Trivial Factors For L-functions of Symmetric Products of Kloosterman Sheaves

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0. Introduction

In this paper, we determine the trivial factors of L-functions of both integral and p-adic symmetric products of Kloosterman sheaves.

Let $\mathbf{F}_q$ be a finite field of characteristic p with q elements, let l be a prime number distinct from p, and let $\psi : \mathbf{F}_q \to \overline{\mathbf{Q}}_l^*$ be a nontrivial additive character. Fix an algebraic closure $\mathbf{F}$ of $\mathbf{F}_q$. For any integer k, let $\mathbf{F}_{q^k}$ be the extension of $\mathbf{F}_q$ in $\mathbf{F}$ with degree k. Let $n \geq 2$ be a positive integer. If λ lies in $\mathbf{F}_{q^k}$, we define the $(n-1)$-variable Kloosterman sum by

$$\mathrm{Kl}_n(\mathbf{F}_{q^k}, \lambda) = \sum_{x_1 \cdots x_n = \lambda,\ x_i \in \mathbf{F}_{q^k}} \psi(\mathrm{Tr}_{\mathbf{F}_{q^k}/\mathbf{F}_q}(x_1 + \cdots + x_n)).$$

Such character sums can be studied via either p-adic methods or l-adic methods. In [D1] Théorème 7.8, Deligne constructs a lisse $\overline{\mathbf{Q}}_l$-sheaf of rank n on $\mathbf{A}^1_{\mathbf{F}_q} - \{0\}$ pure of weight $n-1$, which we denote by Kl_n and call the Kloosterman sheaf, with the property that for any $x \in (\mathbf{A}^1_{\mathbf{F}_q} - \{0\})(\mathbf{F}_{q^k}) = \mathbf{F}^*_{q^k}$, we have

$$\mathrm{Tr}(F_x, \mathrm{Kl}_{n,\bar{x}}) = (-1)^{n-1}\mathrm{Kl}_n(\mathbf{F}_{q^k}, x),$$

where F_x is the geometric Frobenius element at the point x. Let η be the generic point of $\mathbf{A}^1_{\mathbf{F}_q}$. The Kloosterman sheaf gives rise to a Galois representation

$$\mathrm{Kl}_n : \mathrm{Gal}(\overline{\mathbf{F}_q(T)}/\mathbf{F}_q(T)) \to \mathrm{GL}((\mathrm{Kl}_n)_{\bar{\eta}})$$

unramified outside 0 and ∞. From the p-adic point of view, the Kloosterman sheaf is given by an ordinary overconvergent F-crystal of rank n over $\mathbf{A}^1_{\mathbf{F}_q} - \{0\}$. See Sperber [S].

For each positive integer k, denote the L-function of the k-th symmetric product of the Kloosterman sheaf by $L(k,n,T)$:

$$L(k,n,T) := L(\mathbf{A}^1_{\mathbf{F}_q} - \{0\}, \mathrm{Sym}^k\mathrm{Kl}_n, T) \in 1 + T\mathbf{Z}[[T]].$$

We call it simply the k-th symmetric product L-function. This is a rational function whose reciprocal zeros and poles are Weil q-numbers by theorems of Grothendieck and Deligne. Let $j : \mathbf{A}^1_{\mathbf{F}_q} - \{0\} \to \mathbf{P}^1_{\mathbf{F}_q}$ be the canonical open immersion. By definition, we have the following relation between the L-functions $L(k,n,T)$ and $L(\mathbf{P}^1_{\mathbf{F}_q}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n)), T)$:

$$\begin{aligned} & L(k,n,T) \\ = \; & L(\mathbf{P}^1_{\mathbf{F}_q}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n)), T)\det(1 - F_0T, (\mathrm{Sym}^k(\mathrm{Kl}_n)_{\bar\eta})^{I_0})\det(1 - F_\infty T, (\mathrm{Sym}^k(\mathrm{Kl}_n)_{\bar\eta})^{I_\infty}), \end{aligned}$$

where I_0 (resp. I_∞) is the inertia subgroup at 0 (resp. ∞), and F_0 (resp. F_∞) is the geometric Frobenius element at 0 (resp. ∞). Here we use the fact that

$$(\mathrm{Sym}^k(\mathrm{Kl}_n)_{\bar\eta})^{I_0} = (j_*(\mathrm{Sym}^k(\mathrm{Kl}_n)))_{\bar 0},\ (\mathrm{Sym}^k(\mathrm{Kl}_n)_{\bar\eta})^{I_\infty} = (j_*(\mathrm{Sym}^k(\mathrm{Kl}_n)))_{\overline{\infty}}.$$

We call $\det(1 - F_0T, (\mathrm{Sym}^k(\mathrm{Kl}_n)_{\bar\eta})^{I_0})$ (resp. $\det(1 - F_\infty T, (\mathrm{Sym}^k(\mathrm{Kl}_n)_{\bar\eta})^{I_\infty})$) the local factor at 0 (resp. ∞) of $L(k,n,T)$. On the other hand, by Grothendieck's formula for L-functions, we have

$$\begin{aligned} & L(\mathbf{P}^1_{\mathbf{F}_q}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n)), T) \\ = \; & \frac{\det(1 - FT, H^1(\mathbf{P}^1_{\mathbf{F}}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n))))}{\det(1 - FT, H^0(\mathbf{P}^1_{\mathbf{F}}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n))))\det(1 - FT, H^2(\mathbf{P}^1_{\mathbf{F}}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n))))}. \end{aligned}$$

So we get the factorization

$$\begin{aligned} & L(k,n,T) \\ = \; & \frac{\det(1 - FT, H^1(\mathbf{P}^1_{\mathbf{F}}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n))))\det(1 - F_0T, ((\mathrm{Sym}^k\mathrm{Kl}_n)_{\bar\eta})^{I_0})\det(1 - F_\infty T, ((\mathrm{Sym}^k\mathrm{Kl}_n)_{\bar\eta})^{I_\infty})}{\det(1 - FT, H^0(\mathbf{P}^1_{\mathbf{F}}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n))))\det(1 - FT, H^2(\mathbf{P}^1_{\mathbf{F}}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n))))}. \end{aligned}$$

The first factor $\det(1 - FT, H^1(\mathbf{P}^1_{\mathbf{F}}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n))))$ is called the non-trivial factor. It is pure of weight $k(n-1)+1$ by [D2] 3.3.1. All other factors on the right-hand side of the above expression are called trivial factors. The zeros of these trivial factors give rise to the trivial zeros or poles of $L(k,n,T)$.

The aim of this paper is to determine all the trivial factors of $L(k,n,T)$ and their variation with k as k varies p-adically. As a consequence, we obtain some partial information on the non-trivial factor and its variation with k as well. In the case $n=2$, the trivial factor problem for $L(k,2,T)$ was first studied by Robba ([R]) via Dwork's p-adic cohomology. Robba determined the trivial factors for $L(k,2,T)$ assuming $p > k/2$.

In [FW], we studied in detail the behavior of the Kloosterman representation at ∞ based on the work of Deligne and Katz. As a consequence, we completely determined the trivial factor

$$\det(1 - F_\infty T, ((\mathrm{Sym}^k\mathrm{Kl}_n)_{\bar\eta})^{I_\infty}).$$

See Theorem 2.5 in [FW] for the precise statement. The trivial factor $\det(I-F_0T,(\mathrm{Sym}^k(\mathrm{Kl}_n)_{\bar\eta})^{I_0})$ is easy to determine for $n=2$. But in [FW] we were unable to determine it for $n>2$. We solve this problem in the present paper. Our result is as follows.

