}\sum_{k|d}\mu(\frac{d}{k})\sum_{i=1}^{\infty}\alpha_i^k. \tag{2.1}$$

If $n=1$, $N_d=0$ for $d>1$. If $n=2$, then for all positive integers $d>1$,

$$N_d=\frac{1}{d}\sum_{k|d}\mu(\frac{d}{k})q^k. \tag{2.2}$$

Proof. It is easy to see that

$$M_d=\frac{q^{\binom{n+d-1}{n-1}}-1}{q-1}. \tag{2.3}$$

Define

$$Z(t)=\prod_{P(x)}(1-t^{\deg P(x)})^{-1},$$

where the product is over all irreducible polynomials in n variables over F_q. Because any homogeneous polynomial over F_q can be uniquely factored as a product of irreducible homogeneous polynomials over F_q, one checks that

$$\begin{aligned} Z(t) &= \prod_{d=1}^{\infty}(1-t^d)^{-N_d} && (2.4)\\ &= \exp(\sum_{d=1}^{\infty}\frac{W_d}{d}t^d) && (2.5)\\ &= \sum_{d=0}^{\infty}M_d t^d, \end{aligned}$$

where W_d is the following weighted sum (each irreducible polynomial of degree k is counted k times)

$$W_d = \sum_{k|d} kN_k. \quad (2.6)$$

According to (2.3), one computes that

$$\begin{aligned} Z(t) &= \frac{1}{q-1}(-\frac{1}{1-t}+\sum_{d=0}^{\infty}q^{\binom{n+d-1}{n-1}}t^d)\\ &= \frac{1}{1-t}\sum_{d=0}^{\infty}\frac{q^{\binom{n+d-1}{n-1}}-q^{\binom{n+d-2}{n-1}}}{q-1}t^d \quad (2.7)\\ &= \frac{Q(t)}{1-t}, \end{aligned}$$

where $Q(t)$ is defined by (2.7). If $n>2$, (2.7) shows that $Q(t)$ is a p-adic entire function with integral coefficients and constant term 1. The p-adic Weierstrass factorization theorem shows that there is a sequence of p-adic integers α_i approaching zero such that

$$Z(t) = \frac{\prod_{i=1}^{\infty}(1-\alpha_i t)}{1-t}. \quad (2.8)$$

This and (2.5) give the following p-adic formula:

$$W_d = 1-\sum_{i=1}^{\infty}\alpha_i^d. \quad (2.9)$$

To obtain the formula for N_d, we apply Möbius inversion and deduce the theorem for $n>2$. The case $n\le 2$ is similar. One computes that for $n=2$,

$$Z(t) = \frac{1}{q-1}(\frac{q}{1-qt}-\frac{1}{1-t}) = \frac{1}{(1-t)(1-qt)}. \quad (2.10)$$

If $n=1$, then $Z(t)=1/(1-t)$. The theorem is proved.

Theorem 2.2 (p-adic estimate). Let $n>2$. In Theorem 2.1, we order the α_i so that $\mathrm{ord}_q\alpha_i$ is increasing, where ord_q is the q-adic valuation such that $\mathrm{ord}_q q=1$. Then for all $d\ge 1$,

$$\mathrm{ord}_q\alpha_d = \binom{n+d-2}{n-1}-\binom{n+d-3}{n-1}. \quad (2.11)$$

Proof. By (2.7), the Newton polygon of the entire function $Q(t)$ is the convex closure in $\mathbf{R}^2$ of the following vertices

$$(d, \binom{n+d-2}{n-1}), \quad d \geq 0. \tag{2.12}$$

Theorem 2 follows from this.

Next, we derive an explicit combinatorial formula for N_d. This type of formula will be used in a future article [WA] to prove the rationality of a different type of zeta functions of algebraic cycles (roughly speaking, we will fix the degree and let the field F_q varies).

Theorem 2.3 (Combinatorial Formula). For each positive integer m, we have

$$N_m = \sum_{d|m} \frac{-d}{m}\mu(\frac{m}{d}) \sum_{k_1+2k_2+\cdots+dk_d=d, k_i\geq 0} \frac{(-1)^{k_1+k_2+\cdots+k_d}}{k_1+k_2+\cdots+k_d} \binom{k_1+k_2+\cdots+k_d}{k_1 \ \ k_2 \ \ \cdots \ \ k_d} M_1^{k_1}\cdots M_d^{k_d},$$

where M_d is given by (2.3) (the inner summation is similar to Waring's formula for power sums in terms of elementary symmetric functions).

Proof. Taking the logarithm of (2.5), one finds that

$$\begin{aligned} \sum_{d=1}^{\infty} \frac{W_d}{d} t^d &= \log(1 + \sum_{d=1}^{\infty} M_d t^d) \\ &= -\sum_{k=1}^{\infty} \frac{(-1)^k}{k} (\sum_{d=1}^{\infty} M_d t^d)^k. \end{aligned} \tag{2.13}$$

Comparing the cofficients of (2.13), we obtain

$$W_d = -d \sum_{k_1+2k_2+\cdots+dk_d=d, k_i\geq 0} \frac{(-1)^{k_1+k_2+\cdots+k_d}}{k_1+k_2+\cdots+k_d} \binom{k_1+k_2+\cdots+k_d}{k_1 \ \ k_2 \ \ \cdots \ \ k_d} M_1^{k_1}\cdots M_d^{k_d}.$$

Möbius inversion then gives Theorem 2.3.

Remark. If we let M_d be the number of non-associated polynomials of total degree d in n variables over F_q (not necessarily homogeneous), and let N_d be the number of irreducible ones, then the same method shows that Theorem 2.3 remains true with M_d given by

$$M_d = \frac{q^{\binom{n+d}{n}} - q^{\binom{n+d-1}{n}}}{q-1}. \tag{2.14}$$

If $n > 1$, the p-adic formula for N_d is of the form (for all $d > 0$)

$$N_d = -\frac{1}{d} \sum_{k|d} \mu(\frac{d}{k}) \sum_{i=1}^{\infty} \alpha_i^k, \tag{2.15}$$

where the α_d are p-adic numbers whose q-adic valuations are given by

$$\mathrm{ord}_q \alpha_d = \binom{n+d-1}{n-1} - \binom{n+d-2}{n-1}. \tag{2.16}$$

Thus, we have an explicit formula and a p-adic formula for the number of irreducible polynomials of degree d in n variables. In some sense, this answers the question of Carlitz and Cohen, which is our first motivation in studying zeta functions of algebraic cycles.

3. Zeta Functions of Algebraic Cycles

In this section, we introduce the zeta functions of algebraic cycles of a projective variety in a given projective space, and we formulate two general conjectures associated with such zeta functions. Some simple properties of the zeta functions will be proved.

We first recall the definition of the degree of a projective variety. Let n and m be positive integers with $n \leq m$. Let X be a closed n dimensional subscheme of the m dimensional projective space $\mathbf{P}^m$ over F_q (the embedding will be fixed). We shall always work over the ground field F_q unless otherwise stated. Let $S(X) = \oplus_{d=0}^{\infty} S_d(X)$ be the homogeneous coordinate ring of X, where $S_d(X)$ consists of the homogeneous elements of degree d and $S(X)$ is finitely generated by $S_1(X)$ as an F_q algebra ($S_0(X) = F_q$). Let

$$l(d) = \dim_{F_q} S_d(X).$$

For large d, $l(d)$ is a polynomial of degree n, called the Hilbert polynomial of X. The degree of X, denoted $\deg X$, is defined to be the coefficient of $d^n/n!$ in this polynomial (see I.7 in [HA2]). If X is irreducible, an alternative definition of $\deg X$ is the intersection number $X \cdot H^n$, where H is a hyperplane in $\mathbf{P}^m$ (see Example 2.5.2 in [FU]). This means that the degree of X is the number of intersection points cutout by the intersection of n sufficiently general hyperplanes.

Fix a closed n dimensional subscheme X of $\mathbf{P}^m$ over F_q. A subvariety (or a prime cycle) over F_q is a closed integral subscheme of X (i.e., a reduced and irreducible closed subscheme). For each integer $0 \leq r \leq n$, we define the zeta function of r-cycles on X to be the following formal power series

$$Z_r(X,t) = \prod_P (1 - t^{\deg P})^{-1}, \tag{3.1}$$

where P runs over all subvarieties of dimension r on X and $\deg P$ is the degree of P viewed as a closed subscheme of $\mathbf{P}^m$. The complete zeta function of X is then defined to be the product of all $Z_r(X,t)$ $(0 \leq r \leq n)$. Let $N_r(d)$ (resp. $M_r(d)$) be the number of prime r-cycles (resp. effective r cycles) of degree d on X. By a theorem of Chow and van der Waerden, the set of effective r-cycles of degree d is parametrized (one-to-one) by an algebraic set (the Chow scheme) in a projective space. Thus, $M_r(d)$ and $N_r(d)$ are all finite. All zeta functions are well defined. They are easily seen to be p-adic analytic in the open unit disk.

