L-Functions of Function Fields

Daqing Wan

Department of Mathematics
University of California
Irvine, CA 92697-3875
dwan@math.uci.edu

Abstract

This is a short expository paper on L-functions of function fields, based on the author's lecture given at the fourth China-Japan number theory conference held in Weihai.

1 Introduction

Let $\mathbb{F}_q$ be a finite field of q elements with characteristic p. Let C be a smooth projective geometrically connected curve defined over $\mathbb{F}_q$ with function field K. Let U be a Zariski open dense subset of C with inclusion map $j : U \hookrightarrow C$. Let $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ denote the absolute Galois group of K. For example, we can take $C = \mathbb{P}^1$, $U = \mathbb{P}^1 - \{0, \infty\}$ and $K = \mathbb{F}_q(t)$.

Let $\pi_1^{\mathrm{arith}}(U)$ denote the arithmetic fundamental group of U. That is,

$$\pi_1^{\mathrm{arith}}(U) = G_K / < I_x >_{x \in |U|},$$

where the denominator denotes the closed normal subgroup generated by the inertial subgroups I_x as x runs over the closed points $|U|$ of U. Let D_x denote the decomposition group of G_K at x. One has the following exact sequence

$$1 \to I_x \to D_x \to \mathrm{Gal}(\bar{k}_x/k_x) \to 1,$$

where k_x denotes the residue field of K at x. The Galois group $\mathrm{Gal}(\bar{k}_x/k_x)$ is topologically generated by the geometric Frobenius element Frob_x:

$$\mathrm{Frob}_x^{-1} : \alpha \to \alpha^{\# k_x}.$$

Let P_x denote the p-Sylow subgroup of I_x. Then we have the following exact sequence

$$1 \rightarrow P_x \rightarrow I_x \rightarrow I_x^{\mathrm{tame}} = \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \rightarrow 1.$$

Let F_ℓ be a finite extension of $\mathbb{Q}_\ell$, where ℓ is a prime number. Let R_ℓ be the ring of integers in F_ℓ. Let V be a finite dimensional vector space over F_ℓ. Let

$$\rho : G_K \longrightarrow GL(V)$$

be a continuous representation of G_K unramified on U. Equivalently,

$$\rho : \pi_1^{\mathrm{arith}}(U) \longrightarrow GL(V)$$

is a continuous representation of $\pi_1^{\mathrm{arith}}(U)$. The L-function of ρ on U is defined by

$$L(U, \rho) = \prod_{x \in |U|} \frac{1}{\det(I - \rho(\mathrm{Frob}_x) T^{\deg(x)} | V)} \in 1 + TR_\ell[[T]].$$

Similarly, the complete L-function of ρ on C is defined by

$$L(C, \rho) = \prod_{x \in |C|} \frac{1}{\det(I - \rho(\mathrm{Frob}_x) T^{\deg(x)} | V^{I_x})} \in 1 + TR_\ell[[T]],$$

where V^{I_x} is the subspace of V on which I_x acts trivially. These two L-functions differ only by a finite number of Euler factors. We are interested in analytic properties of these L-functions, especially for those representations which come from geometry, that is, arising as the relative ètale cohomology (or its subquotient) with compact support of a family of varieties $f : X \rightarrow U$.

2 ℓ-adic cae: $\ell \neq p$

Since $\ell \neq p$, the restriction of the ℓ-adic representation ρ to P_x is of finite order and thus the representation ρ is almost tame. In fact, by class field theory, ρ itself has finite order up to a twist if ρ has rank one. Thus, there are not too many such ℓ-adic representations. The L-function $L(U, \rho)$ is always a rational function. This follows from Grothendieck's trace formula [7]:

Theorem 2.1 *Let $\mathcal{F}_\rho$ denote the lisse ℓ-adic sheaf on U associated with ρ. Then, there are finite dimensional vector spaces $H_c^i(U \otimes \bar{\mathbb{F}}_q, \mathcal{F}_\rho)$ ($i = 0, 1, 2$) over F_ℓ such that*

$$L(U, \rho) = \prod_{i=0}^{2} \det(I - \mathrm{Frob}_q T | H_c^i(U \otimes \bar{\mathbb{F}}_q, \mathcal{F}_\rho))^{(-1)^{i-1}} \in F_\ell(T).$$

If U is affine, then $H_c^0 = 0$. If ρ does not contain a geometrically trivial component, then $H_c^2 = 0$. Thus, in most cases, it is H_c^1 that is the most interesting.

Corollary 2.2 *Let U be affine. Assume that ρ does not contain a geometrically trivial component. Then, the L-function*

$$L(U, \rho) = \det(I - \mathrm{Frob}_q T | H_c^1(U \otimes \bar{\mathbb{F}}_q, \mathcal{F}_\rho))$$

is a polynomial.

