

DIVISIBILITY OF FROBENIUS EIGENVALUES ON ℓ -ADIC COHOMOLOGY

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ABSTRACT. For a projective variety defined over a finite field with q elements, it is shown that as algebraic integers, the eigenvalues of the geometric Frobenius acting on ℓ -adic cohomology have higher than known q -divisibility beyond the middle dimension. This sharpens both Deligne's integrality theorem [SGA7, Corollary 5.5.3] and the cohomological divisibility theorem [EK05, Theorem 4.1]. Similar lower bound is proved for the Hodge level for a complex projective variety beyond the middle dimension, improving earlier results in this direction. The affine version of our results for the compactly supported cohomology is still open in general.

1. INTRODUCTION

In this small note, we continue to explore the interplay between the Ax-Katz theorem [Ka71] for point count q -divisibility and Deligne's integrality theorem [SGA7] for varieties defined over a finite field $\mathbb{F}_q$ with q elements. This connection was studied earlier in [Es03] for projective complete intersections and [EK05] for both projective and affine varieties. Building on these earlier works, we show that for a projective variety, there is an improved q -divisibility of the eigenvalues of the geometric Frobenius acting on ℓ -adic cohomology beyond the middle dimension. The proof carries over to $\mathbb{C}$ to yield a lower bound for the Hodge level for projective varieties over $\mathbb{C}$, improving earlier results in [Es90], [ENS92], [EW03].

We first describe the affine case and then return to the projective case. Let X be an affine variety in $\mathbb{A}^n$ over $\mathbb{F}_q$, defined by an ideal spanned by r polynomials

$$f_1, \dots, f_r \in \mathbb{F}_q[x_1, \dots, x_n]$$

of positive degree $d_1, \dots, d_r$. To avoid triviality, we assume that r and n are positive integers, thus in particular $\dim(X) \leq n - 1$. For each non-negative integer j , we define another non-negative integer

$$\mu_j(n; d_1, \dots, d_r) = j + \max(0, \lceil \frac{n - j - \sum_{i=1}^r d_i}{\max_{i=1}^r d_i} \rceil) \geq j.$$

This is an increasing function in both j and n , that is

$$\begin{aligned} \mu_{j+1}(n; d_1, \dots, d_r) &\geq \mu_j(n; d_1, \dots, d_r), \\ \mu_j(n+1; d_1, \dots, d_r) &\geq \mu_j(n; d_1, \dots, d_r). \end{aligned}$$

It is clearly a decreasing function in each d_i . The Ax-Katz theorem [Ka71] says that the number $\#X(\mathbb{F}_q)$ of $\mathbb{F}_q$ -rational points on X is divisible by $q^{\mu_0(n; d_1, \dots, d_r)}$ for

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all q . This yields a non-trivial information as soon as $n > \sum_{i=1}^r d_i$. Equivalently, as $\mu_0(n; d_1, \dots, d_r) \leq n$, $\#(\mathbb{A}^n \setminus X)(\mathbb{F}_q)$ is divisible by $q^{\mu_0(n; d_1, \dots, d_r)}$. The bound is achieved, that is the result is sharp, so this q -divisibility result is best possible in general. It has a natural interpretation in terms of zeta functions.

Recall that the zeta function of X is defined by the formal power series

$$Z(X, T) = \exp\left(\sum_{k=1}^{\infty} \frac{\#X(\mathbb{F}_{q^k})}{k} T^k\right) \in 1 + T\mathbb{Z}[[T]] \subset \mathbb{Q}[[T]],$$

which by Dwork's rationality theorem [Dw60] lies in $\mathbb{Q}(T)$. This implies that the reciprocal zeros and poles of $Z(X, T)$ are algebraic integers. From the q -adic radius of convergence for the logarithmic derivative of the zeta function, one deduces that the Ax-Katz theorem is equivalent to the statement that all reciprocal zeros and poles of $Z(X, T)$ are divisible by $q^{\mu_0(n; d_1, \dots, d_r)}$ as algebraic integers.