Alternative expressions for $Z_r(X,t)$ are given by

$$\begin{aligned} Z_r(X,t) &= \prod_{d=1}^{\infty} (1 - t^d)^{-N_r(d)} \\ &= \sum_{d=0}^{\infty} M_r(d) t^d \\ &= \exp\left(\sum_{d=1}^{\infty} \frac{W_r(d)}{d} t^d\right), \end{aligned} \tag{3.2}$$

where $W_r(d)$ is the following weighted sum (each prime r-cycle of degree k is counted k times)

$$W_r(d) = \sum_{k|d} k N_r(k). \tag{3.3}$$

Geometrically, a point counted in $N_r(d)$ consists of d conjugate "non closed points" of dimension r. Thus, $W_r(d)$ can be interpreted as the number of "non closed points" of degree d and dimension r. In particular, $W_0(d)$ is just the number of F_{q^d}-rational points on X and $Z_0(X,t)$ is the classical zeta function of a projective scheme over a finite field. It is well known that $Z_0(X,t)$ is a rational function (Dwork's theorem [DW]). The example in section 2 shows that in general we cannot expect that the $Z_r(X,t)$ are rational functions; in fact, we shall show that $Z_r(X,t)$ is never rational for $0 < r < n$. However, the example does suggest that $Z_r(X)$ is a p-adic meromorphic function. If $Z_r(X,t)$ is p-adic meromorphic, this would imply that there are two sequences of p-adic integers α_i and β_i approaching zero such that for all positive integers d

$$W_r(d) = \alpha_1^d + \alpha_2^d + \cdots - \beta_1^d - \beta_2^d - \cdots.$$

This would generalize the formula for $W_0(d)$, in which there are only finitely many terms, because $Z_0(X,t)$ is rational.

Let $X_1, X_2, \cdots, X_c$ be the distinct closed integral components of X. The zeta function of n cycles

$$Z_n(X,t) = \prod_{i=1}^{c}(1 - t^{\deg X_i})^{-1} \tag{3.4}$$

is a rational function. Thus, the zeta function of top dimension is very simple and well understood.

Since a scheme X and its underlying reduced subscheme $X_{\rm red}$ have the same subvarieties,

$$Z_r(X,t) = Z_r(X_{\rm red}, t), \tag{3.5}$$

for all r. If X_1 and X_2 are closed subschemes of X such that $X = X_1 \cup X_2$, then one checks that

$$Z_r(X,t) = \frac{Z_r(X_1,t)Z_r(X_2,t)}{Z_r(X_1 \cap X_2, t)}, \tag{3.4}$$

where $X_1 \cap X_2$ is the scheme theoretic intersection of X_1 and X_2. These observations show that in studying zeta functions of algebraic cycles we may suppose that X is a variety (closed integral subscheme of $\mathbf{P}^m$). Thus, the homogeneous coordinate ring $S(X)$ of X is an integral domain.

Let H be a hyperplane in $\mathbf{P}^m$. Let P be an irreducible subvariety in X of dimension r. As we know, the degree of P is the intersection number $P \cdot H^r$. If $r = 0$, the number $P \cdot H^0$ is clearly independent of the projective embedding of X. If $r > 0$, however, the degree is dependent on the projective embedding of X. It would be interesting to find the general behaviour of zeta functions as the embedding varies.

Conjecture I (meromorphic continuation). Let X be an n dimensional projective variety in $\mathbf{P}^m$ over F_q. Then for all integers $0 \leq r \leq n$, $Z_r(X,t)$ is a p-adic meromorphic function.

Even for projective space $X = \mathbf{P}^n$ $(n \geq 3)$, we do not have a complete proof of this conjecture. The following simple result shows that the zeta function of divisors is p-adic meromorphic if the homogeneous coordinate ring $S(X)$ is a unique factorization domain. This result is a direct generalization of the polynomial example treated in section 1. In the next section, we shall establish more evidence for the above conjecture in the case of divisors.

Theorem 3.1. Let X be a factorial projective variety in $\mathbf{P}^m$ (i.e., the homogeneous coordinate ring $S(X)$ of X is a unique factorization domain). Then, the zeta function of divisors on X is a p-adic meromorphic function. It is rational if and only if X is a curve.

Proof. Since $S(X)$ is a unique factorization domain, all prime divisors are principal. Thus, all effective divisors are principal. If $f \in S_d(X)$, then intersection theory shows that

$$\deg(\mathrm{div}(f)) = d \times \deg X, \tag{3.6}$$

where $\mathrm{div}(f)$ is the divisor defined by the hypersurface section $f = 0$. Let

$$Z^*(X,t) = \prod_f (1 - t^{\deg(f)})^{-1} = \sum_{d=0}^{\infty} M_d t^d, \tag{3.7}$$

where f runs through all irreducible homogeneous elements of $S(X)$ and M_d is the number of homogeneous elements of degree d in $S(X)$ (two such elements are considered the same if they differ by a constant). Then (3.6) shows that

$$Z_{n-1}(X,t) = Z^*(X, t^{\deg(X)}). \tag{3.8}$$

Since $S(X)$ is a unique factorization domain, it is integrally closed. Thus, two non-zero elements of $S_d(X)$ define the same divisor if and only if they differ by an element of F_q. Let $l(d)$ be the dimension of $S_d(X)$ as an F_q vector space; then

$$M_d = \frac{q^{l(d)} - 1}{q - 1}. \tag{3.9}$$

The Hilbert-Serre theorem states that $l(d)$ is a polynomial of degree n for large d. Substituting (3.9) into (3.7), we deduce that $Z^*(X,t)$ is a p-adic meromorphic function. Because the degree of $l(d)$ is n, it follows that $Z_{n-1}(X,t)$ is rational if and only if $n=1$. The theorem is proved.

Remark 3.2. X is factorial if and only if X is projectively normal ($S(X)$ is integrally closed) and the group $A^1(X)$ of Weil divisor classes is generated by a hyperplane section.

Theorem 3.3. Let X/F_q be an n-dimensional projective variety in $\mathbf{P}^m$. Then $Z_r(X,t)$ is never rational for $0<r<n$.

Proof. The simple reason is that for $0<r<n$ there are too many r-cycles of a given degree, and this prevents $Z_r(X,t)$ from being rational. To prove the theorem, we may assume $n>1$. By (3.2),

$$Z_r(X,t)=\sum_{d=0}^{\infty} M_r(d)t^d.$$

If $Z_r(X,t)$ were rational, we would have the following upper bound

$$\mid M_r(d) \mid \leq q^{cd}, \tag{3.10}$$

for some constant c.

We first treat the case $r=n-1$. Observe that the integral closure of F_q in the quotient field of $S(X)$ is a finite extension F_{q^λ} of F_q. Each homogeneous element f in $S_d(X)$ determines an effective divisor of degree $d\times \deg X$. Because X is projective, two different such elements determine the same divisor if and only if they differ by an element in the quotient field of $S(X)$ which is integral over F_q. Such an element is then in F_{q^λ}. Thus, the following lower bound holds

$$M_{n-1}(d\times \deg X)\geq \frac{q^{l(d)}-1}{q^\lambda-1}, \tag{3.11}$$

where $l(d)$ is the dimension of $S_d(X)$ as an F_q vector space. By the Hilbert-Serre theorem, $l(d)$ is a polynomial in d of degree $n\geq 2$. This and (3.11) show that $M_{n-1}(d)$ grows faster than q^{cd} for any constant c. Thus, $Z_{n-1}(X,t)$ cannot be rational.

For $1\leq r\leq n-2$, we choose a subvariety Y of X of dimension $r+1$ and consider only those effective r-cycles contained in Y. The above argument for divisors implies that $M_r(d)$ grows faster than q^{cd} for any constant c. The theorem is proved.

The proof shows that for $0<r<n$, $Z_r(X,t)$ is not complex analytic in any neighborhood of the origin.

Next, we formulate a conjecture relating the order of pole of the zeta function at $t=1$ to the rank of the group of algebraic cycles modulo rational equivalence.

Let X be an n dimensional projective variety in $\mathbf{P}^m$ over F_q. Let r be an integer between 0 and n. A prime r-cycle V is a closed integral subvariety of dimension r on X which is defined over F_q. An r-cycle over F_q is an element of the free abelian group $C_r(X)$ generated by the prime r-cycles. We write cycles in the form $C=\sum_i n_iV_i$, where the V_i are prime cycles, the n_i are integers, and only finitely many n_i are different from zero. Define the degree $\deg C$ of C by

$$\deg C=\sum_i n_i \deg V_i, \tag{3.12}$$

where $\deg V_i$ is the degree of V_i viewed as a projective subvariety of $\mathbf{P}^m$. If all $n_i\geq 0$, C is called effective. An r-cycle is also called an $(n-r)$-cocycle. An $(n-1)$-cycle is also called a divisor.

We now recall the definition of various equivalences on an algebraic scheme X (separated and of finite type) over an arbitary field F. All varieties here are assumed to be defined over F.