This is the ℓ-adic function field analogue of Artin's entireness conjecture.

Fix an embedding $\iota : \bar{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$. A representation ρ is called ι-pure of weight $w \in \mathbb{R}$ if each eigenvalue of Frob_x acting on V has absolute value $q^{\deg(x)w/2}$ for all $x \in |U|$. A representation ρ is called ι-mixed of weights at most w if each irreducible subquotient of ρ is ι-pure of weights at most w. If ρ is ι-pure of weight w for every embedding ι, then ρ is called pure of weight w. Similarly, if ρ is ι-mixed of weights at most w for every ι, then ρ is called mixed of weights at most w. The fundamental theorem of Deligne [2] on the Weil conjectures implies

Theorem 2.3 *If ρ is geometric, then ρ is mixed with integral weights. Furthermore, if ρ is mixed of weights at most w, then $H_c^i(U \otimes \bar{\mathbb{F}}_q, \mathcal{F}_\rho)$ is mixed of weights at most $w + i$.*

The ℓ-adic function field Langlands conjecture as established by Lafforgue [9] implies

Theorem 2.4 *If ρ is irreducible, then ρ is geometric up to a twist and hence pure of some weight.*

Thus, in the ℓ-adic case with $\ell \neq p$, essentially all ℓ-adic representations are geometric from L-function point of view.

3 p-adic case

In the case $\ell = p$, the restriction of the p-adic representation ρ to P_x can be infinite and thus ρ can be very wildly ramified. The L-function $L(U, \rho)$ is naturally more complicated and cannot be rational in general. One can ask for its p-adic meromorphic continuation. The function $L(U, \rho)$ is trivially p-adic analytic in the open unit disc $|T|_p < 1$ as the coefficients are in the ring R_p. It was shown in [11] that $L(U, \rho)$ is not p-adic meromorphic in general, disproving a conjecture of Katz [8]. However, one can show that $L(U, \rho)$ is p-adic meromorphic on the closed unit disc $|T|_p \leq 1$. Its zeros and poles on the closed unit disc are explained by p-adic ètale cohomology of ρ. This was proved by Emerton-Kisin [4], confirming a conjecture of Katz [8]. That is,

Theorem 3.1 *For any p-adic representation ρ of $\pi_1^{\mathrm{arith}}(U)$, the quotient*

$$\frac{L(U,\rho)}{\prod_{i=0}^{2}\det(I-\mathrm{Frob}_q T|H_c^i(U\otimes\bar{\mathbb{F}}_q,\mathcal{F}_\rho))^{(-1)^{i-1}}}$$

has no zeros and poles on the closed unit disc $|T|_p \leq 1$.

In the case that ρ has rank one, this was first proved by Crew [1]. Note that $H_c^2(U\otimes\bar{\mathbb{F}}_q,\mathcal{F}_\rho)=0$ since U is a curve and $\ell = p$.

For geometric p-adic representation, the following meromorphic continuation was conjectured by Dwork [3] and proved by Wan [12] [13].

Theorem 3.2 *If the p-adic representation ρ is geometric, then the L-function $L(U,\rho)$ is p-adic meromorphic everywhere.*

Unlike the ℓ-adic case, most p-adic representations are not geometric. It seems very difficult to classify geometric p-adic representations, even in the rank one case. This can be viewed as the p-adic function field Langlands program, which is still wide open even for rank one case.

We remark that the Artin entireness conjecture fails for L-functions $L(U,\rho)$ of geometric p-adic representations, even for non-trivial rank one ρ. From the proof in [12], one can understand why this is the case. The geometric p-adic representation ρ is an effective object in the category of geometric p-adic representations, but it ceases to be effective in the larger category of L-functions of infinite rank nuclear overconvergent σ-modules, where the L-function is known to be p-adic meromorphic. It is an virtual object!

For example, if the p-adic representation ρ is of rank one and ρ is the slope zero piece of a rank n geometric ℓ-adic representation ψ_ℓ of $\pi_1^{\mathrm{arith}}(U)$, then

$$L(U,\rho)=\prod_{i=0}^{n}\lim_{k\to\infty}L(U,\mathrm{Sym}^{1+(p-1)p^k-i}\psi_\ell\otimes\wedge^i\psi_\ell)^{(-1)^{i-1}i}.$$

Each of the above limit exists as a formal p-adic power series and they can be viewed as the L-function of some infinite rank ℓ-adic representation with p-adic values. These L-functions of infinite symmetric powers are p-adic entire (Artin conjecture holds), not just p-adic meromorphic. The L-function $L(U,\rho)$ itself is an alternating product of p-adic entire functions and thus it is only p-adic meromorphic. For two important examples with much greater details, see the recent papers by Fu-Wan [5][6] for the n-dimensional Kloosterman case, and by Roja-Leon and Wan [10] for the n-dimensional toric Calabi-Yau hypersurface case.

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