Theorem 0.1. We have

$$\det(I-F_0T,(\mathrm{Sym}^k(\mathrm{Kl}_n)_{\bar\eta})^{I_0}) = \prod_{u=0}^{[\frac{k(n-1)}{2}]}(1-q^uT)^{m_k(u)},$$

where $m_k(u)$ is determined by

$$\frac{(1-x^n)\cdots(1-x^{n+k-2})(1-x^{n+k-1})}{(1-x^2)\cdots(1-x^{k-1})(1-x^k)} = \sum_{u=0}^{\infty} m_k(u)x^u.$$

We have

$$m_k(u) = c_k(u) - c_k(u-1),$$

where $c_k(u)$ is the number of elements of the set

$$\{(i_0,\ldots,i_{n-1})|i_j\geq 0,\ i_0+i_1+\cdots+i_{n-1}=k,\ 0\cdot i_0+1\cdot i_1+\cdots+(n-1)\cdot i_{n-1}=u\}.$$

The trivial poles of $L(k,n,T)$ can be derived from Katz's global monodromy theorem and Grothendieck's formula for L-functions. For completeness, we include this deduction by working out the relevant representation theory which should be well-known to experts.

Denote by G the Zariski closure of the image of $\mathrm{Gal}(\overline{\mathbf{F}(T)}/\mathbf{F}(T))$ under the representation

$$\mathrm{Kl}_n : \mathrm{Gal}(\overline{\mathbf{F}_q(T)}/\mathbf{F}_q(T)) \to \mathrm{GL}((\mathrm{Kl}_n)_{\bar\eta}).$$

By [K] 11.1, we have

$$G = \begin{cases} \mathrm{Sp}(n) & \text{if } n \text{ is even,} \\ \mathrm{SL}(n) & \text{if } n \text{ is odd, and } p\neq 2. \\ \mathrm{SO}(n) & \text{if } n \text{ is odd, } n\neq 7 \text{ and } p=2, \\ \mathrm{G}_2 & \text{if } n=7 \text{ and } p=2. \end{cases}$$

If pn is even, we have $(-1)^n=1$ in $\mathbf{F}_q$. By [K] 4.2.1, we then have a perfect pairing

$$\mathrm{Kl}_n\otimes\mathrm{Kl}_n \to \overline{\mathbf{Q}}_l(1-n).$$

When n is even, we have $G=\mathrm{Sp}(n)$, the pairing is alternating, and Kl_n is isomorphic to the standard representation of $\mathrm{Sp}(n)$. When n is odd and $p=2$, we have $G=\mathrm{SO}(n)$ or $G=\mathrm{G}_2$, the pairing is symmetric, and Kl_n is isomorphic to the standard representation of $\mathrm{SO}(n)$ or G_2. (The standard representation of G_2 is defined to be the unique irreducible representation of dimension 7.) When pn is odd, Kl_n is isomorphic to the standard representation of $\mathrm{SL}(n)$. In the Appendix of this paper, we will prove the following result.

Lemma 0.2. Let $\mathfrak{g}$ be one of the following Lie algebras

$$\mathfrak{sl}(n), \mathfrak{sp}(n), \mathfrak{so}(n), \mathfrak{g}_2$$

and let V be the standard representation of $\mathfrak{g}$. In the case where $\mathfrak{g} = \mathfrak{sl}(n)$ or $\mathfrak{sp}(n)$, the representation $\mathrm{Sym}^k V$ is irreducible, and in the case where $\mathfrak{g} = \mathfrak{so}(n)$ or $\mathfrak{g}_2$, the representation $\mathrm{Sym}^k V$ contains exactly one copy of the trivial representation if k is even, and contains no trivial representation if k is odd.

By [D2] 1.4.1, we have

$$\begin{aligned} H^0(\mathbf{P}^1_{\bar{\mathbf{F}}}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n))) &= ((\mathrm{Kl}_n)_{\bar{\eta}})^{\mathrm{Gal}(\overline{\mathbf{F}_q(t)}/\mathbf{F}_q(t))} = ((\mathrm{Kl}_n)_{\bar{\eta}})^G, \\ H^2(\mathbf{P}^1_{\bar{\mathbf{F}}}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n))) &= ((\mathrm{Kl}_n)_{\bar{\eta}})_{\mathrm{Gal}(\overline{\mathbf{F}_q(t)}/\mathbf{F}_q(t))}(-1) = ((\mathrm{Kl}_n)_{\bar{\eta}})_G(-1). \end{aligned}$$

Combined with [K] 11.1 and Lemma 0.2, we get the following.

Theorem 0.3. We have

$$\det(1 - FT, H^0(\mathbf{P}^1_{\bar{\mathbf{F}}}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n)))) = \begin{cases} 1 & \text{if } n \text{ is even, or } k \text{ is odd, or } pn \text{ is odd,} \\ 1 - q^{\frac{k(n-1)}{2}}T & \text{if } p = 2,\ k \text{ is even and } n \text{ is odd,} \end{cases}$$

$$\det(1 - FT, H^2(\mathbf{P}^1_{\bar{\mathbf{F}}}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n)))) = \begin{cases} 1 & \text{if } n \text{ is even, or } k \text{ is odd, or } pn \text{ is odd,} \\ 1 - q^{\frac{k(n-1)+2}{2}}T & \text{if } p = 2,\ k \text{ is even and } n \text{ is odd.} \end{cases}$$

In a different but related direction, the p-adic limit of $L(k, n, T)$ when k goes to infinity in a fixed p-adic direction was shown to be a p-adic meromorphic function in [W1]. This idea was the key in proving Dwork's unit root conjecture for the Kloosterman family. See [W1], [W2] and [W3]. To be precise, for a p-adic integer s, we choose a sequence of positive integers k_i which approaches s as p-adic integers but goes to infinity as complex numbers. Then we define the p-adic s-th symmetric product L-function to be

$$L_p(s, n, T) = \lim_{i \to \infty} L(k_i, n, T) \in 1 + T\mathbf{Z}_p[[T]].$$

This limit exists as a formal p-adic power series and is independent of the choice of the sequence k_i. It is a sort of two variable p-adic L-function. Note that even when s is a positive integer, $L_p(s, n, T)$ is very different from $L(s, n, T)$. It was shown in [W1] that $L_p(s, n, T)$ is a p-adic meromorphic function by a uniform limiting argument. Alternatively, it was shown in [W2] that

$$L_p(s, n, T) = L(M_s(\infty), T),$$

where $M_s(\infty)$ is an infinite rank nuclear overconvergent σ-module on $\mathbf{A}^1_{\mathbf{F}_q} - \{0\}$. This gives another proof that $L_p(s, n, T)$ is p-adic meromorphic. Combining the above results on trivial factors of $L(k, n, T)$ with the p-adic limiting argument in [W1], we prove the following more precise result.

Theorem 0.4. Let d_j be the coefficient of x^j in the power series expansion of

$$\frac{1}{(1-x^2)(1-x^3)\cdots(1-x^{n-1})}.$$

For each p-adic integer s, we have the factorization

$$L_p(s,n,T) = A_p(s,n,T)\prod_{i=0}^{\infty}(1-q^iT)^{d_i},$$

where $A_p(s,n,T)$ is a p-adically entire function, (i.e., it has no poles). In particular, the p-adic series $L_p(s,n,T)$ is p-adically entire, and it has a zero at $T = q^{-j}$ with multiplicity at least d_j for each non-negative integer j.