For any $r+1$-dimensional subvariety V of X, and any non-zero rational function f on V (i.e., $f\in R(V)^*$), define an r-cycle $\mathrm{div}(f)$ on X by

$$\mathrm{div}(f)=\sum \mathrm{ord}_W(f)W,$$

where the sum is over all codimension one subvarieties W of V and ord_W is the order function on $R(V)^*$ determined by the one-dimensional local ring $O_{W,V}$ (see Chapter 1 in [FU]). A cycle C in $C_r(X)$ is rationally equivalent to zero if and only if there are a finite number of $(r+1)$-dimensional subvarieties V_i of X and $f_i \in R(V_i)^*$ such that

$$C = \sum \mathrm{div}(f_i).$$

In the case $r = n-1$, rational equivalence coincides with linear equivalence.

An alternate definition is as follows: Let $X \times \mathbf{P}^1$ be the Cartesian product of X with the projective line $\mathbf{P}^1$. A cycle C in $C_r(X)$ is rationally equivalent to zero if and only if there are $(r+1)$-dimensional subvarieties $V_1, \cdots, V_t$ of $X \times \mathbf{P}^1$ such that the projections from V_i to $\mathbf{P}^1$ are dominant and

$$C = \sum_{i=1}^{t} V_i(0) - V_i(\infty).$$

A cycle C in $C_r(X)$ is algebraically equivalent to zero if and only if there are nonsingular projective curves $C(i)$ defined over the gound field and $(r+1)$-dimensional subvarieties V_i of $X \times C(i)$ $(i = 1, \cdots, t)$ such that the projections from V_i to $C(i)$ are dominant and

$$C = \sum_{i=1}^{t} V_i(u_i) - V_i(v_i),$$

where u_i and v_i are two rational points on $C(i)$.

A cycle C in $C_r(X)$ is torsion equivalent to zero if some integral multiple of C is algebraically equivalent to zero.

A cycle C in $C_r(X)$ is l-adic homologically equivalent to zero if its l-adic cohomology class in $H_{2r}(X, Q_l)$ is zero, where l is a fixed prime number different from the characteristic of the ground field F.

A cycle C in $C_r(X)$ is numerically equivalent to zero if for all subvarieties D of codimension r on X, the intersection number $C \cdot D$ is zero.

All of the above definitions give equivalence relations. Let

$$C_r^{\mathrm{num}} \supset C_r^{\mathrm{hom}} \supset C_r^{\tau} \supset C_r^{\mathrm{alg}} \supset C_r^{\mathrm{rat}}$$

denote the groups of r-cycles which are, respectively, numerically, l-adic homologically, torsion, algebraically and rationally equivalent to zero. Let

$$A_r^{\mathrm{num}} \subset A_r^{\mathrm{hom}} \subset A_r^{\tau} \subset A_r^{\mathrm{alg}} \subset A_r^{\mathrm{rat}} = A_r(X)$$

be the corresponding quotient groups of $C_r(X)$. The groups $A_r(X)$ are the finest among all cycle groups. In general, they are very difficult to compute. The direct sum $A(X) = A^{\mathrm{rat}}(X) = \oplus_{r=0}^{n} A_r(X)$ is the Chow ring with multiplication given by intersection product. This is a very mysterious ring; only in a few cases has its structure been determined. Similarly, one can define the Chow rings $A^{\mathrm{num}}(X)$, $A^{\mathrm{hom}}(X)$, $A^{\tau}(X)$ and $A^{\mathrm{alg}}(X)$ of numerical equivalence, homological equivalence, torsion equivalence and algebraic equivalence. One often uses the alternate notation $A^r(X) = A_{n-r}(X)$. Similarly, instead of $A^{\mathrm{num}}(X)$, one often writes $A_{\mathrm{num}}(X)$, etc. For an abelian group A, not necessarily finitely generated, we define the rank $r(A)$ of A to be the maximal number of Z-linearly independent elements in A. The rank $r(A)$ is the dimension of the Q-vector space $A \otimes Q$. Any torsion group has rank 0. The group of rational numbers (a divisible group) has rank 1. In general, $r(A)$ can be infinite.

Conjecture II (rank and pole). Let X/F_q be a smooth projective variety of dimension n in $\mathbf{P}^m$. Then, for all r, the rank of $A_r(X)$ equals the order of pole of $Z_r(X,t)$ at $t=1$.

This conjecture means that $Z_r(X,t)$ is p-adic meromorphic on the closed unit disk and its order of pole at $t=1$ equals the rank of $A_r(X)$. In particular, the conjecture implies that the rank is finite. The last property has the possibility of being true because we are working over a finite field. It would not hold over the complex numbers $\mathbf{C}$. For example, over the complex numbers, the group $C^{\mathrm{alg}}_{n-1}(X)/C^{\mathrm{rat}}_{n-1}(X)$ is an abelian variety (torus). If its dimension is non-zero, then the torus contains a copy of $\mathbf{C}$. Hence, the Z-rank of $A_{n-1}(X)$ is infinite. However, it is known that $A^{\mathrm{num}}_r(X)$ and $A^{\mathrm{hom}}_r(X)$ are finitely generated over any field. This follows from the finite dimensionality of l-adic cohomology, see [HA1]. Quillen's theorem on higher K-theory shows that $A^r(X)$ is isomorphic to the cohomology group $H^r(X, K_r)$, where K_r is the sheaf of higher K groups on X. Thus, Conjecture II implies that over a finite field, the cohomology groups $H^r(X, K_r)$ have finite rank.

Proposition 3.4. Let X/F_q be a projective variety and $\bar{X}$ be the base extension of X to the algebraic closure $\bar{F}_q$. Then a cycle C in $A_r(X)$ is a torsion element if and only if it is a torsion element of $A_r(\bar{X})$.

Proof. Let C be a cycle in $A_r(X)$. It suffices to prove that if C is zero in $A_r(\bar{X})$, then C is a torsion element in $A_r(X)$. Thus, let C be rationally equivalent to zero over $\bar{F}_q$. By definition, there are a finite number of $(r+1)$-dimensional subvarieties $\bar{V}_i$ of $\bar{X}\times\mathbf{P}^1$ such that the projections from V_i to $\mathbf{P}^1$ are dominant and

$$C = \sum_{i=1}^{t} V_i(0) - V_i(\infty).$$

Let σ be the Frobenius automorphism $a \to a^q$ of $\bar{F}_q$. σ acts naturally on each V_i and has finite order. Thus, for any integer a,

$$C = \sum_{i=1}^{t} \sigma^a(V_i(0) - V_i(\infty)).$$

Choose a positive integer k such that

$$\sum_{j=0}^{k-1} \sigma^j(V_i) = k_i W_i$$

for all i, where the W_i are subvarieties of $\bar{X} \times \mathbf{P}^1$ which can be defined over F_q and the k_i are positive integers. Then

$$kC = \sum_{i=1}^{t} k_i(W_i(0) - W_i(\infty)).$$

This shows that kC is zero in $A_r(X)$. The proposition is proved.

In the case that X is nonsingular in codimension 1, Weil proved a stronger result for divisors, which says that a divisor D over a field is linearly equivalent to zero if and only if it is linearly equivalent to zero after base extension (see Section VI.5 in [LA]).

Corollary 3.5. Let X/F_q be a projective variety. Then $C^{\tau}_r(X)/C^{\mathrm{rat}}_r(X)$ is a torsion group. In particular, $A_r(X)$ has finite rank if and only if $A^{\tau}_r(X)$ does.

Proof. By the definition and Proposition 3.4, we only need to prove that $C^{\mathrm{alg}}_r(\bar{X})/C^{\mathrm{rat}}_r(\bar{X})$ is a torsion group, where $\bar{X}$ is the base extension of X from F_q to its algebraic closure. Now, it is known (see Example 19.1.2 in [FU]) that $C^{\mathrm{alg}}_r(\bar{X})/C^{\mathrm{rat}}_r(\bar{X})$ is generated by classes in $A^{\mathrm{rat}}_r(\bar{X})$ of the form $f_*g^*(\beta)$ for V a subvariety of $\bar{X} \times C$, where C is a projective nonsingular curve over $\bar{F}_q$, β is a zero cycle of degree zero on C (hence a point on the Jacobian variety $J(C)$), f is the projection from V to X and g is the projection from V to C. Since an abelian variety over $\bar{F}_q$ is a torsion group, the corollary is proved.

It would be interesting to know whether $C^{\tau}_r(X)/C^{\mathrm{rat}}_r(X)$ is finite. This is true for divisors, at least under certain non-singularity assumptions.

Corollary 3.6. Let X/F_q be a projective variety. Then $A_0(X)$ has rank 1.