By Grothendieck's trace formula [Gr65], the zeta function has a cohomological interpretation

$$Z(X, T) = \prod_{i \geq 0} P_{2i+1}(T) / \prod_{i \geq 0} P_{2i}(T),$$

where $P_i(T) = \det(I - TF|H_c^i(X))$ and by a (standard) abuse of notation $H_c^i(X) := H_c^i(X \otimes \mathbb{F}_q, \mathbb{Q}_\ell)$. Here ℓ is a prime different from the characteristic p of $\mathbb{F}_q$, H_c^i denotes the i -th ℓ -adic compactly supported cohomology, F is the geometric Frobenius. By the Weil conjectures, that is Deligne's purity theorem [De74], q^μ -divisibility as algebraic integers of the eigenvalues of the geometric Frobenius is equivalent to $q^{\kappa\mu}$ -divisibility of $\#X(\mathbb{F}_{q^\kappa})$ for all $\kappa \geq 1$ if X is smooth proper. In general, from the divisibility of the number of points, one cannot immediately deduce the divisibility of the eigenvalues of the geometric Frobenius as there might be some cancellation in the expression of the zeta function as a rational function. It is then natural to ask what can be said about the q -divisibility on $H_c^j(X)$. This question was first studied in [Es03] for projective complete intersections and then in [EK05] in the general case. We state their general results for both affine and projective varieties. The affine result is

Proposition 1.1. *Let X be an affine variety in $\mathbb{A}^n$ defined by r polynomials $f_1, \dots, f_r \in \mathbb{F}_q[x_1, \dots, x_n]$ of positive degrees $d_1, \dots, d_r$ with $n, r \geq 1$.*

- (i) [EK05, Theorem 2.1 and Theorem 2.2] *For all integers i , the eigenvalues of the geometric Frobenius acting on $H_c^i(X)$ and $H_c^i(\mathbb{A}^n \setminus X)$ are divisible by $q^{\mu_0(n; d_1, \dots, d_r)}$ as algebraic integers.*
- (ii) [EK05, Theorem 2.3] *For all integers $j \geq 0$, the eigenvalues of the geometric Frobenius acting on $H_c^{n-1+j}(X)$ and $H_c^{n+j}(\mathbb{A}^n \setminus X)$ are divisible by $q^{\mu_j(n; d_1, \dots, d_r)}$ as algebraic integers.*

The projective result is

Proposition 1.2. [EK05, Theorem 4.1] *Let $Y \subset \mathbb{P}^n$ be a projective variety defined by r homogeneous polynomials $f_1, \dots, f_r \in \mathbb{F}_q[x_0, x_1, \dots, x_n]$ of positive degrees $d_1, \dots, d_r$ with $n, r \geq 1$.*

- (i) *For all integers i , the eigenvalues of the geometric Frobenius acting on $H_c^i(\mathbb{P}^n \setminus Y)$ are divisible by $q^{\mu_0(n+1; d_1, \dots, d_r)}$ as algebraic integers.*
- (ii) *For all integers $j \geq 0$, the eigenvalues of the geometric Frobenius acting on $H_c^{n+1+j}(\mathbb{P}^n \setminus Y)$ are divisible by $q^{\mu_j(n+1; d_1, \dots, d_r)}$ as algebraic integers.*

In the case when Y is a complete intersection in $\mathbb{P}^n$, part (i) of Proposition 1.2 was first proved in [Es03]. This part (i) gives a cohomological strengthening of the Ax-Katz theorem which in the projective case is the statement that the eigenvalues of the geometric Frobenius on $H_c^i(\mathbb{P}^n \setminus Y)$ are divisible by $q^{\mu_0(n+1; d_1, \dots, d_r)}$ as algebraic integers for all i . The proof uses the Ax-Katz theorem and thus does not reprove it. Part (ii) of Proposition 1.2 further says that for $j \geq 0$, the eigenvalues of the geometric Frobenius on $H_c^{n+1+j}(\mathbb{P}^n \setminus Y)$ are divisible by $q^{\mu_j(n+1; d_1, \dots, d_r)}$ as algebraic integers. This is better than part (i), since $\mu_j(n+1; d_1, \dots, d_r) \geq \mu_0(n+1; d_1, \dots, d_r)$.