We thus obtain infinitely many trivial zeros (if $n > 2$) for the p-adic s-th symmetric product L-function $L_p(s,n,T)$. This suggests that there should be an interesting trivial zero theory for the L-function of any p-adic symmetric product of a pure l-adic sheaf whose p-adic unit root part has rank one. Our result here provides the first evidence for such a theory.

Remark 0.5.. Grosse-Klönne [GK] showed the p-adic meromorphic continuation of $L_p(s,n,T)$ to some $s \in \mathbf{Q}_p$ with $|s|_p < 1+\epsilon$ for some small $\epsilon > 0$. We do not know if Theorem 0.4 can be extended to such non-integral p-adic s.

The paper is organized as follows. In §1, we recall the canonical form of the local monodromy of the Kloosterman sheaf at 0. In §2, we summarize the basic representation theory for $\mathfrak{sl}(2)$. In §3, we prove Theorem 0.1 using results in the previous two sections. In §4, we use Theorem 0.1 and a p-adic limiting argument to prove Theorem 0.4. In section 5, we derive some consequences for the non-trivial factors and its variation with k. In the appendix, we sketch a proof of Lemma 0.2 which implies Theorem 0.3.

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1. The Canonical Form of the Local Monodromy

Let K be a local field with residue field $\mathbf{F}_q$, and let

$$\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(V)$$

be a $\overline{\mathbf{Q}}_l$-representation. Suppose the inertia subgroup I of $\mathrm{Gal}(\overline{K}/K)$ acts unipotently on V. Fix a uniformizer π of K, and consider the l-adic part of the cyclotomic character

$$t_l : I \to \mathbf{Z}_l(1),\ \sigma \mapsto \left(\frac{\sigma(\sqrt[l^n]{\pi})}{\sqrt[l^n]{\pi}}\right).$$

Note that for σ in the inertia group, the l^n-th root of unity $\frac{\sigma(\sqrt[l^n]{\pi})}{\sqrt[l^n]{\pi}}$ does not depends on the choice of the l^n-th root $\sqrt[l^n]{\pi}$ of π. Since the restriction to I is unipotent, there exists a nilpotent homomorphism

$$N : V(1) \to V$$

such that

$$\rho(\sigma) = \exp(t_l(\sigma).N)$$

for any $\sigma \in I$. Fix a lifting $F \in \mathrm{Gal}(\overline{K}/K)$ of the geometric Frobenius element in $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_q)$. We have

$$t_l(F^{-1}\sigma F) = t_l(\sigma)^q.$$

So

$$\begin{aligned}
\exp(t_l(\sigma).N)\rho(F) &= \rho(\sigma)\rho(F) \\
&= \rho(\sigma F) \\
&= \rho(FF^{-1}\sigma F) \\
&= \rho(F)\rho(F^{-1}\sigma F) \\
&= \rho(F)\exp(t_l(F^{-1}\sigma F).N) \\
&= \rho(F)\exp(qt_l(\sigma).N).
\end{aligned}$$

Therefore

$$\rho(F)^{-1}\exp(t_l(\sigma).N)\rho(F) = \exp(qt_l(\sigma).N).$$

Hence

$$\rho(F)^{-1}(t_l(\sigma).N)\rho(F) = qt_l(\sigma).N.$$

Fix a generator ζ of $\mathbf{Z}_l(1)$. Choose $\sigma \in I$ so that $t_l(\sigma) = \zeta$. For convenience, denote $\rho(F)$ by F, and denote the homomorphism

$$V \to V, v \mapsto N(v \otimes \zeta)$$

by N. Then the last equation gives

$$F^{-1}NF = qN,$$

that is,

$$NF = qFN.$$

Now we take K to be the completion of $\mathbf{F}_q(T)$ at 0, let $V = (\mathrm{Kl}_n)_{\bar\eta}$, and let $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(V)$ be the restriction of the representation $\mathrm{Kl}_n : \mathrm{Gal}(\overline{\mathbf{F}_q(T)}/\mathbf{F}_q(T)) \to \mathrm{GL}((\mathrm{Kl}_n)_{\bar\eta})$ defined by the Kloosterman sheaf. In [D1] Théorème 7.8, it is shown that the inertia subgroup I_0 at 0 acts unipotently on $(\mathrm{Kl}_n)_{\bar\eta}$ with a single Jordan block, and the geometric Frobenius F_0 at 0 acts trivially on the invariant $((\mathrm{Kl}_n)_{\bar\eta})^{I_0}$ of the inertia subgroup. With the above notations, this means the nilpotent map N has a single Jordan block, and F acts trivially on $\ker(N)$. By [D2] 1.6.14.2 and 1.6.14.3, the eigenvalues of F are $1, q, \ldots, q^{n-1}$. Let v be a (nonzero) eigenvector of F with eigenvalue q^{n-1}. Using the equation $NF = qFN$, we see $N(v)$ is an eigenvector of F with eigenvalue q^{n-2}. Note that if $n \geq 2$, then $N(v)$ can not be 0. Otherwise v lies in $\ker(N)$ but F does not act trivially on v. This contradicts to the fact that F acts trivially on $\ker(N)$. Similarly, if $n \geq 3$, then $N^2(v)$ is a nonzero eigenvector of F with eigenvalue q^{n-3}, …, and $N^{n-1}(v)$ is a nonzero eigenvector of F with eigenvalue 1, and $N^n(v) = 0$. As $v, N(v), \ldots, N^{n-1}(v)$ are nonzero eigenvectors of F with distinct eigenvalues, they are linearly independent and form a basis of V. We summarize the above results as follows.

Proposition 1.1. Notation as above. For the triple (V, F, N) defined by the Kloosterman sheaf, there exists a basis $e_0, \ldots, e_{n-1}$ of V such that

$$F(e_0) = e_0,\ F(e_1) = qe_1,\ \ldots,\ F(e_{n-1}) = q^{n-1}e_{n-1}$$

and

$$N(e_0) = 0,\ N(e_1) = e_0,\ \ldots,\ N(e_{n-1}) = e_{n-2}.$$

2. Representation of $\mathfrak{sl}(2)$

In this section, we summarize the representation theory of the Lie algebra $\mathfrak{sl}(2)$ of traceless matrices over the field $\overline{\mathbf{Q}}_l$. Denote

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The following result is standard. (See, for example, [FH] §11.1.)

