On the p-adic Riemann Hypothesis for the Zeta function of Divisors

Daqing Wan *
Institute of Mathematics, Chinese Academy of Sciences
Beijing, P.R. China

Department of Mathematics, University of California
Irvine, CA 92697-3875
dwan@math.uci.edu

C. Douglas Haessig †
Department of Mathematics, University of California
Irvine, CA 92697-3875
chaessig@math.uci.edu

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1 Introduction

In this paper, we continue the investigation of the zeta function of divisors introduced in [6] for a projective variety X defined over a finite field $\mathbb{F}_q$ of q elements of characteristic p. Assume that the effective classes in the divisor class group is a finitely generated monoid, then there are four standard conjectures about this zeta function: p-adic meromorphic continuation, rank and order relation, p-adic Riemann hypothesis, and the simplicty of zeros. When the divisor class group is of rank one, the first two conjectures were proved in [6]. The aim of this paper is to prove the remaining two conjectures under the same assumption. We shall also give an example which provides evidence for the validity of all the four conjectures in the higher rank case. As initiated in [6], we derive the p-adic meromorphic continuation of these zeta functions via a Riemann-Roch approach. After this, we use the theory of Newton polygons to investigate the zeros, specifically the p-adic Riemann hypothesis and the simplicity conjecture.

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Let us start by recalling how analytic information about a zeta function leads to arithmetic information. For example, if N_k denotes the number of $\mathbb{F}_{q^k}$-rational points on X, then we know from Dwork's theorem that the associated zeta function is rational:

$$Z_0(X,T) = exp\left(\sum_{k=1}^{\infty} N_k T^k/k\right) = \frac{\prod(1-\alpha_i T)}{\prod(1-\beta_j T)}.$$

Taking the logarithmic derivative of both sides and equating coefficients, we immediately obtain a formula for N_k in terms of the (reciprocal) zeros and poles of the zeta function:

$$N_k = \sum \beta_j^k - \sum \alpha_i^k.$$

For example, for a curve of genus g defined over $\mathbb{F}_q$, we have $N_k = q^k + 1 - \sum_{i=1}^{2g} \alpha_i^k$ where 1 and q are the (reciprocal) poles of the zeta function and the α_i are the (reciprocal) zeros.

Thus rationality, and more generally, meromorphic continuation implies a nice formula (in terms of the zeros and poles) for the number of objects the zeta function is counting. This is a general principal. For another example, the meromorphic continuation of the Riemann zeta function means there is a formula, in terms of zeros and poles, for the number of primes in an interval $[0,t]$, which led to the prime number theorem. Similarly, the conjectural meromorphic continuation of the Hasse-Weil zeta function would yield a formula, in terms of zeros and poles, for the number of closed points on an arithmetic scheme with norm at most t, see [9].

The zeta function of algebraic r-cycles introduced in [6] is concerned with a formula for the number of irreducible subvarieties of a fixed dimension r. This zeta function will not be complex analytic in general, unlike the above classical zeta functions. However, the conjectural p-adic meromorphic continuation immediately yields a p-adic formula in terms of the p-adic zeros and poles of this zeta function as we shall see below.

Let $\mathbb{F}_q$ be a finite field with q elements, q a power of a prime p. Let X be a projective n-dimensional integral scheme defined over $\mathbb{F}_q$. Let $0 \leq r \leq n$ be integers. A prime r-cycle of X is an r-dimensional closed integral subscheme of X defined over $\mathbb{F}_q$. An r-cycle on X is a formal finite linear combination of prime r-cycles. An r-cycle is called effective, denoted $\sum n_i P_i \geq 0$, if each $n_i \geq 0$.

Each prime r-cycle P has an associated graded coordinate ring $\bigoplus_{k=0}^{\infty} S_k(P)$ since X is projective. By a theorem of Hilbert-Serre, for all k sufficiently large, we have $dim_{\mathbb{F}_q} S_k(P)$ equal to a polynomial $a_r k^r +$ (lower terms). Define the degree of P, denoted $deg(P)$, as $r!$ times the leading coefficient a_r. We extend the definition of degree to arbitrary r-cycles by $\deg(\sum n_i P_i) := \sum n_i \deg(P_i)$.

Defining the degree allows us to measure and compare the prime r-cycles. Define the zeta function of algebraic r-cycles on X as

$$Z_r(X,T) := \prod_P (1 - T^{\deg(P)})^{-1}.$$

where the product is taken over all prime r-cycles P in X.

Denote the set of all effective r-cycles of degree d on X by $E_{r,d}(X)$. A theorem of Chow and van der Waerden states that this set has the structure of a projective variety. Since we are over a finite field, $E_{r,d}(X)$ is finite. This means that $Z_r(X,T)$ is a well-defined element of $1+T\mathbb{Z}[[T]]$, and so, converges p-adically in the open unit disk $|T|_p < 1$.

Equivalent forms of this zeta function are

$$\begin{aligned} Z_r(X,T) &= \sum_{d=0}^{\infty} \#E_{r,d}(X)T^d \\ &= \prod_{d=0}^{\infty}(1-T^d)^{-N_d} \\ &= \exp(\sum_{k=1}^{\infty}\frac{T^k}{k}W_k), \end{aligned}$$

where N_d is the number of prime r-cycles of degree d and $W_k := \sum_{d|k} dN_d$ is the weighted number of prime r-cycles of degree dividing k, each prime r-cycle of degree d is counted d times. The p-adic meromorphic continuation of $Z_r(X,T)$ would imply the complete p-adic factorization

$$Z_r(X,T) = \frac{\prod(1-\alpha_i T)}{\prod(1-\beta_j T)}$$

where the products are now infinite with $\alpha_i \to 0$ and $\beta_j \to 0$ in $\mathbb{C}_p$. Again, taking the logarithmic derivative, we obtain a formula for W_d in terms of infinite series:

$$W_d = \sum \beta_j^d - \sum \alpha_i^d.$$

Thus, the p-adic meromorphic continuation implies a well structured formula for the sequence W_k. By Möbius inversion, this gives a well structured formula for the sequence N_k as well.