Our main result is the following projective theorem which improves part (ii) of Proposition 1.2.

Theorem 1.3. *Let $Y \subset \mathbb{P}^n$ be a projective variety defined by r homogeneous polynomials $f_1, \dots, f_r \in \mathbb{F}_q[x_0, x_1, \dots, x_n]$ of positive degrees $d_1, \dots, d_r$ with $n, r \geq 1$. We denote by $H_{\text{prim}}^i(Y) = H^i(Y)/H^i(\mathbb{P}^n)$ the primitive cohomology. Then the eigenvalues of the geometric Frobenius acting on*

$$(i) \ H_{\text{prim}}^{\dim(Y)+j}(Y) \text{ for } 0 \leq j \leq \dim(Y),$$

$$(ii) \ H_c^{\dim(Y)+1+j}(\mathbb{P}^n \setminus Y) \text{ for } 0 \leq j \leq \dim(Y) + 1$$

are divisible by $q^{\mu_j(n+1; d_1, \dots, d_r)}$ as algebraic integers.

This improves part (ii) of Proposition 1.2 because now the higher divisibility starts at $\dim(Y) + 1$, earlier than $n + 1$. Even at the same cohomological degree $n + 1 + j$, Theorem 1.3 would give higher divisibility than Proposition 1.2 does, since

$$H_c^{n+1+j}(\mathbb{P}^n \setminus Y) = H_c^{\dim(Y)+1+(j+n-\dim(Y))}(\mathbb{P}^n \setminus Y).$$

In the non-trivial case where $\max_i d_i > 1$, one checks that for $0 \leq j \leq \dim(Y)$,

$$\frac{\dim(Y) + j}{2} \geq \mu_j(n+1; d_1, \dots, d_r),$$

so we can replace the primitive cohomology in Theorem 1.3 (i) by the ordinary cohomology. We state this remark as a corollary.

Corollary 1.4. *In Theorem 1.3, if $\max_i d_i > 1$, then for all $j \geq 0$, the eigenvalues of the geometric Frobenius acting on $H^{\dim(Y)+j}(Y)$ are divisible by $q^{\mu_j(n+1; d_1, \dots, d_r)}$ as algebraic integers.*

Since $\mu_j(n+1; d_1, \dots, d_r) \geq j$, this corollary also improves Deligne's integrality theorem [SGA7, Corollary 5.5.3] which says that the eigenvalues of the geometric Frobenius acting on $H^{\dim(Y)+j}(Y)$ are divisible by q^j as algebraic integers.

In this note, we have improved the projective Proposition 1.2 beyond middle cohomological dimension. It would be interesting to know if the affine Proposition 1.1 can be similarly improved beyond middle cohomological dimension. We state this as an open problem.

Question 1.5. *Let X be an affine variety in $\mathbb{A}^n$ defined by $f_1, \dots, f_r \in \mathbb{F}_q[x_1, \dots, x_n]$ of positive degrees $d_1, \dots, d_r$ with $n, r \geq 1$. Is it true that the eigenvalues of the geometric Frobenius acting on*

$$(i) \ H_c^{\dim(X)+j}(X) \text{ for } j \geq 0,$$

$$(ii) \ H_c^{\dim(X)+1+j}(\mathbb{A}^n \setminus X) \text{ for } 0 \leq j \leq \dim(X) + 1$$

are divisible by $q^{\mu_j(n; d_1, \dots, d_r)}$ as algebraic integers?