Proposition 2.1. Let V be a finite dimensional irreducible $\overline{\mathbf{Q}}_l$-representation of $\mathfrak{sl}(2)$. Then there exists a (nonzero) eigenvector v of H such that $Xv = 0$. Such a vector is called a *highest weight vector* for the representation V. Let $n = \dim(V) - 1$. For any highest weight vector v, we have

$$Hv = nv.$$

We call n the *weight* of the representation. Moreover, the set $\{v, Yv, \ldots Y^n v\}$ is a basis of V, and we have

$$\begin{aligned} H(Y^i v) &= (n-2i)Y^i v \ (i = 0, 1, \ldots, n), \\ X(Y^i v) &= i(n-i+1)Y^{i-1} v \ (i = 0, 1, \ldots, n), \\ Y(Y^i v) &= Y^{i+1} v \ (i = 0, 1, \ldots, n-1), \\ Y(Y^n v) &= 0. \end{aligned}$$

Remark 2.2. The trivial representation $V_0 = \overline{\mathbf{Q}}_l$ of $\mathfrak{sl}(2)$ is the irreducible representation of weight 0. Let $V_1 = \overline{\mathbf{Q}}_l^2$ be the standard representation of $\mathfrak{sl}(2)$ on which $\mathfrak{sl}(2)$ acts as the multiplication of matrices on column vectors. It is the irreducible representation of weight 1, and $f_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a highest weight vector. Let $V_n = \mathrm{Sym}^n(V_1)$ be the n-th symmetric product of V_1. It is the irreducible representation of weight n, and f_0^n is a highest weight vector.

Let V_n be the irreducible representation of $\mathfrak{sl}(2)$ of weight n. Note that the eigenvalues $n, n-2, n-4, \ldots, -n$ of H form an unbroken arithmetic progression of integers with difference -2, and each eigenvalue has multiplicity 1. Moreover, the space $\ker(X)$ has dimension 1 and coincides with the eigenspace of H corresponding to the eigenvalue n. For any integer w, let V_n^w be the eigenspace of H corresponding to the eigenvalue w. We then have

$$\dim(V_n^w) = \begin{cases} 1 & \text{if } w \equiv n \bmod 2 \text{ and } -n \leq w \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we have

$$\begin{aligned} V_n \cap \ker(X) &= V_n^n, \\ V_n^w \cap \ker(X) &= \begin{cases} V_n^w & \text{if } w = n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In general, any finite dimensional representation V of $\mathfrak{sl}(2)$ is a direct sum of irreducible representations. Let

$$V = m_0 V_0 \oplus m_1 V_1 \oplus \cdots \oplus m_k V_k$$

be the isotypic decomposition of V. For any integer w, let V^w be the eigenspace of H corresponding to the eigenvalue w. If w is non-negative, then we have

$$V^w = m_w V_w^w \oplus m_{w+2} V_{w+2}^w \oplus \cdots$$

and

$$\dim(V^w) = m_w + m_{w+2} + \cdots.$$

Moreover, we have

$$\begin{aligned}\ker(X) &= (m_0V_0 \cap \ker(X)) \oplus (m_1V_1 \cap \ker(X)) \oplus \cdots \oplus (m_kV_k \cap \ker(X)) \\ &= m_0V_0^0 \oplus m_1V_1^1 \oplus \cdots \oplus m_kV_k^k.\end{aligned}$$

and hence

$$\ker(X) \cap V^w = m_wV_w^w.$$

It follows that

$$\ker X = (\ker(X) \cap V^0) \oplus (\ker(X) \cap V^1) \oplus \cdots \oplus (\ker(X) \cap V^k)$$

and

$$\dim(\ker(X) \cap V^w) = m_w = \dim(V^w) - \dim(V^{w+2}).$$

We summarize these results as follows.

Proposition 2.3. Let V be a finite dimensional $\overline{\mathbf{Q}}_l$-representation of $\mathfrak{sl}(2)$. For any integer w, let V^w be the eigenspace of H corresponding to the eigenvalue w. Then we have

$$\ker X = (\ker(X) \cap V^0) \oplus (\ker(X) \cap V^1) \oplus \cdots,$$

and for any non-negative w, we have

$$\dim(\ker(X) \cap V^w) = \dim(V^w) - \dim(V^{w+2}).$$

3. The Local Factor at 0

In this section, we calculate the local factor

$$\det(I - F_0t, (\mathrm{Sym}^k(\mathrm{Kl}_n))^{I_0})$$

at 0 of the L-function of the k-th symmetric product of the Kloosterman sheaf. Let (V, N, F) be the triple defined in §1 corresponding to the Kloosterman sheaf. Then the above local factor is simply

$$\det\left(I - Ft, \ker(N : \mathrm{Sym}^k(V) \to \mathrm{Sym}^k(V))\right).$$

Let $V_1 = \overline{\mathbf{Q}}_l^2$ be the standard representation of $\mathfrak{sl}(2)$. Set

$$f_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; f_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We have

$$H(f_0) = f_0,\ H(f_1) = -f_1,$$
$$X(f_0) = 0,\ X(f_1) = f_0.$$

Let $V_{n-1} = \mathrm{Sym}^{n-1}(V_1)$, and set

$$e_i = \frac{1}{i!} f_0^{n-1-i} f_1^i \ (i = 0, 1, \ldots, n-1).$$

We have

$$H(e_0) = (n-1)e_0,\ H(e_1) = (n-3)e_1,\ \ldots,\ H(e_{n-1}) = -(n-1)e_{n-1}$$

and

$$X(e_0) = 0,\ X(e_1) = e_0,\ \ldots,\ X(e_{n-1}) = e_{n-2}.$$

Comparing with Proposition 1.1, we can identify V_{n-1} with V coming from the triple (V, F, N) defined by the Kloosterman sheaf such that N is identified with X, and the eigenspace of F with eigenvalue q^i is identified with the eigenspace of H with eigenvalue $n - 2i - 1$.

Consider the k-th symmetric product $\mathrm{Sym}^k(V_{n-1})$. It has a basis

$$\{e_0^{i_0} e_1^{i_1} \cdots e_n^{i_{n-1}} | i_j \geq 0, i_0 + i_1 + \cdots + i_{n-1} = k\}.$$

We have

$$H(e_0^{i_0} e_1^{i_1} \cdots e_{n-1}^{i_{n-1}}) = ((n-1) \cdot i_0 + (n-3) \cdot i_1 + \cdots + (-(n-1)) \cdot i_{n-1}) e_0^{i_0} e_1^{i_1} \cdots e_{n-1}^{i_{n-1}}.$$

So $e_0^{i_0} e_1^{i_1} \cdots e_{n-1}^{i_{n-1}}$ is an eigenvector of H with eigenvalue $(n-1) \cdot i_0 + (n-3) \cdot i_1 + \cdots + (-(n-1)) \cdot i_{n-1}$. It is also an eigenvector F with eigenvalue

$$q^{0 \cdot i_0 + 1 \cdot i_1 + \cdots + (n-1) \cdot i_{n-1}} = q^{\frac{1}{2}((n-1)k - ((n-1) \cdot i_0 + (n-3) \cdot i_1 + \cdots + (-(n-1)) \cdot i_{n-1}))}.$$

Here we use the fact that

$$\begin{aligned} & 2(0 \cdot i_0 + 1 \cdot i_1 + \cdots + (n-1) \cdot i_{n-1}) + ((n-1) \cdot i_0 + (n-3) \cdot i_1 + \cdots + (-(n-1)) \cdot i_{n-1}) \\ = \ & (n-1)(i_0 + i_1 + \cdots + i_{n-1}) \\ = \ & k(n-1). \end{aligned}$$

This equality also shows that

$$(n-1) \cdot i_0 + (n-3) \cdot i_1 + \cdots + (-(n-1)) \cdot i_{n-1} \equiv k(n-1) \bmod 2.$$