If $Z_r(X,T)$ is p-adic meromorphic, we can adjoin all the reciprocal zeros α_i's and all the reciprocal poles β_j's to $\mathbb{Q}_p$. The resulting field extension of $\mathbb{Q}_p$ is called the **splitting field** of $Z_r(X,T)$ over $\mathbb{Q}_p$. This splitting field is automatically a Galois extension (possibly of infinite degree) over $\mathbb{Q}_p$ by the Weierstrass factorization of $Z_r(X,T)$ over $\mathbb{Q}_p$ and the fact that we are in characteristic zero.

Now, if $r=0$, then $Z_r(X,T) \in \mathbb{Q}(T)$ is rational and it satisfies the Riemann hypothesis by the Weil Conjectures. If $r = n = dim(X)$, then $Z_n(X,T)$ is trivially rational, its zeros and poles are roots of unity, and thus satisfies the Riemann hypothesis as well. In particular, $Z_r(X,T)$ is well understood if $n \leq 1$. So, we will assume $n = dim(X) \geq 2$ and $1 \leq r \leq n-1$.

Let $CH_r(X)$ be the Chow group of r-cycles on X; that is, the free abelian group generated by the prime r-cycles on X modulo the rational equivalence. Let $\text{EffCone}_r(X)$ be the set of effective r-cycle classes in $CH_r(X)$. It is conjectured that $CH_r(X)$ is a finitely generated abelian group, note that our base field is a finite field. This is known if $r = n-1$. In general, $\mathrm{EffCone}_r(X)$ may not be a finitely generated monoid. The following conjectures only apply to those X for which $\mathrm{EffCone}_r(X)$ is a finitely generated monoid. It is an interesting but independent question to determine when $\mathrm{EffCone}_r(X)$ is a finitely generated monoid.

Assumption Assume that $\mathrm{EffCone}_r(X)$ is a finitely generated monoid. Then, we have the following conjectures.

Conjecture I (p-adic Meromorphic Continuation)
$Z_r(X,T)$ is p-adic meromorphic in T,

Conjecture II (Order and Rank)
$-\mathrm{ord}_{T=1}(Z_r(X,T)) = \mathrm{rank}\ CH_r(X)$,

Conjecture III (p-adic Riemann Hypothesis)
The splitting field of $Z_r(X,T)$ over $\mathbb{Q}_p$ is a finite extension of $\mathbb{Q}_p$,

Conjecture IV (Simplicity of Zeros and Poles)
All zeros and poles, except for finitely many, are simple.

A slightly weaker version of the p-adic Riemann hypothesis says that the slopes of the zeros and poles are rational numbers with bounded denominator, that is, the splitting field of $Z_r(X,T)$ over $\mathbb{Q}_p$ has finite ramification degree. For a characteristic p analogue of this type of Riemann hypothesis, see [7] and Goss [2]. The above stronger formulation of the p-adic Riemann hypothesis is motivated by Goss' corresponding formulation for the characteristic p L-function. Note that in the characteristic p case, the infinite primes need to be taken into account, see [3] for the new reformulation. In the present p-adic case, the infinite primes do not enter into the picture, and so the original formulation in [2] does seem to be the right one to look at.

As we shall see in this paper, some positive results supporting the above conjectures come from the case when $r = dim(X) - 1$, the divisor case. We call $Z_{dim(X)-1}(X,T)$ the zeta function of divisors. We have

Theorem 1. *Let X be a normal, connected scheme over $\mathbb{F}_q$ of dimension $n \geq 2$. Assume the rank of $CH_{n-1}(X)$ is one. Then all the above four conjectures are true for $Z_{n-1}(X,T)$.*

This extends Theorem 4.1 in [6] which had proven Conjectures I and II under the same rank one assumption. Evidence to support Conjecture I and II in the higher rank case is also given in [6] Theorem 6.1. This states that when the effective cone is a finitely generated monoid, then the zeta function of divisors is p-adic meromorphic on the *closed* unit disk and satisfies Conjecture II. In this paper, we also give an example which satisfies for all four conjectures, in the higher rank case. This is the first higher rank example for which Conjecture III and IV are proved. We have

Theorem 2. *Consider the zeta function of divisors for the quadric surface $xw = zy$ in $\mathbb{P}^3_{\mathbb{F}_q}$. Then all four conjectures hold, except possibly the simplicity conjecture in the case $p = 2$.*

The method used to investigate the zeta function of algebraic cycles comes from rewriting the zeta function in terms of cycle classes. We then need to estimate the number of effective cycles in a cycle class. In the divisor case, these are linear equivalence classes, and counting effective divisors in a linear equivalence class comes down to knowing the dimension of a certain vector space. This dimension is best studied by seeing how the dimension varies within a family. This family comes from a Riemann-Roch approach which (informally) asks 'how does the dimension vary as the linear equivalence classes vary'. See section 2 for a more precise description of the method.