Remark 1.6. The projective Theorem 1.3 can already be used to treat one non-trivial case. Let Y be the projective variety in $\mathbb{P}^n$ defined the vanishing of the homogeneous polynomials

$$g_i(x_0, x_1, \dots, x_n) := x_0^{d_i} f_i\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right), \quad 1 \leq i \leq r.$$

Let Y_∞ be the hyperplane section $Y \cap \{x_0 = 0\}$ at infinity, so

$$\mathbb{P}^n \setminus Y = (\mathbb{A}^n \setminus X) \sqcup (\mathbb{P}^{n-1} \setminus Y_\infty).$$

By the excision sequence

$$\dots \rightarrow H_c^j(\mathbb{A}^n \setminus X) \rightarrow H_c^j(\mathbb{P}^n \setminus Y) \rightarrow H_c^j(\mathbb{P}^{n-1} \setminus Y_\infty) \rightarrow \dots$$

the affine case for $\mathbb{A}^n \setminus X$ is reduced to the projective case for both $\mathbb{P}^n \setminus Y$ and $\mathbb{P}^{n-1} \setminus Y_\infty$ in case $\dim(Y) = \dim(X)$. Thus, Question 1.5 has a positive answer if $\dim(Y) = \dim(X)$.

In general, our inductive proof shows that Question 1.5 has a positive answer if for a general hyperplane A in $\mathbb{A}^n$, the affine Gysin map

$$H_c^{i-2}(A \setminus A \cap X)(-1) \xrightarrow{\text{Gysin}} H_c^i(\mathbb{A}^n \setminus X)$$

was surjective for $i - 1 > \dim(X)$. Its projective version is true. If X is a local complete intersection in $\mathbb{A}^n$, the affine Gysin lemma follows from the Weak Lefschetz Theorem for perverse sheaves [Del93, A.5] as observed by Dingxin Zhang. Thus, Question 1.5 also has a positive answer if X is a local complete intersection in $\mathbb{A}^n$.

An immediate application of the above remark is a cohomological strengthening of the polar result in [Wa02, Theorem 1.2b] for affine complete intersections. A non-zero number α is called a reciprocal zero (resp. reciprocal pole) of the rational function $Z(X, T)$ if $1/\alpha$ is a zero (resp. pole) of $Z(X, T)$. In the case that X is an affine complete intersection in $\mathbb{A}^n$, then $H_c^i(X) = 0$ for all $i < \dim(X)$ and the cohomological formula for the zeta function reduces to

$$Z(X, T)^{(-1)^{\dim(X)-1}} = \prod_{j=0}^{\dim(X)} \det(I - TF|H_c^{\dim(X)+j}(X))^{(-1)^j}.$$

This implies that the reciprocal poles of $Z(X, T)^{(-1)^{\dim(X)-1}}$ are among the Frobenius eigenvalues on $H_c^{\dim(X)+j}(X)$ for odd j , and hence are divisible as algebraic integers by

$$q^{\min_{j \geq 0} \mu_{2j+1}(n; d_1, \dots, d_r)} = q^{\mu_1(n; d_1, \dots, d_r)}.$$

This recovers the polar result in [Wa02, Theorem 1.2b] according to which if X is an affine complete intersection in $\mathbb{A}^n$, all reciprocal poles of $Z(X, T)^{(-1)^{\dim(X)-1}}$ are divisible by $q^{\mu_1(n; d_1, \dots, d_r)}$ as algebraic integers, where we note that

$$\mu_1(n; d_1, \dots, d_n) = 1 + \mu_0(n-1; d_1, \dots, d_n) \geq \mu_0(n; d_1, \dots, d_n).$$

The proof of Theorem 1.3 does not contain any new geometric or cohomological idea. It depends on earlier results in [Es03] and [EK05]. For the proof of Theorem 1.3 (ii), we mimic the method in [Es03, Section 2]. The whole point is to have the right numbers which allow an inductive argument and apply the result in [EK05] as our starting point (the case $j = 0$). In Section 3, we mention the Hodge theoretic analogues of our divisibility result.