For each non-negative integer w, let

$$D_k(w) = \{(i_0, \ldots, i_{n-1}) | i_j \geq 0,\ i_0 + i_1 + \cdots + i_{n-1} = k,\ (n-1) \cdot i_0 + (n-3) \cdot i_1 + \cdots + (-(n-1)) \cdot i_{n-1} = w\},$$

and let $d_k(w)$ be the number of elements of $D_k(w)$. We have $d_k(w) = 0$ if $w \not\equiv k(n-1) \bmod 2$ or if $w > k(n-1)$. Note that

$$\{e_0^{i_0} e_1^{i_1} \cdots e_{n-1}^{i_{n-1}} | (i_0, i_1, \ldots, i_{n-1}) \in D_k(w)\}$$

is a basis of the eigenspace $(\mathrm{Sym}^k(V_{n-1}))^w$ of H with eigenvalue w. By Proposition 2.3, we have

$$\ker(X) = \bigoplus_{w=0}^{k(n-1)} \ker(X) \cap (\mathrm{Sym}^k(V_{n-1}))^w$$

and

$$\dim(\ker(X) \cap (\mathrm{Sym}^k(V_{n-1}))^w) = d_k(w) - d_k(w+2).$$

Now $(\mathrm{Sym}^k(V_{n-1}))^w$ is also the eigenspace of F on $\mathrm{Sym}^k(V)$ with eigenvalue $q^{\frac{k(n-1)-w}{2}}$. So we have

$$\det\Big(I - Ft, \ker(N : \mathrm{Sym}^k(V) \to \mathrm{Sym}^k(V))\Big) = \prod_{w=0}^{k(n-1)} (1 - q^{\frac{k(n-1)-w}{2}} t)^{d_k(w) - d_k(w+2)}.$$

As $d_k(w) = 0$ if $w \not\equiv k(n-1) \bmod 2$ or if $w > k(n-1)$, we have

$$\det\Big(I - Ft, \ker(N : \mathrm{Sym}^k(V) \to \mathrm{Sym}^k(V))\Big) = \prod_{u=0}^{[\frac{k(n-1)}{2}]} (1 - q^u t)^{d_k(k(n-1)-2u) - d_k(k(n-1)-2u+2)}.$$

Set

$$c_k(u) = d_k(k(n-1) - 2u)$$

so that we have

$$\det\Big(I - Ft, \ker(N : \mathrm{Sym}^k(V) \to \mathrm{Sym}^k(V))\Big) = \prod_{u=0}^{[\frac{k(n-1)}{2}]} (1 - q^u t)^{c_k(u) - c_k(u-1)}.$$

In the following, we find an expression for $c_k(u) - c_k(u-1)$.

Note that $c_k(u)$ is the number of elements of the set

$$\{(i_0, \ldots, i_{n-1}) | i_j \geq 0,\ i_0 + i_1 + \cdots + i_{n-1} = k,\ 0 \cdot i_0 + 1 \cdot i_1 + \cdots + (n-1) \cdot i_{n-1} = u\}.$$

Taking power series expansion, we get

$$\frac{1}{(1-y)(1-xy)\cdots(1-x^{n-1}y)} = \sum_{k=0}^{\infty} \sum_{u=0}^{\infty} c_k(u) x^u y^k.$$

Since

$$(1-y)\frac{1}{(1-y)(1-xy)\cdots(1-x^{n-1}y)} = (1-x^n y)\frac{1}{(1-xy)\cdots(1-x^n y)},$$

we have

$$(1-y)(\sum_{k,u} c_k(u) x^u y^k) = (1-x^n y)(\sum_{k,u} c_k(u) x^u (xy)^k),$$

that is,

$$\sum_{k,u} c_k(u)x^u y^k - \sum_{k,u} c_k(u)x^u y^{k+1} = \sum_{k,u} c_k(u)x^{u+k}y^k - \sum_{k,u} c_k(u)x^{n+u+k}y^{k+1}.$$

Comparing the coefficients of y^k, we get

$$\sum_u c_k(u)x^u - \sum_u c_{k-1}(u)x^u = (\sum_u c_k(u)x^u)x^k - (\sum_u c_{k-1}(u)x^u)x^{n+k-1},$$

that is,

$$\sum_u c_k(u)x^u = \frac{1-x^{n+k-1}}{1-x^k}\sum_u c_{k-1}(u)x^u.$$

Applying this expression repeatedly, we get

$$\begin{aligned}\sum_u c_k(u)x^u &= \frac{1-x^{n+k-1}}{1-x^k}\sum_u c_{k-1}(u)x^u \\ &= \frac{(1-x^{n+k-2})(1-x^{n+k-1})}{(1-x^{k-1})(1-x^k)}\sum_u c_{k-2}(u)x^u \\ &= \cdots \\ &= \frac{(1-x^n)\cdots(1-x^{n+k-2})(1-x^{n+k-1})}{(1-x)\cdots(1-x^{k-1})(1-x^k)}.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_u (c_k(u)-c_k(u-1))x^u &= \sum_u c_k(u)x^u - x\sum_u c_k(u)x^u \\ &= (1-x)\sum_u c_k(u)x^u \\ &= (1-x)\frac{(1-x^n)\cdots(1-x^{n+k-2})(1-x^{n+k-1})}{(1-x)\cdots(1-x^{k-1})(1-x^k)} \\ &= \frac{(1-x^n)\cdots(1-x^{n+k-2})(1-x^{n+k-1})}{(1-x^2)\cdots(1-x^{k-1})(1-x^k)}.\end{aligned}$$

So $c_k(u)-c_k(u-1)$ is the coefficients of x^u in the power series expansion of $\frac{(1-x^n)\cdots(1-x^{n+k-2})(1-x^{n+k-1})}{(1-x^2)\cdots(1-x^{k-1})(1-x^k)}$. This proves Theorem 0.1 in the introduction.

4. p-adic Symmetric Product L-functions

Let s be a p-adic integer. Recall that the p-adic s-th symmetric product L-function is the p-adic limit

$$L_p(s,n,T) = \lim_{i\to\infty} L(\mathrm{Sym}^{k_i}(\mathrm{Kl}_n),T),$$

where k_i is any sequence of positive integers going to infinity as complex numbers and approaching to s as p-adic integers. Since Kl_n is pure of weight $n-1$, for each positive integer k_i, Grothendieck's formula for L-functions implies that we can write

$$L(k_i,n,T) := L(\mathbf{A}^1_{\mathbf{F}_q} - \{0\}, \mathrm{Sym}^{k_i}(\mathrm{Kl}_n), T) = \frac{P(k_i,n,T)}{((1-q^{(n-1)k_i/2}T)(1-q^{((n-1)k_i+2)/2}T))^{e_i}},$$

where

$$P(k_i,n,T) = \det(1-FT, H^1(\mathbf{P}^1_{\mathbf{F}}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n))))\det(1-F_0T, ((\mathrm{Sym}^k\mathrm{Kl}_n)_{\bar\eta})^{I_0})\det(1-F_\infty T, ((\mathrm{Sym}^k\mathrm{Kl}_n)_{\bar\eta})^{I_\infty}),$$

and e_i is the multiplicity of the geometrically trivial representation in $\mathrm{Sym}^{k_i}(\mathrm{Kl}_n)$. In fact, by Theorem 0.3, we know that $e_i = 0$ unless $p = 2$, k_i even and n odd, in which case we have $e_i = 1$. Taking the limit, we deduce that

$$L_p(s,n,T) = \lim_{i\to\infty} P(k_i,n,T).$$

Fix a positive integer r. By the results in [W1] (Theorem 5.7 and Lemma 5.10), the number of zeros and poles of the L-function $L(k_i,n,T)$ as k_i varies is uniformly bounded in the disk $|T|_p < p^r$. In particular, the number of zeros of the polynomial $P(k_i,n,T)$ (the numerator of $L(k_i,n,T)$) as k_i varies is uniformly bounded in the disk $|T|_p < p^r$. Under the condition $k \geq n$, we have

$$\frac{(1-x^n)\cdots(1-x^{n+k-2})(1-x^{n+k-1})}{(1-x^2)\cdots(1-x^{k-1})(1-x^k)} = \frac{(1-x^{k+1})\cdots(1-x^{n+k-2})(1-x^{n+k-1})}{(1-x^2)\cdots(1-x^{n-2})(1-x^{n-1})}.$$