Proof. Let $\bar{X} = X \otimes \bar{F}_q$ be the base extension of X from F_q to its algebraic closure. First assume that X is absolutely irreducible, i.e., $\bar{X}$ is irreducible. Let $B_0(\bar{X})$ be the kernel of the degree map $A_0(\bar{X}) \to \mathbf{Z}$. $B_0(\bar{X})$ is generated by the images $f_*(J(C))$, where $f : C \to X$, C is a non-singular projective curve over $\bar{F}_q$ and $J(C)$ is the Jacobian variety of C (see Example 1.6.6 in [FU]). Since an abelian variety over $\bar{F}_q$ is a torsion group, $A_0(\bar{X})$ has rank 1. The corollary then follows from Proposition 3.4. If X is not absolutely irreducible, one uses the above proof together with a Galois argument.

We now prove

Theorem 3.7. Conjectures I-II hold for $r = 0$ and $r = n$.

Proof. It remains to prove Conjecture II for $r = 0$. Let X/F_q be smooth and projective. Weil's conjectures show that there are integral polynomials $P_i(t)$ such that

$$Z_0(X,t) = \prod_{i=0}^{2n} P_i(t)^{(-1)^{i+1}},$$

where $P_0(t) = 1 - t$ and each reciprocal root of $P_i(t)$ has absolute value $q^{i/2}$. Thus, $t = 1$ is a simple pole of $Z_0(X,t)$. The theorem is proved.

4. Zeta Functions of Divisors: Rank One Case

In this section, we prove Conjectures I and II for the zeta function of divisors in the case when $A^1(X)$ is finitely generated with rank 1.

Let X be an n-dimensional projective variety in $\mathbf{P}^m$ with homogeneous coordinate ring $S(X)$. Let $K(X)$ be the function field of X (i.e., the quotient field of $S(X)$). Let $C^1(X/F_q)$ be the group of divisors. Each nonzero element $f \in K(X)$ determines a principal divisor, denoted $\mathrm{div}(f)$. Intersection theory shows that a principal divisor has degree 0 (see I.7 in [HA2]). Two divisors are linearly equivalent if their difference is a principal divisor. Let $A^1(X/F_q)$ be the group of linear equivalence classes of divisors in $C^1(X/F_q)$. Then the degree function extends to $A^1(X/F_q)$. An alternative definition of the degree of a divisor D in $C^1(X/F_q)$ is given by the intersection number $D \cdot H^{n-1}$, where H is a hyperplane in $\mathbf{P}^m$. The two definitions coincide, see (Example 2.5.2 in [FU]). Assume that $A^1(X/F_q)$ is a finitely generated abelian group (by a theorem of Lang-Néron [LN], this is true if $\bar{X}$ is nonsingular in codimension 1). By analogy with the Dedekind zeta function of a number field or the Birch-Swinnerton-Dyer conjecture for elliptic curves over a number field, a fundamental question in characteristic p is then to relate the rank ρ of $A^1(X)$ and the order h of the torsion in $A^1(X)$ to the order of pole of $Z_{n-1}(X,t)$ at $t = 1$ and its special value (for p-adic analogues of this question, see Chapter 4 in [KO2]). We use the letter h to denote the torsion order because it is the higher dimensional analogue of the class number. The following result shows that we have a complete understanding of the above question in the case when $A^1(X)$ is finitely generated of rank 1. In particular, Conjectures I and II hold in this case.

Theorem 4.1. Let X be an n-dimensional projective variety in $\mathbf{P}^m$, $n \geq 2$. Assume that $A^1(X/F_q)$ is finitely generated of rank 1 and torsion order h. Then the zeta function $Z_{n-1}(X,t)$ of divisors is a p-adic meromorphic function and has a simple pole at $t = 1$ with residue $\frac{h}{\mu(q^\lambda - 1)}$, where μ is the smallest positive degree of an element of $A^1(X)$ and λ is the positive integer such that F_{q^λ} is the integral closure of F_q in the quotient field of $S(X)$.

We first give some preliminaries. Let F_{q^λ} be the integral closure of F_q in the quotient field $K(X)$ of $S(X)$. If X is projectively normal, then $\lambda = 1$. More generally, if X is normal, then $\lambda = 1$. Becasue in this case, certain d-uple embedding of X will be projectively normal and the function field $K(X)$ is independent of embeddings.

If D is a divisor on X, define an F_q-vector space as follows:

$$L(D) = \{f \in K(X) \mid f = 0 \text{ or } \mathrm{div}(f) + D > 0\}. \tag{4.1}$$

Set $l(D) = \dim_{F_q} L(D)$. $L(D)$ can also be viewed as an F_{q^λ}-vector space, whose dimension is denoted by $l^*(D) = l(D)/\lambda$.

Lemma 4.2. Let X/F_q be a projective variety. Let D be a divisor on X. Then $L(D)$ is a finite dimensional vector space over F_q and the number of effective divisors on X linearly equivalent to D is $(q^{l(D)} - 1)/(q^\lambda - 1)$.

Proof. Let $\{f_i\}$ be a basis of the F_q-vector space $L(D)$. Every effective divisor D_1 linearly equivalent to D has the form $\mathrm{div}(\sum_i a_i f_i) + D$ with $a_i \in F_q$ not all zero. Now $\mathrm{div}(f) = \mathrm{div}(g)$ if and only if $\mathrm{div}(f/g) = 0$.

Since X is a projective variety, this can happen if and only if f/g is integral over F_q. That is, f/g is an element of F_{q^λ}. Thus, $\{a_i\}$ and $\{b_i\}$ determine the same effective divisor D_1 if and only if $b_i = ca_i$ for some nonzero element c in F_{q^λ}. Since the number of effective divisors of a given degree is finite, it follows that $l(D)$ is finite and the number of effective divisors on X linearly equivalent to D is $(q^{l(D)} - 1)/(q^\lambda - 1)$. The lemma is proved.

Lemma 4.3. Let D be a Cartier divisor (i.e., locally principal) on an n-dimensional projective variety X over F_q. Let C be an ample Cartier divisor on X over F_q. There is an integer d_0 such that for all $d \geq d_0$, $l(D + dC)$ is a polynomial in d of degree n with leading coefficient $C^n/n!$, where C^n is the self intersection number.

Proof. On an integral scheme X, there is a one-to-one correpondence between Cartier divisor classes and isomorphy classes of invertible sheaves. Lemma 4.3 then follows from the general Riemann-Roch theorem (Example 18.3.6 in [FU]) and the vanishing theorem for cohomology (see III. 5.3 in [HA2]).

Proof of Theorem 4.1. Let M_d and N_d be the number of effective divisors and prime divisors of degree d, respectively. Then,

$$Z_{n-1}(X, t) = \sum_{d=0}^{\infty} M_d t^d. \tag{4.2}$$

Since $A^1(X)$ has rank 1, an element is torsion if and only if its degree is zero. Let $D_1, \cdots, D_h$ be h mutually inequivalent divisors of degree 0. Let μ be the smallest positive integer such that there is a divisor D^* of degree μ. If μ does not divide d, then $M_d = 0$. If μ divides d, then any divisor of degree d is equivalent to $D_i + (d/\mu)D^*$ for some i. In this case,

$$M_d = \sum_{i=1}^{h} \frac{q^{l(D_i + \frac{d}{\mu}D^*)} - 1}{q^\lambda - 1}. \tag{4.3}$$

Thus, we can write

$$\begin{aligned} Z_{n-1}(X, t) &= \frac{1}{q^\lambda - 1} \sum_{d=0}^{\infty} \sum_{i=1}^{h} (q^{l(D_i + dD^*)} - 1) t^{d\mu} \\ &= \frac{1}{q^\lambda - 1} (\sum_{d=0}^{\infty} \sum_{i=1}^{h} q^{l(D_i + dD^*)} t^{d\mu} - \frac{h}{(1 - t^\mu)}). \end{aligned} \tag{4.4}$$

Let H_X be a hyperplane section on X over F_q. H_X is ample with positive degree $d(X)$. By the choice of μ, we have $\mu \mid d(X)$. Thus, H_X is linearly equivalent to $D_i + \frac{d(X)}{\mu} D^*$ for some i. Since $hD_i = 0$, it follows that hH_X is linearly equivalent to $\frac{d(X)h}{\mu} D^*$. This proves that aD^* is an ample effective Cartier divisor for some positive integer a.

Fix such an integer a and write $d = j + ak$, where $0 \leq j < a$. For each pair (i, j), $l(D_i + jD^* + kaD^*)$ is unbounded as k varies, because aD^* is a multiple of the hyperplane section and all components of $D_i + jD^*$ are components of some hypersurface section. Thus, we may assume that $l(D_i + jD^* + baD^*) > \lambda$ for some positive integer b and for all pairs (i, j). This means that there is a non-constant function f_{ij} in the vector space $L(D_i + jD^* + baD^*)$. Then, because aD^* is effective,

$$\begin{aligned} l(D_i + jD^* + (d + b)aD^*) &= l(D_i + jD^* + baD^* + daD^*) \\ &\geq l(daD^*). \end{aligned} \tag{4.5}$$

Since aD^* is an ample Cartier divisor, Lemma 4.3 shows that for large d, $l(daD^*)$ is a polynomial in d of degree n. This and (4.4) show that $Z_{n-1}(X, t)$ is a p-adic meromorphic function. That $Z_{n-1}(X, t)$ has a simple pole at $t = 1$ with the given residue follows from (4.4). The theorem is proved.