Finally, we use the theory of Newton polygons to study the p-adic Riemann hypothesis and the simplicity of zeros. See section 3 for a description of this connection.

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2 The generalized Riemann-Roch problem

A general strategy to investigate the zeta function of divisors comes from a generalized Riemann-Roch problem. In this section, this strategy is recalled and the generalized Riemann-Roch problem is easily solved for the rank one case. In the case when we are not dealing with codimension one cycles a similar method has been proposed in [6] section 7, but essentially nothing could be proved.

Let X be a normal, connected scheme over $\mathbb{F}_q$ of dimension $n \geq 2$. Let $CH_{n-1}(X)$ denote the Chow group of $(n-1)$-cycles and define the effective cone, denoted $\mathrm{EffCone}(X)$, as the effective divisor classes in $CH_{n-1}(X)$. The effective cone helps in determining properties of the zeta function by gathering divisors of the same degree; recall, the degree depends only on the divisor class. That is, we may rewrite the zeta function of divisors as

$$Z_{n-1}(X,T) = \sum_{D \in \mathrm{EffCone}(X)} M_D T^{\deg(D)}$$

where M_D is the number of effective divisors linearly equivalent to D. The numbers M_D are studied using the following $\mathbb{F}_q$-vector space defined as follows. Denote the function field of X by $\mathbb{F}_q(X)$. Let D be a divisor on X. Define

$$L(D) := \{f \in \mathbb{F}_q(X) | f = 0 \text{ or } \mathrm{div}(f) + D \geq 0\}.$$

Note, $L(D)$ is a finite dimensional $\mathbb{F}_q$-vector space; we will denote its dimension by $l(D)$.

Proposition 1. *Consider the projective space obtained from $L(D)\setminus\{0\}$ modulo the equivalence: $f \sim g$ if there is a $\lambda \in \mathbb{F}_q^*$ so that $f = \lambda g$. This space is in one-to-one correspondence with the set M_D.*

See [6] Lemma 4.2 for a proof. Thus, we may conclude that

$$Z_{n-1}(X,T) = \sum_{D\in \mathrm{EffCone}(X)} \frac{q^{l(D)}-1}{q-1} T^{\deg(D)}.$$

As demonstrated in Section 6 of [6], when the effective cone is finitely generated, by using a simplicial decomposition Δ of the effective cone, one may reduce the zeta function to functions of the form

$$Z_\Delta(X,T) = \sum_{a_1,\ldots,a_t\geq 0} \frac{q^{l(E+a_1D_1+\cdots+a_tD_t)}-1}{q-1} T^{\deg(E+a_1D_1+\cdots+a_tD_t)}.$$

This decomposition focuses our attention on the behaviour of $l(E + a_1D_1 + \cdots + a_tD_t)$ as $a_1 + \cdots + a_t$ tends to infinity. Simply understanding $l(kD)$ as k tends to infinity is difficult; this is known as the Riemann-Roch problem. See Hartshorne [4], Problem II.7.6. Zariski worked on this problem for surfaces in [10]; also see Cutkosky and Srinivas [1]. We state this as a general question which may be considered a generalized Riemann-Roch problem:

Question. For effective divisors E and $D_1,\ldots,D_t$, what is the behaviour of $l(E + a_1D_1 + \cdots + a_tD_t)$ as $a_1 + \cdots + a_t$ tends to infinity?

We will need to understand this behaviour when $CH_{n-1}(X)$ is of rank one. In this case, every divisor of positive degree is ample and the following proposition provides us with the needed understanding.

Proposition 2. *Let D be an ample divisor and D' an arbitrary divisor. Then for all k sufficiently large, $l(D'+kD)$ is a polynomial in k of degree $n = dim(X)$, with leading coefficient $D^n/n!$ where D^n is the self-intersection number of D.*

Proof. From Riemann-Roch, we know

$$\sum_{i\geq 0}(-1)^i \dim\, H^i(X, D'+kD) = \frac{D^n}{n!}k^n + (\text{lower terms in } k).$$

Since D is ample, $H^i(X, D' + kD) = 0$ for all $i \geq 1$ and k sufficiently large. The proposition follows since $H^0(X, D' + kD) = L(D' + kD)$. □

3 p-adic Riemann Hypothesis

In this section, we discuss a method used to investigate the p-adic Riemann Hypothesis using Newton polygons.

Once we know that $Z_r(X,T)$ is p-adic meromorphic, we may write it as a quotient of p-adic entire functions, that is,

$$Z_r(X,T) = \frac{\sum_{i=0}^{\infty} a_i T^i}{\sum_{j=0}^{\infty} b_j T^j}$$

with $a_i, b_j \in \mathbb{Z}_p$ and $a_0 = b_0 = 1$. The Newton polygon of $\sum_{i\geq 0} a_i T^i$ is the convex hull of the points $(i, ord_q(a_i))$. Since this series defines an entire function, the Newton polygon grows faster than any linear function; in particular, there are no infinitely long line segments. Newton polygons are exciting because they encode arithmetic information about the zeros of the series $\sum a_i T^i$. Specifically, if there is a line segment of horizontal length h and slope s, then there will be exactly h reciprocal zeros of this series having q-adic order s.

Now, if the horizontal lengths of the line segments creating the Newton polygon are bounded (in particular, the slopes have bounded denominator), then by the p-adic Weierstrass preparation theorem, we may write

$$\sum_{i\geq 0} a_i T^i = \prod_j f_j(T)$$

where each $f_j \in 1 + T\mathbb{Z}_p[T]$, $deg(f_j)$ is uniformly bounded and the reciprocal zeros of $f_j(T)$ approch to zero as j grows.