It follows that $m_k(j) = d_j$ for $j \leq k$, where $m_k(j)$ is defined in Theorem 0.1, and d_j is defined in Theorem 0.4. So we have $m_{k_i}(j) = d_j$ for all $1 \leq j \leq r$ provided that $k_i \geq \max(r,n)$. Then by Theorem 0.1, we can write

$$P(k_i,n,T) = B_r(k_i,n,T)\prod_{j=0}^{r}(1-q^jT)^{d_j},$$

where $B_r(k_i,n,T) \in 1+T\mathbf{Z}[T]$ is a polynomial in T. Furthermore, the number of the zeros of $B_r(k_i,n,T)$ in the disk $|T|_p < p^r$ is uniformly bounded as k_i varies. This implies that the limit

$$C_r(s,n,T) := \lim_{i\to\infty} B_r(k_i,n,T) = \frac{L_p(s,n,T)}{\prod_{j=0}^{r}(1-q^jT)^{d_j}}$$

exists and is p-adically analytic in the disk $|T|_p < p^r$. In particular, $L_p(s,n,T)$ is p-adically analytic in the disk $|T|_p < p^r$ and has a zero at $T = q^{-j}$ with multiplicity at least d_j for $0 \leq j \leq r$. As we can take r to be an arbitrarily large integer, we deduce that

$$A_p(s,n,T) := \lim_{r\to\infty} C_r(s,n,T) = \frac{L_p(s,n,T)}{\prod_{j=0}^{\infty}(1-q^jT)^{d_j}}$$

is p-adically entire. This proves Theorem 0.4.

Note that Theorem 2.5 in [FW] shows that for $(n, p) = 1$, the limit of the local factors at infinity disappears and hence has no contribution to the zeros of p-adic s-th symmetric product L-function. This together with the above proof implies that

$$A_p(s, n, T) = \lim_{i \to \infty} \det(1 - FT, H^1(\mathbf{P}^1_{\mathbf{F}}, j_*(\mathrm{Sym}^{k_i}(\mathrm{Kl}_n)))),$$

that is, $A_p(s, n, T)$ is the p-adic limit of the non-trivial factor of $L(\mathbf{A}^1_{\mathbf{F}_q} - \{0\}, \mathrm{Sym}^{k_i}(\mathrm{Kl}_n), T)$ as k_i approaches to s. It is a p-adic entire function. Its zeros are called non-trivial zeros of $L_p(s, n, T)$. Some partial results on the distribution of the zeros of $L_p(s, n, T)$ were obtained in [W2].

Remark. The same proof shows that the entireness property for $L_p(s, n, T)$ can be extended to any p-adic s-th symmetric product L-function of a lisse pure positive weight l-adic sheaf whose p-adic unit part has rank one. The Kloosterman sheaf is just the first such example. The ordinary family of Calabi-Yau hypersurfaces is another important example, see [RW] for a complete treatment.

5. Variation of the non-trivial factor

In this section, we derive some consequences for the non-trivial factor

$$K_q(k, n, T) := \det(1 - FT, H^1(\mathbf{P}^1_{\mathbf{F}}, j_*(\mathrm{Sym}^k(\mathrm{Kl}_n)))) \in 1 + T\mathbf{Z}[T].$$

This is a polynomial with integer coefficients, pure of weight $k(n-1)+1$. Its degree can be computed explicitly by the degree formula for $L(k, n, T)$ (Theorem 0.1 in [FW]) and the degree formulas for the trivial factors of $L(k, n, T)$ as implicit in Theorem 2.5 in [FW], Theorem 0.1 and Theorem 0.3.

In the simplest case $n = 2$ and $q = p$, the polynomial $K_p(k, n, T)$ is the Kloosterman analogue of the p-th Hecke polynomial acting on weight $k+2$ modular forms. It would be interesting to understand how the polynomial $K_p(k, n, T)$ varies as p varies while k is fixed or as k varies while p is fixed.

For fixed k and n, the polynomial $K_p(k, n, T)$ should be the p-th Euler factor of a motive $M_{k,n}$ over $\mathbf{Q}$. It would be interesting to construct explicitly this motive (its underlying scheme) or its corresponding compatible system of Galois representation or its automorphic interpretation. In the case when $n = 2$ and $k \leq 4$, it is easy, see [CE]. In the case $n = 2$ and $k = 5, 6$, the polynomial $K_p(k, n, T)$ has degree 2 and is known to be the Euler factor at p of an explicit modular form, see [PTV] for the case $k = 5$ and [HS] for the case $k = 6$.

Just like the case for $L(k, n, T)$, we are interested in how the polynomial $K_p(k, n, T)$ varies as k varies p-adically. The first simple result is a p-adic continuity result.

Proposition 5.1. Let k_1, k_2 and k_3 be positive integers such that $k_1 = k_2 + p^m k_3$ with k_1 not divisible by p. Then we have the congruence

$$K_p(k_1, n, T) \equiv K_p(k_2, n, T) (\text{mod } p^{\min(m, k_2/2)}).$$

Proof: Let $q = p$. The Frobenius eigenvalues of the Kloosterman sheaf at each closed point are all divisible by p except for exactly one eigenvalue which is a p-adic 1-unit (i.e., congruent to 1 modulo p). From this and the Euler product definition of the L-function $L(k, n, T)$, we deduce the slightly stronger congruence:

$$L(k_1, n, T) \equiv L(k_2, n, T) (\text{mod } p^{\min(m, k_2)}).$$

To prove the proposition, it remains to check that the same congruence in the proposition holds for the trivial factors. This follows from the explicit results stated in Theorem 2.5 in [FW], Theorem 0.1 and Theorem 0.3.

Let s be a p-adic integer. Choose a sequence of positive integers k_i going to infinity as complex numbers and approaching s as p-adic integers. The above congruence for $K_p(k, n, T)$ implies that the limit

$$A_p(s, n, T) := \lim_{i \to \infty} K_p(k_i, n, T)$$

exists and it is exactly the non-trivial factor $A_p(s, n, T)$ in Theorem 0.4. It follows that $A_p(s, n, T)$ is a p-adic entire function. It would be interesting to determine the p-adic Newton polygon of the entire function $A_p(s, n, T)$. This would give exact information on the distribution of the zeros of $A_p(s, n, T)$.

The rigid analytic curve in the (s, T) plane defined by the equation $A_p(s, n, T) = 0$ is the Kloosterman sum analogue of the eigencurve in the theory of p-adic modular forms studied by Coleman-Mazur [CM]. It would be interesting to study the properties of the rigid analytic curve $A_p(s, n, T) = 0$ and its relation to p-adic automorphic forms.