Example 4.4. There are many varieties X such that $A^1(X) = Z$ (cf. [LI]). Results of this type are called Noether theorems or Noether-Lefschetz theorems. Examples are Grassmann varieties (Igusa's analogue of the Severi theorem), nonsingular complete intersections of dimension ≥ 3 (Grothendieck's analogue of the Lefschetz theorem [GR]) and sufficiently general complete intersections of dimension 2 (Deligne's analogue of the Noether theorem, see Exposé XIX in [DK]).

Remark 4.5. It is clear that Theorem 4.1 remains true if X is a non-singular curve. Let $\mathrm{Pic}(X/F_q)$ be the group of isomorphy classes of invertible sheaves on X/F_q. This is also the group of Cartier divisor classes because X is integral. One may define the zeta function of Cartier divisors (or invertible sheaves) as an Euler product over all effective prime Cartier divisors. This is a factor of $Z_{n-1}(X,t)$. Then Theorem 4.1 remains true for the zeta function of Cartier divisors with $A^1(X)$ replaced by $\mathrm{Pic}(X)$. The proof is the same (in fact, even easier). If X is locally factorial, then $\mathrm{Pic}(X)$ is isomorphic to $A^1(X/F_q)$. It follows from Kleiman's generalization of the Lang-Néron theorem that $\mathrm{Pic}(X/F_q)$ always has finite rank (Proposition IV.4 in [KL]).

From (4.4), one sees that the Newton polygon of the numerator of $Z_{n-1}(X,t)$ is related to the Riemann-Roch problem.

5. Effective Cone and the Riemann-Roch Problem

In this section, we establish some properties of interior effective divisors, which will be used later on.

Let D be an effective divisor on an n-dimensional projective variety X defined over any field F. The Riemann-Roch problem is to determine the behaviour of $l(dD)$ as d goes to infinity. This is well understood in the case when D is ample. In that case, the general Riemann-Roch theorem and the cohomology vanishing theorem show that $l(dD)$ is a polynomial in d of degree n with leading coefficient equal to $D^n/n! > 0$, where D^n is the self-intersection number.

Zariski ([ZA2]) proved that if X is a nonsingular surface over an algebraically closed field and D is an effective divisor on X, then there are finitely many polynomials $p_1(x), \cdots, p_l(x)$ such that for all large d, $l(dD) = p_{\omega(d)}(d)$, where $\omega(d)$ is a function of d with values in the set $\{1, \cdots, l\}$. These polynomials differ at most in their constant term. It is an open question whether or not $\omega(d)$ is always periodic. In higher dimensions, no such result is known.

Here, we are interested in giving a lower bound for the Riemann-Roch problem. Let X/F be a projective variety and D an effective divisor on X. Following Zariski [ZA2], we define two rings as follows:

$$R[D] = \cup_{d=0}^{\infty} L(dD), \quad R^*[D] = \oplus_{d=0}^{\infty} L(dD).$$

$R^*[D]$ is a graded ring. If $R^*[D]$ is finitely generated over F, then $l(dD)$ (the dimension of the degree d part) is a polynomial for large d. This fails in general, even in the case of surfaces, as Zariski's counter example shows. The question of whether or not $R[D]$ is finitely generated is a generalization of Hilbert's 14th problem. Finite generation holds for surfaces (Zariski's theorem [ZA1]), but fails for three-folds (counter examples were given by Rees and later by Zariski [ZA2]).

Define the transcendence degree $t(D)$ of D to be the transcendence degree of $R[D]$. $t(D)$ takes values between 0 and n. The significance of $t(D)$ is the following simple result, whose proof is immediate.

Proposition 5.1. There is a positive constant c such that for all large d, $l(dD) \geq cd^{t(D)}$.

We now assume that $A^1(X)$ has finite rank ρ. Then, $A^1(X)/A^1(X)_{\mathrm{tor}}$ is a lattice L of full rank in the ρ-dimensional Euclidean space $A^1(X) \otimes \mathbf{R}$. Let $\Delta(X)$ be the effective cone generated by the effective divisor classes. This cone is ρ-dimensional. If σ is a face of Δ of any dimension, we say that an effective divisor D is an interior point of σ if its image in $A^1(X) \otimes \mathbf{R}$ is an interior point of σ, and we say that D is in σ if this image is in σ.

Proposition 5.2. (i). Let σ be an open face of $\Delta(X)$. Then all effective divisors in σ have the same transcendence degree. (ii). The interior face of Δ has transcendence degree n. (iii). The hyperplane section H_X is an interior point of Δ.

Proof. Let D_1 and D_2 be two effective divisors in σ and let L_1 and L_2 be their lattice points in σ. Since σ is open, for all large integers a, $aL_1 - L_2 = L(a)$ is also a lattice point in σ. Thus, for a certain positive integer d we have $adD_1 = dD_2 + D(a)$, where $D(a)$ is an effective divisor. This shows that $t(D_1) \geq t(D_2)$. By symmetry, we see that $t(D_1) = t(D_2)$, which proves (i).

Given an effective divisor D, $aH_X - D$ is an effective divisor class for all large integers a, because H_X is very ample. This proves that H_X is an interior point, and (iii) follows. Clearly, $t(H_X) = n$. Thus (ii) follows from (i). The proposition is proved.

By part (i) of this proposition, we can define the transcendental degree $t(\sigma)$ of any open face σ in Δ to be the transcendence degree of any of its lattice points. An interesting question is to determine $t(\sigma)$ for each open face σ of $\Delta(X)$. This is trivial if $\rho(X) = 1$. In general, it might be a difficult problem. For our later application, we would like to know the answers to the following much weaker questions: When is the minimum value $t(D)$ on $\Delta(X)$ positive? When is the minimum of $t(D)$ greater than 1?

Example 5.3. Let $a_1, \cdots, a_\rho$ be ρ positive integers. Let X be the product of the projective spaces $\mathbf{P}^{a_i}$. Let D_i be the inverse image in X of a hyperplane in $\mathbf{P}^{a_i}$. D_i is an effective divisor on X. It is well-known that $A^1(X)$ is the free abelian group of rank ρ generated by the D_i. Furthermore, it is not difficult to prove that the effective cone $\Delta(X)$ is also generated by the D_i. Now, it is clear that $t(D_i)$ is at least a_i (simply choose a_i algebraically independent functions on $\mathbf{P}^{a_i}$). Thus, the minimum transcendence degree on $\Delta(X)$ is at least the minimum of the a_i.

Corollary 5.4. Let D be an effective divisor on a projective variety X of dimension n. Suppose that D is an interior point of $\Delta(X)$. Then there are positive numbers $c_1 < c_2$ such that for large d we have

$$c_1 d^n \leq l(dD) \leq c_2 d^n.$$

Proof. The lower bound follows from the above two propositions. The upper bound is true because $D < aH_X$ for some hyperplane section H_X and some positive integer a.

Corollary 5.5. Let X/F_q be a projective variety. Let Δ_a be the cone generated by the ample divisors, and let Δ° be the interior cone of Δ. Then $\Delta^\circ \supset \Delta_a$.

Proposition 5.2 shows that the very ample divisors are interior points of Δ. If D is ample, then some multiple of D is very ample. The corollary follows.

A related result is a theorem of Kleiman [KL] which says that over an algebraically closed field, Δ_a is the interior of the closed pseudo ample cone in $A^1_{\text{num}}(X)$.

Remark 5.6. The inclusion in Corollary 5.5 can be strict. For example, let $f : X' \to X$ be the blowing up at a point of a nonsingular surface X. Let E be the exceptional curve on X'. Then, $A^1(X') = f^*A^1(X) \oplus \mathbf{Z}$, where $\mathbf{Z}$ is generated by E. For any effective divisor D on X which is an interior point of $\Delta(X)$, the effective divisor $f^*D + E$ on X' is an interior point of $\Delta(X')$. But $f^*D + E$ is not ample because $(f^*D + E) \cdot E = E^2 = -1$. If we consider the quadric surface defined by $xy = zw$, then it is easy to check that the interior of the effective cone coincides with the ample cone. Both of them are two dimensional open cones.

If Δ is open, Corollary 5.4 gives a good estimate for $l(dD)$ for high multiples of any effective divisor. If $\rho > 1$, $l(dD)$ does not tend to have a good uniform lower bound. This suggests that Δ should not be open. It would be interesting to know whether or not $\Delta(X)$ is closed. More precisely,

Question 5.7. Let X/F be a non-singular projective variety. Is it true that the monoid $E^1(X)$ of effective divisor classes in $\mathrm{Num}(X)$ is finitely generated?