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2. PROOFS

2.1. Proof of Theorem 1.3. We prove Theorem 1.3 (ii). We argue by induction on $\dim(Y)$. If $\dim(Y) = -1$ that is $Y = \emptyset$, then $r > n$ thus $\mu_0(n+1; d_1, \dots, d_r) = 0$ and there is nothing to prove. We assume that $\dim(Y) \geq 0$. If $j = 0$, the theorem is already true by Proposition 1.2 (i). We assume that $\dim(Y) \geq 0$ and $j \geq 1$. If $n = 1$, then $\dim(Y)$ must be zero and $j = 1$. In this case, Theorem 1.3 (ii) is true as well since $\mu_1(2, d_1, \dots, d_r) = 1$ and $H_c^2(\mathbb{P}^1 \setminus Y)$ has the unique Frobenius eigenvalue q . Thus, we assume that $\dim(Y) \geq 0$, $j \geq 1$ and $n \geq 2$ below.

We copy the argument of [Es03, Section 2]. By replacing $\mathbb{F}_q$ by a finite extension, which preserves the divisibility, there is by [Es03, Theorem 2.1] a linear hyperplane $\iota : A \hookrightarrow \mathbb{P}^n$ such that the Gysin homomorphism

$$H^{i-2}(A, \iota^* \mathcal{F})(-1) \xrightarrow{\text{Gysin}} H_A^i(\mathbb{P}^n, \mathcal{F})$$

is a Frobenius equivariant isomorphism, where $\mathcal{F} = j_! \mathbb{Q}_\ell$, $j : \mathbb{P}^n \setminus Y \rightarrow \mathbb{P}^n$. On the other hand, one has an exact sequence

$$H^{i-1}(Y \setminus Y \cap A) \rightarrow H^i(\mathbb{P}^n \setminus A, \mathcal{F}) = H^i(\mathbb{P}^n \setminus A, Y \setminus Y \cap A) \rightarrow H^i(\mathbb{P}^n \setminus A)$$

so Artin vanishing on the left and right terms implies

$$H^i(\mathbb{P}^n \setminus A, \mathcal{F}) = 0 \text{ for } i - 1 > \dim(Y).$$

Thus the composite

$$\begin{aligned} H_c^{i-2}(A \setminus A \cap Y)(-1) &= H^{i-2}(A, \iota^* \mathcal{F})(-1) \xrightarrow{\text{Gysin}} H_A^i(\mathbb{P}^n, \mathcal{F}) \\ &\xrightarrow{\text{excision}} H^i(\mathbb{P}^n, \mathcal{F}) = H_c^i(\mathbb{P}^n \setminus Y) \end{aligned}$$

is surjective and Frobenius equivariant for $i - 1 > \dim(Y)$. We now assume this inequality and set $i = \dim(Y) + 1 + j$ so $j \geq 1$. Since $\dim(Y) \geq 0$ and A is in general position, we have $\dim(A \cap Y) = \dim(Y) - 1$. By our induction hypothesis, the theorem is true for the lower dimensional projective variety $A \cap Y$ in $A = \mathbb{P}^{n-1}$ defined by the vanishing of r homogeneous polynomials of degrees $d_1, \dots, d_r$ with $(n-1), r \geq 1$. It follows that for $0 \leq j-1 \leq \dim(A \cap Y) + 1$, the eigenvalues of the geometric Frobenius on $H_c^{\dim(A \cap Y) + 1 + (j-1)}(A \setminus A \cap Y)(-1)$ are divisible by

$$q^{1 + \mu_{j-1}(n; d_1, \dots, d_r)} = q^{\mu_j(n+1; d_1, \dots, d_r)}$$

as algebraic integers. So for $1 \leq j \leq \dim(A \cap Y) + 2 = \dim(Y) + 1$, the divisibility on $H_c^{\dim(Y) + 1 + j}(\mathbb{P}^n \setminus Y)$ is also by $q^{\mu_j(n+1; d_1, \dots, d_r)}$. This finishes the proof of Theorem 1.3 (ii).