Let us now relate this back to the p-adic Riemann hypothesis. Recall, the p-adic Riemann hypothesis states that the splitting field of $\sum_{i\geq 0} a_i T^i$ over $\mathbb{Q}_p$ is a finite extension of $\mathbb{Q}_p$. This finitness is a consequence of the above bounded degree factorization and the following finitness lemma.

Lemma 1. *Let d be a positive integer. Let $E(d)$ denote the set of all $f \in \mathbb{Q}_p[T]$ with $deg(f) \leq d$. Let K be the field extension obtained by adjoining to $\mathbb{Q}_p$ every zero of every $f \in E(d)$. Then $[K : \mathbb{Q}_p] < \infty$.*

The proof of the lemma follows easily from the fact that for each positive integer n, there are only finitely many extensions of $\mathbb{Q}_p$ of degree n in a fixed algebraic closure of $\mathbb{Q}_p$; see page 132 of [5].

4 The Rank one case

In this section, we prove that when $CH_{n-1}(X)$ is of rank one, then all four conjectures hold true for $Z_{n-1}(X,T)$.

Theorem 1. *Let X be a normal, connected projective scheme over $\mathbb{F}_q$ of dimension $n \geq 2$. Assume $CH_{n-1}(X)$ is of rank one. Then all four conjectures hold true. That is, $Z_{n-1}(X,T)$ is p-adic meromorphic, $-ord_{T=1} Z_{n-1}(X,T) = 1$, the p-adic Riemann hypothesis holds, and almost all zeros and poles are simple.*

Remark. Conjectures I and II of this theorem were already proven in [6]. The proof below of this theorem will refines the arguments given there. It will also prove another result from [6] about the special value of $Z_{n-1}(X,T)$ at $T = 1$. We state this as a corollary.

Corollary 1. *Let h be the number of torsion elements in $CH_{n-1}(X)$. With*

$$\mu := \min\{deg(D) | D \text{ a divisor of } X,\ deg(D) > 0\},$$

we have

$$Res\ Z_{n-1}(X,T)|_{T=1} = \frac{h}{\mu(q-1)}.$$

Proof. First, notice that the degree map $deg : CH_{n-1}(X) \to \mathbb{Z}$ sends any torsion element to zero. Also, the image of deg is $\mathbb{Z}\mu$ for some positive integer μ. Choose $D \in CH_{n-1}(X)$ such that $deg(D) = \mu$. Then,

$$CH_{n-1}(X) = \mathbb{Z}D \oplus \{D_1, \ldots, D_h\}$$

where $CH_{n-1}(X)_{\mathrm{tors}} = \{D_1, \ldots, D_h\}$. By construction, if μ does not divide d, then there are no effective divisors of degree d. That is, all divisors must have degree $d = k\mu$ for some k.

Let M_d be the number of effective divisors of degree $d := k\mu$. Then, using Proposition 1,

$$\begin{aligned} M_d &= \#E_{r,d}(X) \\ &= \sum_{\substack{D' \geq 0 \\ \deg(D') = k\mu}} 1 \ = \sum_{i=1}^{h} \sum_{\substack{D' \geq 0 \\ D' \sim D_i + kD}} 1 \\ &= \sum_{i=1}^{h} \frac{q^{l(D_i+kD)} - 1}{q-1}. \end{aligned}$$

Now, D is ample since X is projective and $CH_{n-1}(X)$ has rank one. So, we may bring in Proposition 2. This tells us that for each torsion element D_i, for all k sufficiently large,

$$l(D_i + kD) := dim_{\mathbb{F}_q} L(D_i + kD)$$

will agree with a degree $n \geq 2$ polynomial in k with leading coefficient $D^n/n!$. Consequently, we see

$$\begin{aligned} Z_{n-1}(X,T) &= \sum_{d=0}^{\infty} M_d T^d \\ &= \sum_{k=0}^{\infty} \sum_{i=1}^{h} \frac{q^{l(D_i+kD)} - 1}{q-1} T^{\mu k} \\ &= \underbrace{\sum_{i=1}^{h} \frac{1}{q-1} \sum_{k=0}^{\infty} q^{l(D_i+kD)} T^{\mu k}}_{\text{p-adic entire}} \quad - \quad \underbrace{\frac{h}{(q-1)(1-T^{\mu})}}_{\text{rational}}. \end{aligned}$$

From this, we deduce that $Z_{n-1}(X,T)$ is p-adic meromorphic and satisfies the Order and Rank Conjecture. Further, we see that the residue at $T = 1$ is as mentioned in the corollary.

Now, substitute T^μ with T into the above and rewrite $Z_{n-1}(X,T)$ as

$$
\begin{aligned}
&= \frac{1}{(q-1)(1-T)} \sum_{i=1}^{h} \left[\left(\sum_{k=0}^{\infty} q^{l(D_i+kD)} T^k \right) (1-T) - 1 \right] \\
&= \frac{1}{(q-1)(1-T)} \sum_{k=0}^{\infty} T^k \underbrace{\sum_{i=1}^{h} \left[q^{l(D_i+kD)} - q^{l(D_i+(k-1)D)} \right]}_{A_k}
\end{aligned}
$$

Define A_k as the finite sum indicated above. To get at the p-adic Riemann Hypothesis, it suffices to show the horizontal length of each side of the Newton polygon of the p-adic entire function $\sum_{k=0}^{\infty} A_k T^k$ is uniformly bounded. See section 3 for an explanation of this.