An affirmative answer to this question would have important applications. For example, in the next section we shall prove that if the monoid of effective divisors in $A^1(X/F_q)$ is finitely generated, then Conjecture II is true for the zeta function of divisors. For another application, let us consider a projective nonsingular surface X over an algebraically closed field. Nakai's criterion says that a divisor D on X is ample if and only if $D^2 > 0$ and $D \cdot C > 0$ for all irreducible curves C in X. If, now, the monoid of effective divisor classes is finitely generated, then it is generated by a finite number of prime divisors C_i. Applying Nakai's criterion, one concludes that a divisor D is ample if and only if $D^2 > 0$ and $D \cdot C_i > 0$. This would prove that the determination of whether or not a divisor is ample can be achieved in a finite amount of time.

6. Zeta Functions of Divisors: The General Case

Our main purpose in this section is to prove that Conjecture II is true for divisors if the monoid of effective divisor classes is finitely generated. In fact, we also obtain information about the torsion part of $A^1(X)$.

Let X/F_q be a projective variety. Let $E^1(X)$ be the monoid generated by all effective prime divisors modulo linear equivalence. Equivalently, $E^1(X)$ is the monoid of effective divisor classes. $E^1(X)$ is a torsion-free monoid, because the degree of an effective divisor is positive. Let $D_1, \cdots, D_r$ be effective divisors. They are said to be Z-linearly independent if there are no non-negative integers a_i and b_j, not all zero, such that $\sum_i a_i D_i = \sum_j b_j D_j$, where $i \neq j$. The rank of $E^1(X)$ is defined to be the maximum number of Z-linearly independent elements in $E^1(X)$. It is easy to check that the rank of $E^1(X)$ is the same as that of $A^1(X)$. Even if $E^1(X)$ is finitely generated, it may not have a basis. Let $A^1(X)$ be finitely generated with rank ρ. Then $A^1(X) \otimes \mathbf{R}$ is a ρ dimensional Euclidean space. Let $\Delta(X)$ be the cone generated by the images of $E^1(X)$. Δ is a ρ dimensional cone in $A^1(X) \otimes \mathbf{R}$. $A^1(X)/A^1(X)_{\rm tor}$ (the free part) is a lattice L of full rank. That $E^1(X)$ is finitely generated means that the cone $\Delta(X)$ is closed with finitely many faces. In this case, $A^1(X)$ is a finitely generated abelian group.

To describe the special value $\lim_{t\to 1^-}(1-t)^\rho Z_{n-1}(X,t)$, we need to define the regulator $R(X)$. For each minimal parallelepiped B_i of the lattice L lying in the cone Δ with the origin as vertex, let $D_{i,j}$ $(1 \leq j \leq \rho)$ be the remaining vertices which generate the parallelepiped. Let $d_{i,j}$ be the degree of $D_{i,j}$. These are positive integers because they are in the cone Δ. A set B of such B_i is called a fundamental cover of Δ if the B_i are distinct and the cones generated by the B_i cover Δ. Since Δ is closed with finitely many faces, such a fundamental cover contains only finitely many B_i. In general, there are infinitely many fundamental covers of Δ. One can give a partial ordering on the set of all fundamental covers. Then there is an initial element with respect to this partial ordering. Choose a fundamental cover $B = \{B_i\}$ of Δ and define the regulator by

$$R(X) = \sum_i \frac{1}{\prod_{j=1}^{\rho} d_{i,j}}.$$

It can be proved directly that $R(X)$ is independent of the choice of the fundamental cover B. However, we shall not do this, because independence will automatically follow from the proof of Theorem 6.1. In particular, the regulator is well defined.

If $\Delta(X)$ is a simplicial cone (this is the case if $\rho \leq 2$), there are ρ uniquely determined lattice points P_j generating $\Delta(X)$ such that each P_j is minimal. The regulator $R(X)$ then admits the decomposition $R^*/\prod d_j$, where d_j is the degree of P_j and R^* is the number of lattice points in the parallelepiped generated by the P_j. Thus, R^* is intrinsic and independent of the embedding of X. The geometric invariant R^* can be interpreted as the volume of a fundamental domain in Δ (suitably normalized). Thus, R^* is the analogue of the classical regulator of a number field. The other factor $\prod d_j$ in R can be viewed as an analogue of the discriminant of a number field. This is an algebraic invariant, which depends on the projective embedding of X. Theorem 6.1 below shows that if $\Delta(X)$ is simplicial, we have a complete analogue of the Dedekind residue formula. On the other hand, if $\Delta(X)$ is not simplicial, it seems that the regulator R does not admit such a decomposition into a product of a geometric quantity and an algebraic quantity (they are somehow mixed together).

Now we can state our second main result.

Theorem 6.1. Let X/F_q be a projective variety. Let the monoid $E^1(X)$ be finitely generated over $Z_{\geq 0}$. Let $A^1(X)$ have rank ρ and torsion order h. Then $Z_{n-1}(X,t)$ is a p-adic meromorphic function on the closed unit disk $|\, t \,| \leq 1$ with a pole of order ρ at $t=1$. Furthermore, the special value is given by

$$\lim_{t\to 1^-}(1-t)^\rho Z_{n-1}(X,t) = -\frac{1}{q^\lambda - 1} hR(X),$$

where λ as before is the integer such that F_{q^λ} is the algebraic closure of F_q in the quotient field $K(X)$ of $S(X)$ (thus, $q^\lambda - 1$ is the number of roots of unity in $K(X)$).

Lemma 6.2. Let $l(a_1, \cdots, a_s)$ be an increasing function from $\mathbf{Z}_{\geq 0}^s$ to $\mathbf{Z}_{\geq 0}$, i.e., for all i,

$$l(a_1, \cdots, a_i, \cdots, a_s) \leq l(a_1, \cdots, a_i + 1, \cdots, a_s). \tag{6.1}$$

Define the following formal power series

$$f(t) = \sum_{a_i=0}^{\infty} q^{l(a_1,\cdots,a_s)} t^{a_1 d_1+\cdots+a_s d_s}. \tag{6.2}$$

Then $f(t)$ is p-adic meromorphic on the closed unit disk with a pole of order at most s at $t=1$. The order equals s if and only if the function $l(a)$ is bounded, in which case $f(t)$ is a rational function.

Proof. We first assume that $l(a)$ is bounded with maximum value k. Let $l(b_1,\cdots,b_s)=k$. Because $l(a)$ is increasing, we have

$$f(t) = q^k \frac{t^{b_1 d_1}\cdots t^{b_s d_s}}{(1-t^{d_1})\cdots(1-t^{d_s})} + \sum{}' q^{l(a_1,\cdots,a_s)} t^{\sum a_i d_i}, \tag{6.3}$$

where the summation $\sum'$ runs over all nonnegative integers a_i with at least one of the a_i smaller than b_i. This reduces us to a finite number of summations over $s-1$ copies of $Z_{\geq 0}$. By induction, the second term of (6.3) is a rational function with a pole of order at most $s-1$. Thus, $f(t)$ is a rational function with a pole of order s at $t=1$.

Next, we assume that $l(a)$ is unbounded. It suffices to prove that for any high power q^h of q, the reduction of $f(t)$ modulo q^h is a meromorphic function on the closed unit disk with a pole of order at most $s-1$ at $t=1$. Fix a positive integer h and let $l(b_1,\cdots,b_r)\geq h$ for some integers b_i. Since $l(a)$ is an increasing function in the variables a_i, it follows that

$$f(t) \equiv \sum{}' q^{l(a_1,\cdots,a_s)} t^{\sum_i a_i d_i} \ (\mathrm{mod} q^h), \tag{6.4}$$

where the summation $\sum'$ runs over all nonnegative integers a_i with at least one of the a_i smaller than b_i. This reduces $f(t)$ to a finite number of summations over $s-1$ copies of $Z_{\geq 0}$. By induction, each of the summations is meromorphic on the closed unit disk with a pole of order at most $s-1$. The proposition is proved.

In certain cases, a better meromorphic continuation of $f(t)$ can be obtained. We need the following simple lemma, whose proof is obvious.

Lemma 6.3. In the notation of Lemma 6.2, assume $l(ke_i)\geq ck$ for some constant c and all large k, where e_i is the vector a with $a_i=1$ and $a_j=0$ for $j\neq i$. Then $f(t)$ is p-adic analytic in a disk $|\, t\,| < 1+\epsilon$. If we assume that $\lim_{k\to\infty} l(ke_i)/k=\infty$, then $f(t)$ is a p-adic entire function.

Proof of Theorem 6.1. Fix a decomposition of the abelian group $A^1(X)=L\oplus T$, where L is a free abelian group of rank ρ containing the hyperplane section H_X and T is the torsion group of order h with elements $T_1=0,\cdots,T_h$. The degree of T_i is zero. Consider the ρ-dimensional vector space $L\otimes \mathbf{R}=\mathbf{R}^\rho$, in which L is a lattice of full rank. A lattice point is effective if the divisor class representing it is effective. The set of all linear combinations of effective lattice points with nonnegative real coefficients forms a ρ-dimensional closed cone Δ with vertex at the origin. Because $E^1(X)$ is a finitely generated monoid, Δ has finitely many faces. Note that there may be lattice points in Δ which are not effective. However, such lattice points can be written as linear combinations of effective lattice points with non-negative rational coefficients; thus, some multiple of such a lattice point will be an effective lattice point. Any effective divisor is linearly equivalent to $P+T_j$ for some j and a lattice point P in Δ.