To prove Theorem 1.3 (i), one applies the excision sequence which gives the isomorphism of Frobenius modules

$$H_{\text{prim}}^i(Y) := H^i(Y)/H^i(\mathbb{P}^n) \cong H_c^{i+1}(\mathbb{P}^n - Y), \quad 0 \leq i \leq 2 \dim(Y) \leq 2(n-1).$$

This implies that for $0 \leq j \leq \dim(Y)$, Frobenius eigenvalues on

$$H_{\text{prim}}^{\dim(Y) + j}(Y) = H^{\dim(Y) + j}(Y)/H^{\dim(Y) + j}(\mathbb{P}^n) \cong H_c^{\dim(Y) + 1 + j}(\mathbb{P}^n - Y)$$

are divisible by $q^{\mu_j(n+1, d_1, \dots, d_r)}$ as algebraic integers. The theorem is proved.

3. HODGE LEVEL

The divisibility as algebraic integers of the Frobenius eigenvalues on ℓ -adic cohomology suggests by a vast generalization of Tate conjecture a similar divisibility of the corresponding motive by the Tate motive. The motivic divisibility in turn implies a similar lower bound for the Hodge level, which we sketch in this section.

For a non-empty separated finite type scheme X over $\mathbb{C}$, the compactly supported cohomology group $H_c^i(X)$ has a mixed Hodge structure [De71] with the decreasing Hodge filtration Fil :

$$H_c^i(X) = \text{Fil}^0 H_c^i(X) \supseteq \text{Fil}^1 H_c^i(X) \supseteq \text{Fil}^2 H_c^i(X) \supseteq \cdots.$$

The Hodge level of $H_c^i(X)$ is the largest integer κ such that

$$\text{Fil}^\kappa H_c^i(X) = H_c^i(X).$$

Theorem 3.1. *Let $Y \subset \mathbb{P}^n$ be a projective variety defined over $\mathbb{C}$ by r homogeneous polynomials of positive degrees $d_1, \dots, d_r$ with $n, r \geq 1$. We denote by $H_{\text{prim}}^i(Y) = H^i(Y)/H^i(\mathbb{P}^n)$ the primitive cohomology. The Hodge level of*

- (i) $H_{\text{prim}}^{\dim(Y)+j}(Y)$ for $0 \leq j \leq \dim(Y)$,
- (ii) $H_c^{\dim(Y)+1+j}(\mathbb{P}^n \setminus Y)$ for $0 \leq j \leq \dim(Y) + 1$

is at least $\mu_j(n+1; d_1, \dots, d_r)$.

The proof of this theorem is the same as that of Theorem 1.3. Indeed one has Artin's vanishing theorem, excision, purity and base change. One does the induction in the same way. It reduces to the case $j = 0$. For $j = 0$, the Hodge level lower bound is proved in [ENS92] in the projective case. The latter extends earlier results in [DD90] for projective hypersurfaces and in [Es90] for projective complete intersections.

Remark 3.2. In the complete intersection case, Theorem 3.1 (ii) was first proved in [EW03, Theorem 2.3] using a different geometric argument. The complete intersection condition comes from a vanishing theorem for Zariski sheaf cohomology in [Es90]. This suggests that it might be possible to refine the geometric approach in [DD90], [Es90], [EW03] to prove Theorem 3.1 as well for any projective variety Y in $\mathbb{P}^n$.

In the affine case, we expect a similar improvement and we state this as an open problem.

Question 3.3. *Let X be an affine variety in $\mathbb{A}^n$ defined over $\mathbb{C}$ by r polynomials of positive degrees $d_1, \dots, d_r$ with $n, r \geq 1$. Is it true that the Hodge level of*

- (i) $H_c^{\dim(X)+j}(X)$ for $j \geq 0$,
- (ii) $H_c^{\dim(X)+1+j}(\mathbb{A}^n \setminus X)$ for $0 \leq j \leq \dim(X) + 1$

is at least $\mu_j(n; d_1, \dots, d_r)$?

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