Since the $l(D_i + kD)$ are polynomials of the same degree for large k, without loss, we may assume that for all k sufficiently large,

$$l(D_1 + kD) \le l(D_2 + kD) \le \cdots \le l(D_h + kD).$$

For convenience, with each $1 \le i \le h$, write

$$c_i(k) := l(D_i + kD) - l(D_1 + kD) \ge 0,$$

and

$$a_i(k) := l(D_i + kD) - l(D_1 + (k-1)D) \ge 0.$$

Note that $a_i(k)$ tends to infinity as $k \to \infty$ since it is a polynomial in k of degree $n-1 \ge 1$ with the positive leading coefficient $D^n/(n-1)!$. Let m be the integer such that for all k sufficiently large,

$$c_i(k) = \begin{cases} \text{bounded (constant)} & \text{if } 1 \le i \le m \\ \text{unbounded} & \text{if } m+1 \le i \le h \end{cases}.$$

Now,

$$
\begin{aligned}
A_k &= \sum_{i=1}^{h} [q^{l(D_i+kD)} - q^{l(D_i+(k-1)D)}] \\
&= \sum_{i=1}^{h} [q^{l(D_1+(k-1)D)+a_i(k)} - q^{l(D_1+(k-1)D)+c_i(k-1)}]
\end{aligned}
$$

which we may write as

$$A_k = \left(\sum_{i=1}^{h} q^{a_i(k)} - \sum_{i=1}^{m} q^{c_i(k-1)} - \sum_{i=m+1}^{h} q^{c_i(k-1)} \right) q^{l(D_1+(k-1)D)}.$$

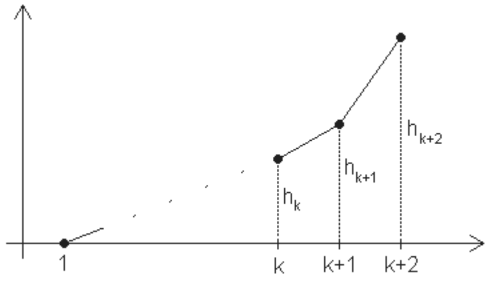

Figure 1: All large slopes are in $\mathbb{Z}$.

Let

$$b_k := \mathrm{ord}_q \sum_{i=1}^{m} q^{c_i(k-1)}.$$

By construction, for all k sufficiently large, the b_k are constant. Now, since $a_i(k)$ and $c_{m+1}(k), \ldots, c_h(k)$ grow as an unbounded polynomial, we see from the above that for all k sufficiently large,

$$h_k := \mathrm{ord}_q A_k = b_k + l(D_1 + (k-1)D).$$

Define

$$\Delta h := (h_{k+2} - h_{k+1}) - (h_{k+1} - h_k).$$

If we can show Δh is strictly positive, then Figure 1 accurately displays the Newton polygon; this implies that the horizontal length of each side (except for the first finitely many sides) of the Newton polygon will have length 1. Now, recall that $l(D_1+kD)$ is a degree $n \geq 2$ polynomial with initial coefficient $D^n/n!$. Also, recall how we defined $a_i(k)$. Then for $n \geq 2$, Δh is strictly positive for all k sufficiently large since

$$\begin{aligned}\Delta h &= a_1(k+1) - a_1(k) \\ &= n(n-1)\frac{D^n}{n!}k^{n-2} + (\text{poly of degree } \leq n-3).\end{aligned}$$

Since all except perhaps finitely many line segments comprising the Newton polygon have horizontal length one, we see that almost all the zeroes are in $\mathbb{Q}_p$ and are simple zeros. Changing back to the variable T^μ and using Lemma 1 of section 3, we see that adjoining the zeroes and poles of $Z_{n-1}(X,T)$ to $\mathbb{Q}_p$ gives a finite extension of $\mathbb{Q}_p$. Furthermore, almost all zeroes are simple since $T^\mu - a$ $T^\mu - b$ clearly have no common roots if a and b have different absolute values. This finishes the proof of the theorem.

5 The quadric surface example

In this section, since the Riemann-Roch problem is easily and explicitly solved for the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$, we will obtain an explicit description of the zeta function of divisors for this quadric surface. Proving the conjectures from this description, while technical, uses only freshman calculus (for details, see section 6).

The method used to determine the Riemann-Roch problem on this surface and calculate the degree of its subvarieties explicitly is by the surface's intersection theory. Let us recall the details of this.

Intersection theory of $\mathbb{P}^1 \times \mathbb{P}^1$. Let $[x : y : z : w]$ be the coordinates for $\mathbb{P}^3_{\mathbb{F}_q}$ and let X be the quadric surface defined by $xw = yz$.