Let H_X be the lattice point corresponding to the hyperplane section on X. Let L_X be the half line emanating from the origin and passing through H_X. We fix a simplicial decomposition of Δ. This means that we decompose Δ into a finite union of closed ρ-dimensional simplicial cones Δ_i such that $\Delta_i\cap\Delta_j$ $(i\neq j)$ has dimension less than ρ. By the definition of the zeta function, Lemma 4.3 and the inclusion-exclusion principle, we have

$$Z_{n-1}(X,t) = \sum_{j=1}^{h}\sum_{P\in\Delta} \frac{q^{l(T_j+P)}-1}{q^\lambda-1} t^{\deg(P)},$$

$$= \sum_{j=1}^{h} \sum_{i} \sum_{P \in \Delta_i} \frac{q^{l(T_j+P)} - 1}{q^{\lambda} - 1} t^{\deg(P)} + R(t), \tag{6.5}$$

$$= \sum_{j=1}^{h} \sum_{i} G_{j,i}(t) + R(t),$$

where P denotes a lattice point in Δ and $R(t)$ is a finite alternating sum of power series similar to the $G_{j,i}(t)$ with summation over cones of smaller dimension.

We claim that each $G_{j,i}(t)$ is p-adic meromorphic on the closed unit disk with a pole of order ρ at $t = 1$ such that the value of $(1-t)^{\rho} G_i(t)$ at $t = 1$ is a positive rational number. The same proof will also show that $R(t)$ is p-adic meromorphic on the closed unit disk with a pole of order at most $\rho - 1$ at $t = 1$. This is sufficient to prove the first part of the theorem.

Without loss of generality, we consider $G_{1,1}(t)$. Let M_1 be the monoid of lattice points in Δ_1. Let $\{D_1, \cdots, D\rho\}$ be ρ effective lattice points of Δ_1 which span the cone Δ_1. Thus, the D_i are actually effective divisors (if this is not true of the D_i, then replace them by the multiples hD_i). Let M_1' be the monoid generated by the D_i. Then the quotient monoid M_1/M_1' is finite. Choose representatives E_k such that each lattice point of Δ_1 can be uniquely expressed as $E_k + P$ for some k and $P \in M_1'$. Let $d_i = \deg D_i$. Then to prove the claim it is sufficient to prove it for the following power series:

$$\begin{aligned}(q^{\lambda} - 1)G(t) &= \sum_{a_i=0}^{\infty} (q^{l(T_1+E+\sum_i a_i D_i)} - 1) t^{e+\sum_i a_i d_i} \\ &= \sum_{a_i=0}^{\infty} q^{l(T_1+E+\sum_i a_i D_i)} t^{e+\sum_i a_i d_i} - \sum_{a_i=0}^{\infty} t^{e+\sum_i a_i d_i} \\ &= F_1(t) - F_2(t),\end{aligned} \tag{6.6}$$

where E is one of the E_k and $e = \deg(E)$.

One computes that

$$F_2(t) = \frac{t^e}{(1-t^{d_1}) \cdots (1-t^{d_{\rho}})}. \tag{6.7}$$

This is a rational function with a pole of order ρ at $t = 1$ such that the value of $(1-t)^{\rho} F_2(t)$ at $t = 1$ equals $1/\prod_i d_i$, which is a positive rational number. To prove the claim for $G(t)$, it suffices to prove that $F_1(t)$ is meromorphic on the closed unit disk with a pole of order less than ρ at $t = 1$.

Because the D_i are effective, $l(T_1 + E + \sum_i a_i D_i)$ is an increasing function in the a_i. Since $\sum_i D_i$ is an interior point of Δ, Proposition 5.2 shows that there is a positive integer b such that $T_1 + E + b\sum_i D_i$ is an effective divisor class. Corollary 5.4 then implies that $l(T_1 + E + \sum_i a_i D_i)$ is unbounded. The desired conclusion about $F_1(t)$ follows from Lemma 6.2. This proves that $Z_{n-1}(X, t)$ is p-adic meromorphic on the closed unit disk with a pole of order ρ at $t = 1$.

From (6.5)-(6.7) and the above proof, one sees that the special value is given by $-\frac{1}{q^{\lambda}-1} hR'(X)$, where

$$R'(X) = \sum_{i} \frac{\mid M_i/M_i' \mid}{d_{i,1} \cdots d_{i,\rho}},$$

M_i is the monoid of lattice points in Δ_i and M_i' is a submonoid of M_i generated by ρ effective lattice points of degree $d_{i,j}$ such that $\mid M_i/M_i' \mid$ is finite. The above proof also shows that $R'(X)$ is the special value of the following rational function

$$Z(\Delta, t) = \sum_{P \in \Delta} t^{\deg P}.$$

Thus, $R'(X)$ is of a combinatorial nature, and it is independent of the choice of simplicial decomposition of Δ. Using a finer simplicial decomposition of Δ, if necessary, we may assume that we can choose $M_i' = M_i$. Such a fine simplicial decomposition corresponds uniquely to a fundamental cover B of Δ. Thus, $R'(X) = R(X)$.

Because the special value is independent of the choice of simplicial decomposition, it follows that the regulator is well defined. The theorem is proved.

Example 6.4. Let $\pi : X \longrightarrow C$ be a ruled surface over a nonsingular curve C with invariant $e \geq 0$ (see Proposition V.2.20 in [HA2]). Let $C_0 \subset X$ be a section and let f be a fibre. Then $A^1(X) = \mathbf{Z} \oplus \pi^* A^1(C)$ is finitely generated of rank 2. Furthermore, the effective cone $\Delta(X)$ is the two dimensional closed simplicial cone spanned by C_0 and f, because the intersection class of any irreducible curve on X is contained in the cone spanned by C_0 and f. Thus Theorem 4.1 holds for a ruled surface with invariant $e \geq 0$. Note that the ruled surface $C \times \mathbf{P}^1$ has invariant $e = 0$.

Assuming further conditions on the Riemann-Roch problem, we have the following stronger meromorphic continuation.

Theorem 6.5. Let X/F_q be a projective variety. Let $E^1(X)$ be finitely generated over $Z_{\geq 0}$ with a set of generators D_i. Suppose that for each i, $l(bD_i)$ is unbounded. Then $Z_{n-1}(X,t)$ is p-adic meromorphic in a disk $\mid t \mid < 1 + \epsilon$.

Proof. Since $l(bD_i)$ is unbounded, there are nonconstant functions $f_i \in L(D_i)$. Using the powers of the f_i, we see that $l(bD_i) \geq ck$ for some positive number c. This shows that $l(kD) \geq ck$ for large k, where D is any effective divisor. The theorem follows from Lemma 6.3 and the proof of Theorem 6.1.

Similarly, using Lemma 6.3, one can prove

Theorem 6.6. Let X/F_q be a projective variety. Assume that $E^1(X)$ is finitely generated over $Z_{\geq 0}$ with a set of generators D_i. Suppose that for each i, $l(kD_i)/k$ is unbounded. Then $Z_{n-1}(X,t)$ is p-adic meromorphic.

Examples 6.7. Let X be the nonsingular quadric surface defined by $xy = zw$ in $\mathbf{P}^3$. Then it is well known that $A^1(X) = Z \oplus Z$ is generated by D_{xw} and D_{yw}, where D_{xw} is the prime divisor defined by $x = w = 0$. Furthermore, all effective divisors are of the form $aD_{xw} + bD_{yw}$ with nonnegative integral coefficients. Thus, $E^1(X)$ is generated by D_{xw} and D_{yw} of rank 2. One computes that

$$\mathrm{div}(\frac{z}{x}) = D_{zy} - D_{xw} > -D_{xw}$$
$$\mathrm{div}(\frac{z}{y}) = D_{zx} - D_{yw} > -D_{yw}.$$

Thus, the conditions of Theorem 6.5 are satisfied with $\rho = 2$. Another example of Theorem 6.5 is any abelian variety X such that $E^1(X)$ is finitely generated. Over an abelian variety X, $l(kD)$ is always unbounded for any effective divisor D, see [MU, Chapter II].

The quadric surface X is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. More generally, let $a_1, \cdots, a_\rho$ be ρ positive integers. Let X be the product of the projective spaces $\mathbf{P}^{a_i}$. By Example 5.4, the monoid $E^1(X)$ of effective divisor classes is finitely generated of rank ρ. The minimum transcendence degree of the effective cone $\Delta(X)$ is at least the minimum of the a_i, which is positive. Thus X satisies the conditions of Theorem 6.5. If all of the a_i are greater than 1, then X satisfies the conditions of Theorem 6.6. If some of the a_i is 1, the conclusion of Theorem 6.6 also holds (by a direct computation similar to section 2). Another example of Theorem 6.6 is the product $X = \prod_{i=1}^{\rho} X_i$, where each X_i is a non-singular complete intersection of dimension at least 3. Grothendieck's theorem implies that $E^1(X)$ is the free monoid of rank ρ.