6. Appendix

In this section, we sketch a proof of Lemma 0.2 in the introduction. The main reference for this section is [FH].

First recall the dimension formula for irreducible representations of simple Lie algebras. Let $\mathfrak{g}$ be a simple Lie algebra (over $\overline{\mathbf{Q}}_l$). Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and let R be the set of roots. We have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \bigoplus (\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha).$$

For each $\alpha \in R$, let H_α be the unique element in $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\alpha(H_\alpha) = 2$. The weight lattice Λ_W is the lattice in $\mathfrak{h}^*$ generated by those linear functionals β with the property $\beta(H_\alpha) \in \mathbf{Z}$ for all $\alpha \in R$. Fix an ordering of R. Let R^+ be the set of positive roots, and let $\mathcal{W}$ be the Weyl chamber. Set

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha.$$

For any $\lambda \in \Lambda_W \cap \mathcal{W}$, the dimension of the irreducible representation Γ_λ with highest weight λ is given by

$$\dim(\Gamma_\lambda) = \prod_{\alpha \in R^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} = \prod_{\alpha \in R^+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)},$$

where $(\ ,\ )$ is the Killing form on $\mathfrak{h}^*$, and

$$\langle \beta, \alpha \rangle = \beta(H_\alpha) = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$$

for any $\beta \in \mathfrak{h}^*$ and $\alpha \in R$. See [FH] Corollary 24.6.

For each pair $1 \leq i, j \leq n$, let E_{ij} be the $(n \times n)$-matrix whose only nonzero entry is on the i-th row and j-th column, and this nonzero entry is 1. For each $1 \leq i \leq n$, let L_i be the linear functional on the space of diagonal matrices with the property

$$L_i(E_{jj}) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Consider the Lie algebra $\mathfrak{sl}(n)$ of traceless $(n \times n)$-matrices. Let $\mathfrak{h}$ be the space of diagonal matrices in $\mathfrak{sl}(n)$. It is a Cartan subalgebra of $\mathfrak{sl}(n)$. The set of roots of $\mathfrak{sl}(n)$ are

$$R = \{L_i - L_j | i \neq j\}$$

and

$$H_{L_i - L_j} = E_{ii} - E_{jj} \ (i \neq j).$$

Choose an ordering of roots so that

$$R^+ = \{L_i - L_j | i < j\}$$

is the set of the positive roots. We have

$$\rho = \sum_{i=1}^{n} (n - i) L_i.$$

(To deduce this formula, we use the fact that $L_1 + \cdots + L_n = 0$ for $\mathfrak{sl}(n)$.) By the dimension formula, for any

$$\lambda = \lambda_1 L_1 + \cdots + \lambda_n L_n$$

lying in the intersection of the weight lattice and the Weyl chamber, the dimension of the irreducible representation Γ_λ of $\mathfrak{sl}(n)$ with highest weight λ is

$$\begin{aligned}\dim(\Gamma_\lambda) &= \prod_{\alpha\in R^+} \frac{\langle\lambda+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle} \\ &= \prod_{i<j} \frac{(\lambda+\rho)(E_{ii}-E_{jj})}{\rho(E_{ii}-E_{jj})} \\ &= \prod_{i<j} \frac{(\sum_i(\lambda_i+(n-i)L_i))(E_{ii}-E_{jj})}{(\sum_i(n-i)L_i)(E_{ii}-E_{jj})} \\ &= \prod_{i<j} \frac{\lambda_i-\lambda_j+j-i}{j-i}\end{aligned}$$

In the case where $\lambda = kL_1$, we have

$$\lambda_i = \begin{cases} k & \text{if } i=1, \\ 0 & \text{if } i\geq 2. \end{cases}$$

So we have

$$\dim(\Gamma_{kL_1}) = \prod_{1<j} \frac{k+j-1}{j-1} = \begin{pmatrix} k+n-1 \\ n-1 \end{pmatrix}.$$

Note that the dimension of Γ_{kL_1} is exactly the dimension of $\mathrm{Sym}^k(V)$, where V is the standard representation of $\mathfrak{sl}(n)$. Since the weights of V are $L_1,\ldots,L_n$, the representation $\mathrm{Sym}^k(V)$ has a highest weight kL_1. So we must have

$$\mathrm{Sym}^k(V) = \Gamma_{kL_1}.$$

In particular, $\mathrm{Sym}^k(V)$ is irreducible.

Now suppose $n = 2m$ is an even number and consider the Lie algebra $\mathfrak{sp}(n)$ of matrices of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, D are $(m\times m)$-matrices, B and C are symmetric and $A^t + D = 0$. Let $\mathfrak{h}$ be the space of diagonal matrices in $\mathfrak{sp}(n)$. It is a Cartan subalgebra of $\mathfrak{sp}(n)$. The set of roots of $\mathfrak{sp}(n)$ are

$$R = \{\pm L_i \pm L_j | 1 \leq i,j \leq m\} - \{0\}.$$

Choose an ordering of roots so that

$$R^+ = \{L_i - L_j | i<j\} \cup \{L_i + L_j | i \leq j\}$$

is the set of the positive roots. Using the dimension formula, one can show that for any

$$\lambda = \lambda_1 L_1 + \cdots + \lambda_m L_m$$

lying in the intersection of the weight lattice and the Weyl chamber, the dimension of the irreducible representation Γ_λ of $\mathfrak{sp}(n)$ with highest weight λ is

$$\dim(\Gamma_\lambda) = \prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{i<j} \frac{\lambda_i + \lambda_j + 2m + 2 - i - j}{2m + 2 - i - j} \prod_{i} \frac{\lambda_i + m + 1 - i}{m + 1 - i}.$$

From this formula, we deduce

$$\dim(\Gamma_{kL_1}) = \begin{pmatrix} k + 2m - 1 \\ 2m - 1 \end{pmatrix}.$$

Note that the dimension of Γ_{kL_1} is exactly the dimension of $\mathrm{Sym}^k(V)$, where V is the standard representation of $\mathfrak{sp}(n)$. As above, this implies

$$\mathrm{Sym}^k(V) = \Gamma_{kL_1}.$$

In particular, $\mathrm{Sym}^k(V)$ is irreducible.

Now consider the cases where $\mathfrak{g} = \mathfrak{so}(n)$ or $\mathfrak{g}_2$. In these cases, there is a symmetric non-degenerate $\mathfrak{g}$-invariant bilinear form $Q(\ ,\ )$ on V. Consider the contraction map

$$\begin{array}{rcl} \mathrm{Sym}^k(V) & \to & \mathrm{Sym}^{k-2}(V), \\ v_1 \cdots v_k & \mapsto & \displaystyle\sum_{i<j} Q(v_i, v_j) v_1 \cdots \hat{v}_i \cdots \hat{v}_j \cdots v_k. \end{array}$$

It is an epimorphism of representations of $\mathfrak{g}$. We will show the kernel of the contraction map is irreducible.