Knowing that this surface is the image of the Segre embedding of $\mathbb{P}^1 \times_{\mathbb{F}_q} \mathbb{P}^1$ in $\mathbb{P}^3$ helps us determine its intersection theory. Denote the coordinates of the two projective lines by $[u_0 : u_1]$ and $[v_0 : v_1]$. Then, the Segre embedding is the map

$$[u_0 : u_1] \times [v_0 : v_1] \mapsto [u_0v_0 : u_0v_1 : u_1v_0 : u_1v_1] = [x : y : z : w].$$

Consider the prime divisors

$$D_1 := (x = y = 0) \quad \text{and} \quad D_2 := (x = z = 0).$$

It is well known that $CH_1(X)$, which equals $Pic(X)$ in this case, equals $\mathbb{Z}D_1 \oplus \mathbb{Z}D_2$. Notice, when considering D_1 and D_2 in $\mathbb{P}^1 \times \mathbb{P}^1$, that

$$D_1 = (u_0 = 0) \times \mathbb{P}^1 \quad \text{and} \quad D_2 = \mathbb{P}^1 \times (v_0 = 0).$$

This helps determine their intersection product: $D_1^2 = D_2^2 = 0$ and $(D_1 \cdot D_2) = 1$. To see this, notice that D_1 and D_2 meet (transversally) at only one point, thus, their product is one. If we let $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection morphism on the first coordinate, we see that D_1 is the fibre of the point $u_0 = 0$. Since fibres are disjoint from one another and linearly equivalent, their self-intersection is zero. Thus $D_1^2 = D_2^2 = 0$.

Next, notice that for the hyperplane $(x = 0)$ in $\mathbb{P}^3$, we have $(x = 0) \cap X$ equal to the union of the two divisors $D_1 = (x = y = 0)$ and $D_2 = (x = z = 0)$. This observation shows that $H := D_1 + D_2$ is a hyperplane section. From this, we may compute the degree of a divisor D on X by

$$\deg(D) := (D \cdot H).$$

In the first section of the paper, we defined the degree via the Hilbert polynomial. The one given here is equivalent.

Notes to take away from the intersection theory. For any effective divisor D on this quadric surface, it will be linearly equivalent to $aD_1 + bD_2$ for some $a, b \in \mathbb{Z}_{\geq 0}$. Thus, from the above, we see that $deg(D) = a + b \geq 0$. Also, using Proposition 1, we may calculate the number of effective divisors linearly equivalent to D by using the dimension of $L(aD_1 + bD_2)$. Since this vector space is parameterized by bihomogeneous polynomials of bidegree (a, b), we see that $\mathrm{EffCone}(X) = \mathbb{Z}_{\geq 0}D_1 \oplus \mathbb{Z}_{\geq 0}D_2$ since $l(aD_1 + bD_2) = (a+1)(b+1)$ when $a, b \geq 0$ and 0 otherwise.

So, we may write

$$\begin{aligned} Z_1(X,T) &= \sum_{D \in \text{ EffCone}(X)} \frac{q^{l(D)}-1}{q-1} T^{\deg(D)} \\ &= \sum_{a,b\geq 0} \frac{q^{l(aD_1+bD_2)}-1}{q-1} T^{\deg(aD_1+bD_2)} \\ &= \sum_{a,b\geq 0} \frac{q^{(a+1)(b+1)}-1}{q-1} T^{a+b} \\ &= \frac{1}{q-1}\left[\frac{-1}{(1-T)^2} + \sum_{a,b\geq 0} q^{(a+1)(b+1)}\, T^{a+b}\right]. \end{aligned}$$

After rewriting the sum over a and b (which will be done in section 6), we have the following.

Lemma 2. *The zeta function of divisors of the surface $xw = yz$ may be written as*

$$Z_1(X,T) = \frac{1}{q-1}\left[\frac{-1}{(1-T)^2} + \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}T^{2n}[1+q^{n+1}T]}{1-q^{n+1}T}\right].$$

In this form, we may deduce that $Z_1(X,T)$ is p-adic meromorphic and $-\mathrm{ord}_{T=1}Z_1(X,T) = 2 = \text{ rank } CH_1(X)$. This was essentially known already in [6] Example 6.7. Also, notice the poles $\{1, q^{-1}, q^{-2}, \cdots\}$ already satisfy the p-adic Riemann Hypothesis. Thus, we need only to concentrate on the zeroes. We do this via its Newton polygon, as we did in Theorem 1.

Lemma 3. *For the quadric surface, write the numerator of $Z_1(X,T)$ as $1 + \sum_{i\geq 1} a_i T^i$, where each a_i is a p-adic integer. Then*

$$ord_q(a_i) = \begin{cases} (3/2)k^2 + (1/2)k & \text{if } i = 3k \\ \text{greater than } (3/2)k^2 + (3/2)k + 1 & \text{if } i = 3k+1 \\ (3/2)k^2 + (5/2)k + 1 & \text{if } i = 3k+2. \end{cases}$$

The proof of this will be given in section 6. From this explicit description of the Newton polygon, we may prove the following.

Theorem 2. *The zeta function of divisors for the quadric surface $xw = zy$ satisfies all four conjectures. That is, it is p-adic meromorphic, satisfies the order and rank relation, satisfies the p-adic Riemann hypothesis, and almost all zeros and poles are simple if $p \neq 2$.*

Remark. In the proof, we show the horizontal lengths of the segments of the Newton polygon alternate between one and two.

Proof. We mentioned above that $Z_1(X,T)$ is meromorphic and satisfies the order and rank relation. Further, we have seen that the poles satisfy the p-adic Riemann hypothesis. So, let us do the same with the zeroes.