If X is n-dimensional with $n \geq 2$ and $A^1(X)$ is finitely generated of rank 1, then the minimum weight of $\Delta(X)$ is n. Thus X satisfies the conditions of Theorem 6.6. This shows that Theorem 6.6 is a generalization of Theorem 4.1.

7. Connections with the Higher Codimensional Riemann-Roch Problem

In this last section, we discuss possible approaches which might lead to a proof (at least in certain cases) of the conjectures formulated in this paper. The Riemann-Roch approach used in sections 4 and 5 naturally suggests a higher codimensional Riemann-Roch problem. We show how information about the higher codimensional Riemann-Roch problem can be used to prove Conjecture II.

We start with the various rationality proofs for the zeta function of zero cycles on a curve. Recall that there are three quite independent rationality proofs for the classical zeta function of a smooth curve over a finite field. The first proof, due to Weil, is to use the Riemann-Roch theorem for curves. The second type of proof is p-adic in nature (Dwork, Washnitzer-Monsky, Lubkin, Grothendieck-Berthelot). The third is the l-adic proof (Grothendieck). The last two approaches are cohomological in nature.

For zeta functions of zero cycles in higher dimensions, both the p-adic and the l-adic approaches generalize and work very well. Each approach has its advantages. For example, the l-adic approach has been successfully used to prove the Riemann hypothesis. However, to study p-adic analytic variation one has to use p-adic methods. In the general case, because $Z_r(X,t)$ is expected to be a non-rational p-adic meromorphic function for $0<r<n$, it is almost certain that the l-adic approach would not work for these more general zeta functions. However, we do not know any consideration that would exclude the p-adic approach. Thus, one might hope that $Z_r(X,t)$ can be expressed in terms of the Fredholm determinants of certain nuclear operators (generalizing the Frobenius operators) on an infinite dimensional p-adic Banach space. For this to work, one needs a trace formula of Dwork type. This would generalize Dwork's proof and establish the p-adic meromorphic continuation of $Z_r(X,t)$. One might also hope that there is a p-adic cohomology theory (of Dwork type or some other type) and a fixed point theorem which completely describe all the zeta functions, as in the case of zero cycles. Such a theory, if it exists, is necessarily infinite dimensional. This suggests that in studying the meromorphic continuation of zeta functions of r-cycles for $0<r<n$, one would expect much more complexity than in the case of zero cycles (because of the difficulties of extension from rational functions to meromorphic functions or from finite dimensional vector spaces to infinite dimensional vector spaces).

It is mysterious why the Riemann-Roch approach does not generalize to higher dimensions, even in the case of zero cycles. In section 4, we showed how the general Riemann-Roch theorem can be successfully applied to study zeta functions of divisors in certain cases. Motivated by this, we now discuss a generalization of the Riemann-Roch approach to zeta functions of all cycles, not just zero cycles and divisors. This involves a generalization of the Riemann-Roch problem from divisors to arbitrary cycles.

Let X/F_q be an n-dimensional projective variety in a fixed projective space. For any $0 \leq r \leq n$, write

$$Z_r(X,t) = \sum_{d=0}^{\infty} M_d t^d, \tag{7.1}$$

where M_d is the number of effective r-cycles of degree d on X. Let $E_r(X)$ be the monoid of effective r-cycles modulo rational equivalence.

First, we assume that the monoid $E_r(X)$ is finitely generated. Then one can construct the cone $\Delta_r(X)$ with vertex at the origin that is generated by the effective r-cycle classes. Δ_r is closed and has finitely many faces. A lattice is given by $A_r(X)$ modulo torsion elements. By a simplicial decomposition and a finite quotient argument, as in the proof of Theorem 6.1, we are reduced to the power series

$$G_r(X,t) = \sum_{a_i=0}^{\infty} M(a_1,\cdots,a_s) t^{a_1 d_1 + \cdots + a_s d_s}, \tag{7.2}$$

where $M(a_1,\cdots,a_s)$ is the number of effective r-cycles rationally equivalent to $C_0 + a_1 C_1 + \cdots + a_s C_s$, C_0 is an r-cycle, the C_i $(i \geq 1)$ are effective r-cycles which are independent (as lattice points) and d_i is the degree of C_i.

Let X be a projective variety over a field F. Let $CH_r(d,X)$ be the Chow scheme, which is an algebraic set parametrizing effective r-cycles over F of degree d on X. Given an r-cycle C of degree d on X, let $L(C)$ be the set of effective r-cycles on X rationally equivalent to C. $L(C)$ is either empty or a connected algebraic subset of $CH_r(d,X)$ with the property that any two points of $L(C)$ can be joined by a chain of rational curves. Thus, $L(C)$ is in some sense almost a linear variety, as in the case of divisors. The simple geometric nature of $L(C)$ suggests that over a finite field $F=F_q$ it should be possible to compute or p-adically estimate the number of F_q-rational points $N(C)$ on $L(C)$, and, in particular, determine the behaviour of the function $N(C_0 + d_1 C_1 + \cdots + d_s C_s)$, where C_0 is an r-cycle and the C_i $(i \geq 1)$ are rationally independent effective r-cycles.

In the case when $C=D$ is a divisor, $L(D)$ is a vector space of dimension $l(D)$; and if $F=F_q$, then $N(D) = (q^{l(D)}-1)/(q^{\lambda}-1)$, where λ is a positive integer independent of D. The general Riemann-Roch problem is to determine the function $l(D_0 + d_1 D_1 + \cdots + d_s D_s)$, where D_0 is a divisor and the D_i $(i \geq 1)$ are linearly independent effective divisors. This is equivalent to determining the behaviour of the varieties $L(D_0 + d_1 D_1 + \cdots + d_s D_s)$ as the d_i vary.

In higher codimension, the general Riemann-Roch problem is then to determine the behaviour of the algebraic sets $L(C_0 + d_1 C_1 + \cdots + d_s C_s)$ as the d_i vary, where C_0 is an r-cycle and the C_i $(i \geq 1)$ are rationally independent effective r-cycles. Over a finite field, this means determining the behaviour of the function $N(C_0 + d_1 C_1 + \cdots + d_s D_s)$. The higher codimensional Riemann-Roch problem can be expected to be a fruitful problem in algebraic geometry.

Returning to zeta functions of algebraic cycles, we say that X has the property (P_r) if for all rationally independent effective r-cycles C_i and any r-cycle C_0, there is a constant c_r such that $N(C_0 + \sum d_i C_i)$ either is constant or else is congruent to c_r modulo any power q^h as long as all the d_i are sufficiently large depending on h. Note that $N(\sum d_i C_i)$ is an increasing function of non-negative integers. In the case of divisors, this is always true, where one simply takes $c_{n-1} = -1/(q^\lambda - 1)$.

Imitating the proof of Theorem 6.1, we can prove the following generalization for zeta functions of r-cycles.

Theorem 7.1. Let X/F_q be an n-dimensional projective variety. Let r be an integer with $0 \leq r \leq n$. Assume that X/F_q has property (P_r) and $E_r(X)$ is a finitely generated monoid. Let ρ_r and h_r be the rank and torsion order of $A^r(X)$. Then the zeta function $Z_r(X, t)$ of r-cycles is p-adic meromorphic on the closed unit disk with a pole of order ρ_r at $t = 1$. Its special value at $t = 1$ is given by the formula $c_r h_r R_r(X)$, where $R_r(X)$ is the regulator defined in a manner similar to case of divisors.

Thus, the two major problems left are the finiteness of the number of generators for the monoid $E_r(X)$ and a determination (or good p-adic estimate) of the function $N(C_0 + d_1 C_1 + \cdots + d_s C_s)$. This approach, if it works, would give more information about the zeta function of zero cycles than the p-adic approach or l-adic approach because it also gives the special value at $t = 1$, which involves the torsion order and the regulator.

To see an example, let us consider the zeta function of zero cycles on a projective space $X = \mathbf{P}^n$. In this case,

$$Z_0(\mathbf{P}^n, t) = \frac{1}{(1-t)(1-qt)\cdots(1-q^n t)}.$$

$E_0(X)$ is generated by a point. Thus, $A_0(X)$ has rank 1 and torsion order 1. The regulator $R_0(X)$ is easily seen to be 1. The Riemann-Roch problem for zero cycles is then to compute the number M_d of effective zero cycles of degree d, which is simply the coefficient of t^d in $Z_0(\mathbf{P}^n, t)$. One checks that for large d, M_d is congruent to $c_0 = 1/(1-q)\cdots(1-q^n)$. Thus, both the condition and the conclusion of Theorem 7.1 hold.

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