First consider the case where $n = 2m$ is even, and the Lie algebra is $\mathfrak{so}(n)$ of matrices of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, D are $(m \times m)$-matrices, B and C are skew-symmetric and $A^t + D = 0$. Let $\mathfrak{h}$ be the space of diagonal matrices in $\mathfrak{so}(n)$. It is a Cartan subalgebra of $\mathfrak{so}(n)$. The set of roots of $\mathfrak{so}(n)$ are

$$R = \{\pm L_i \pm L_j | 1 \leq i, j \leq m,\ i \neq j\}.$$

Choose an ordering of roots so that

$$R^+ = \{L_i - L_j | i < j\} \cup \{L_i + L_j | i < j\}$$

is the set of the positive roots. Using the dimension formula, one can show that for any

$$\lambda = \lambda_1 L_1 + \cdots + \lambda_m L_m$$

lying in the intersection of the weight lattice and the Weyl chamber, the dimension of the irreducible representation Γ_λ of $\mathfrak{so}(n)$ with highest weight λ is

$$\dim(\Gamma_\lambda) = \prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{i<j} \frac{\lambda_i + \lambda_j + 2m - i - j}{2m - i - j}.$$

From this formula, we deduce

$$\dim(\Gamma_{kL_1}) = \frac{(k+m-1)(k+2m-3)!}{(m-1)(2m-3)!k!}.$$

Note that

$$\dim(\Gamma_{kL_1}) = \binom{k+2m-1}{k} - \binom{k+2m-3}{k-2} = \dim(\mathrm{Sym}^k(V)) - \dim(\mathrm{Sym}^{k-2}(V)).$$

Since the contraction map $\mathrm{Sym}^k(V) \to \mathrm{Sym}^{k-2}(V)$ is surjective, and its kernel has a highest weight kL_1, it follows that Γ_{kL_1} coincides with the kernel of the contraction map. So we must have

$$\mathrm{Sym}^k(V) = \Gamma_{kL_1} \oplus \mathrm{Sym}^{k-2}(V).$$

Using this expression repeatedly, we get

$$\mathrm{Sym}^k(V) = \bigoplus_{i=0}^{[\frac{k}{2}]} \Gamma_{(k-2i)L_1}.$$

In particular, when k is even, $\mathrm{Sym}^k(V)$ contains one copy of the trivial representation, and when k is odd, it contains no trivial representation.

Next consider the case where $n = 2m+1$ is odd, and the Lie algebra is $\mathfrak{so}(n)$ of matrices of the form

$$\begin{pmatrix} A & B & E \\ C & D & F \\ G & H & 0 \end{pmatrix},$$

where A, B, C, D are $(m \times m)$-matrices, E and F are $(m \times 1)$-matrices, G and H are $(1 \times m)$-matrices, B and C are skew-symmetric, $A^t + D = 0$, $E^t + H = 0$, and $F^t + G = 0$. Let $\mathfrak{h}$ be the space of diagonal matrices in $\mathfrak{so}(n)$. It is a Cartan subalgebra of $\mathfrak{so}(n)$. The set of roots of $\mathfrak{so}(n)$ are

$$R = \{\pm L_i \pm L_j | 1 \le i, j \le m,\ i \ne j\} \cup \{\pm L_i | 1 \le i \le m\}.$$

Choose an ordering of roots so that

$$R^+ = \{L_i - L_j | i < j\} \cup \{L_i + L_j | i < j\} \cup \{L_i\}$$

is the set of the positive roots. Using the dimension formula, one can show that for any

$$\lambda = \lambda_1 L_1 + \cdots + \lambda_m L_m$$

lying in the intersection of the weight lattice and the Weyl chamber, the dimension of the irreducible representation Γ_λ of $\mathfrak{so}(n)$ with highest weight λ is

$$\dim(\Gamma_\lambda) = \prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{i<j} \frac{\lambda_i + \lambda_j + 2m + 1 - i - j}{2m + 1 - i - j} \prod_i \frac{\lambda_i + m + \frac{1}{2} - i}{m + \frac{1}{2} - i}.$$

From this formula, we deduce

$$\begin{aligned}\dim(\Gamma_{kL_1}) &= \frac{(2k+2m-1)(k+2m-2)!}{(2m-1)!k!} \\ &= \begin{pmatrix} k+2m \\ k \end{pmatrix} - \begin{pmatrix} k+2m-2 \\ k-2 \end{pmatrix} \\ &= \dim(\mathrm{Sym}^k(V)) - \dim(\mathrm{Sym}^{k-2}(V)).\end{aligned}$$

The same argument as before shows that

$$\mathrm{Sym}^k(V) = \bigoplus_{i=0}^{[\frac{k}{2}]} \Gamma_{(k-2i)L_1}.$$

In particular, when k is even, $\mathrm{Sym}^k(V)$ contains one copy of the trivial representation, and when k is odd, it contains no trivial representation.

Finally let $n = 7$ and consider the Lie algebra $\mathfrak{g}_2$. The following points on the real plane form the root system R of $\mathfrak{g}_2$:

$$\begin{aligned}&\alpha_1 = (1,0) \\ &\alpha_2 = (\frac{3}{2}, \frac{\sqrt{3}}{2}), \\ &\alpha_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2}), \\ &\alpha_4 = (0, \sqrt{3}), \\ &\alpha_5 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}), \\ &\alpha_6 = (-\frac{3}{2}, \frac{\sqrt{3}}{2}), \\ &\beta_1 = -\alpha_1,\ \beta_2 = -\alpha_2,\ \beta_3 = -\alpha_3,\ \beta_4 = -\alpha_4,\ \beta_5 = -\alpha_5,\ \beta_6 = -\alpha_6.\end{aligned}$$

Moreover, the Killing form induces the canonical inner product on the real plane spanned by the roots. Choose an order on R so that α_i $(i = 1, \ldots, 6)$ are the positive roots. The Weyl chamber $\mathcal{W}$ is the positive cone generated by α_3 and α_4, and the weight lattice Λ_W is the lattice generated by α_1 and α_6. Any element in $\Lambda_W \cap \mathcal{W}$ is of the form

$$\lambda = a\alpha_3 + b\alpha_4 = \left(\frac{1}{2}a, \frac{\sqrt{3}}{2}a + \sqrt{3}b\right),$$

where a and b are non-negative integers. Using the dimension formula, one can show that the dimension of the irreducible representation Γ_λ with highest weight $\lambda = a\alpha_3 + b\alpha_4$ is

$$\dim(\Gamma_\lambda) = \frac{(a+1)(a+b+2)(2a+3b+5)(a+2b+3)(a+3b+4)(b+1)}{120}.$$

In particular, the dimension of the irreducible representation Γ_{α_3} is

$$\dim(\Gamma_{\alpha_3}) = \frac{2 \cdot 3 \cdot 7 \cdot 4 \cdot 5 \cdot 1}{120} = 7.$$

So Γ_{α_3} is the standard representation V. The dimension of the irreducible representation $\Gamma_{k\alpha_3}$ is

$$\begin{aligned}\dim(\Gamma_{k\alpha_3}) &= \frac{(k+1)(k+2)(2k+5)(k+3)(k+4)}{120} \\ &= \binom{k+6}{6} - \binom{k+4}{6} \\ &= \dim(\mathrm{Sym}^k(V)) - \dim(\mathrm{Sym}^{k-2}(V)).\end{aligned}$$

The same argument as before shows that

$$\mathrm{Sym}^k(V) = \bigoplus_{i=0}^{[\frac{k}{2}]} \Gamma_{(k-2i)\alpha_3}.$$

In particular, when k is even, $\mathrm{Sym}^k(V)$ contains one copy of the trivial representation, and when k is odd, it contains no trivial representation. This finishes the proof of the proposition.

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