First, write the numerator as $1+\sum a_i T^i$. Since the Newton polygon of this power series is obtained as the lower convex hull of the points $(i,\ \mathrm{ord}_q(a_i))$, we need to determine which points are the vertices. For convenience, define the points

$$\begin{aligned}
P_0(k) &:= (3k, 3k^2/2 + k/2)\\
P_1(k) &:= (3k+1, 3k^2/2 + 3k/2 + 1)\\
P_2(k) &:= (3k+2, 3k^2/2 + 5k/2 + 1).
\end{aligned}$$

From Lemma 3, we know that

$$\begin{aligned}
&P_0(k) = (3k,\ \mathrm{ord}_q(a_{3k}))\\
&P_1(k) \text{ lies below } (3k+1,\ \mathrm{ord}_q(a_{3k+1}))\\
&P_2(k) = (3k+2,\ \mathrm{ord}_q(a_{3k+2})).
\end{aligned}$$

Fix $k \geq 0$. Denote the slope between two points Q_1 and Q_2 by $\mathrm{Slope}(Q_1 \leftrightarrow Q_2)$. Now, since

$$\begin{aligned}
\mathrm{Slope}(P_0(k) \leftrightarrow P_2(k)) &= k + 1/2\\
\mathrm{Slope}(P_2(k) \leftrightarrow P_0(k+1)) &= k + 1,\\
\mathrm{Slope}(P_0(k) \leftrightarrow P_1(k)) &= k + 1,
\end{aligned}$$

we see that Figure 2 accurately represents the Newton polygon.

For the line segment of horizontal length two, since the midpoint is strictly above this line segment, we see that all the zeros are simple if $p \neq 2$. This is because the roots of $T^2 - a$ is already distinct if $p > 2$ and a is a non-zero element in $\mathbb{F}_p$. If $p = 2$, the simplicity of the zeros does not seem to be obvious and it would require some more work. $\square$

6 Proofs of the lemmas for the quadric surface

Let X be the quadric surface $xw = yz$. In section 5, we saw how we may write $Z_1(X,T)$ as

$$Z_1(X,T) = \frac{1}{q-1}\left[\frac{-1}{(1-T)^2} + \sum_{a,b\geq 0} q^{(a+1)(b+1)}\, T^{a+b}\right].$$

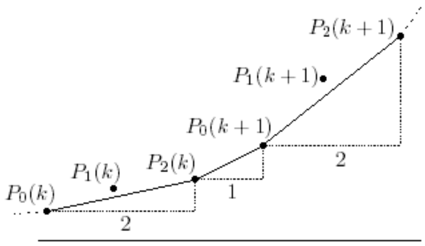

Figure 2: Newton Polygon of $xw = yz$.

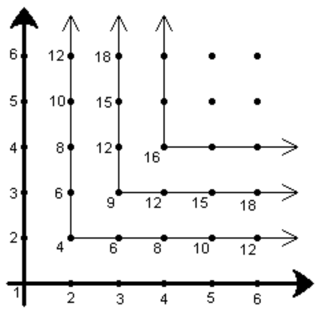

Figure 3: Plot of $(a+1)(b+1)$ on the Effective Cone $\mathbb{Z}_{\geq 0} \oplus \mathbb{Z}_{\geq 0}$.

As promised in Lemma 2, let us see how to rewrite this as

$$Z_1(X,T) = \frac{1}{q-1}\left[\frac{-1}{(1-T)^2} + \sum_{n=0}^{\infty} \frac{q^{(n+1)^2} T^{2n}[1+q^{n+1}T]}{1-q^{n+1}T}\right]. \tag{2}$$

Proof. Let us concentrate on rewriting

$$\sum_{a,b\geq 0} q^{(a+1)(b+1)}\, T^{a+b}.$$

For the following, refer to Figure 3. Since $(a+1)(b+1)$ is symmetric with respect to the coordinates (a,b), we gather together points on the lines $a=0$ and $b=0$; next gather all the points on the lines $a=1$ and $b=1$ that were not on the previous lines; continue this. Double counting the point of intersection of the lines (and then subtracting one off), we obtain

$$\begin{aligned}\sum_{a,b\geq 0} q^{(a+1)(b+1)}\,T^{a+b} &= 2\left[\sum_{n=0}^{\infty} q^{(n+1)}T^{n}\right] - q \\ &\quad + 2\left[\sum_{n=1}^{\infty} q^{2(n+1)}T^{1+n}\right] - q^{2^2}T^2 \\ &\quad + 2\left[\sum_{n=2}^{\infty} q^{3(n+1)}T^{2+n}\right] - q^{3^2}T^4 \\ &\quad + \cdots .\end{aligned}$$

Finally, after simplifying each geometric series, we obtain

$$Z_1(X,T) = \frac{1}{q-1}\left[\frac{-1}{(1-T)^2} + \sum_{n=0}^{\infty}\frac{2q^{(n+1)^2}T^{2n}}{1-q^{n+1}T} - \sum_{n=0}^{\infty} q^{(n+1)^2}T^{2n}\right]$$

which we may rewrite as (2). This completes the lemma. ☐

Lastly, we need to prove our explicit description of the Newton polygon. Recall, Lemma 3 said if we write the numerator of $Z_1(X,T)$ as $1+\sum a_iT^i$, then

$$\operatorname{ord}_q(a_i) = \begin{cases} (3/2)k^2 + (1/2)k & \text{if } i = 3k \\ \text{greater than } (3/2)k^2 + (3/2)k + 1 & \text{if } i = 3k+1 \\ (3/2)k^2 + (5/2)k + 1 & \text{if } i = 3k+2. \end{cases}$$

Proof. For each $n \in \mathbb{Z}_{\geq 0}$, define

$$F_n(T) := q^{(n+1)^2}T^{2n}(1-T)^2[1+q^{n+1}T]\left(\frac{\prod_{k=0}^{\infty}(1-q^{k+1}T)}{1-q^{n+1}T}\right).$$

The numerator of $Z_1(X,T)$ now takes the form

$$-\prod_{n=0}^{\infty}(1-q^{n+1}T) + \sum_{n=0}^{\infty} F_n(T).$$

Let $m \geq 1$. To find the q-adic valuation of the coefficient of T^m in the above series, we compute (or estimate) the valuation of each of the terms in the series, compare them, and choose the smallest. Let us start by computing the q-adic valuation of the coefficient of T^m in each F_n.

Since each F_n has the term

$$T^{2n}(1-T)^2 = T^{2n} - 2T^{2n+1} + T^{2n+2}, \tag{3}$$

there are three cases to consider: when $m \geq 2n+2$, when $m = 2n+1$, and when $m = 2n$.

Case 1: $m \geq 2n+2$**.** Considering the many ways to multiply the T's together in F_n to obtain T^m, we see that using any term in (3) other than T^{2n+2} yields a coefficient of larger valuation. Thus, when $m \geq 2n+2$, the q-adic valuation of the coefficient of T^m in $F_n(T)$ is

$$\begin{aligned} v_m(n) :&= (n+1)^2 + [1+2+\cdots+(m-(2n+2))] \\ &= (n+1)^2 + (1/2)[m-2n-2][m-2n-1]. \end{aligned}$$

Note, the derivative $v'_m(n) = 6n - 2m + 5$. Of the F_n with $m \geq 2n+2$, we wish to find the smallest valuation $v_m(n)$. Since $v_m(n)$ is a convex parabola in the variable n with vertex $m/3 - 5/6$, we need to consider the different residue classes of m modulo 3. If $m = 3k$, then the vertex is at $k - 5/6$. The integer closest to this is $k-1$. Thus, F_{k-1} gives the smallest valuation $v_{3k}(k-1) = 3k^2/2 + k/2$. If $m = 3k+2$, then the vertex is at $k - 1/6$. Thus, F_k produces the smallest valuation $v_{3k+2}(k) = 3k^2/2 + 5k/2 + 1$. Now, if $m = 3k+1$, then the vertex is at $k - 1/2$. So, we see that both F_k and F_{k-1} produce the same (smallest) valuation. However, the sign of T^m in F_k is the opposite of F_{k-1}, and since we are adding the F_n together, they will cancel. So, while the exact value illudes us, we have a lower bound that will do for our purpose: $v_{3k+1}(k) \geq 3k^2/2 + 3k/2 + 1$.

Case 2: $m = 2n+1$**.** Using the term T^{2n} in (3) to obtain T^m we obtain a coefficient with valuation $(n+1)^2 + 1$. Using the term T^{2n+1} yields a valuation

$$\begin{cases} (n+1)^2 & \text{if } q \neq 2^r \\ (n+1)^2 + 1/r & \text{if } q = 2^r. \end{cases} \tag{4}$$

Thus, if $q \neq 2$, we see that using the term T^{2n+1} yields a strictly smaller valuation than using the term T^{2n}. We will consider the case $q = 2$ in a moment. Meanwhile, since $n = (m-1)/2$, we see that (4) takes the form

$$(m+1)^2/4 \quad \text{and} \quad (m+1)^2/4 + 1/r.$$

However, in both instances, we see that if $m = 3k+j$ with $j = 0, 1$, or 2, the valuation is roughly $9k^2/4$. So, for large k, this is larger than the valuations found for the F_n when $m \geq 2n+2$, and so, may be disregarded. One may quickly check that this is true for all k, not just the large ones.

Now, suppose $m = 2n+1$ and $q = 2$. Using only T^{2n} from $T^{2n}(1-T)^2$ in the definition of F_n, we get

$$q^{(n+1)^2} T^{2n}(1-qT)(1-q^2T)\cdots(1-q^{2n}T)(1+q^{2n+1}T)(1-q^{2n+2}T)\cdots. \tag{5}$$

Then, the coefficient of T^{2n+1}, with $q = 2$, is

$$2^{(n+1)^2}(-2-2^2-2^3-\cdots-2^{2n}+2^{2n+1}-2^{2n+2}-\cdots).$$

Consider the same product in (5) but with T^{2n} replaced by $-2T^{2n+1}$. The coefficient of T^m in this product becomes $2^{(n+1)^2}(-2)$. Adding the two coefficients obtained from both products, we see that the coefficient of T^m in F_n is

$$2^{(n+1)^2}(-2^{2n+2}-2^{2n+3}-\cdots).$$

Thus, the 2-adic valuation is $(n+1)^2+(2n+2)$. Since $n = (m-1)/2$, and setting $m = 3k + j$ where $j = 0$, 1 or 2, we see that the valuation is approximately $9k^2/4$. So, for large m (which means large k), the valuation is much larger than the F_n with $m \geq 2n + 2$. Again, one may check this is true for all k.

Case 3: $m = 2n$**.** Then we need to consider $F_{m/2}$. The coefficient of T^m in this has valuation $(m+2)^2/4$. Again, if $m = 3k+j$ with $j = 0, 1$, or 2, then the valuation is roughly $9k^2/4$ which will be larger than the F_n with $m \geq 2n + 2$. And again, one may check this is true for all k.

Lastly, the smallest valuation the term $-\prod_{n=0}^{\infty}(1 - q^{n+1}T)$ contributes to the coefficient of T^m is $m(m + 1)/2$. If $m = 3k + j$, with $j = 0, 1$, or 2, then the valuation is roughly $9k^2/2$. Again, this is too large compared to the valuations obtained by the F_n when $m \geq 2n + 2$. And again(!), one may check this is true for all k. $\square$

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